Category Theory and Universes in Explicit Mathematics.

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(work in progress)

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1. Categories in Explicit Mathematics

2. The category of “Sets” and some of its properties

3. A categorical universe translated back to a universe in Explicit Mathematics
Definition (Language)

The language used to write down the axioms for category theory is built from seven symbols

\[ \text{ob}, \text{mor}, \text{id}, \circ, =_o, =_m, \rightarrow \]

and the following abbreviations.

\[ x =_o y \equiv \langle x, y \rangle \in =_o \]
\[ f =_m g \equiv \langle f, g \rangle \in =_m \]
\[ f \circ g \equiv \circ(f, g) \]
Category Theory in EC

Definition (Category)

Let $u$ be a universe. A Category (relative to $u$) is a six-tuple $\langle \text{ob}, \text{mor}, \text{id}, \circ, =_o, =_m \rangle$ which satisfies the following properties (including (UNIV)):

- (CL) $R(\text{ob}) \land R(\text{mor}) \land R(=_o) \land R(=_m)$
- (UNIV) $\text{ob} \in u \land \text{mor} \in u \land =_o \in u \land =_m \in u$
- (MOR) $(\forall m \in \text{mor})(\exists x, y \in \text{ob})(m = \langle x, y, \pi_2 m \rangle)$
- (EQ$_O1$) $(=_o \subset \text{ob} \times \text{ob}) \land (\forall x \in \text{ob})(x =_o x)$
- (EQ$_M1$) $(=_m \subset \text{mor} \times \text{mor}) \land (\forall f \in \text{mor})(f =_m f)$
- (CMP1) $(\forall f, g \in \text{mor})$

\[ (\pi_1 g =_o \pi_0 f \to (f \circ g) \downarrow \land (f \circ g) \in \text{mor}) \]
- (ID1) $(\forall x \in \text{ob})(\text{id}(x) \in \text{mor}$

\[ \land \pi_0 \text{id}(x) =_o x \land \pi_1 \text{id}(x) =_o x) \]
<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EQO2)</td>
<td>$(\forall x, y \in \text{ob})(x =_o y \rightarrow y =_o x)$</td>
</tr>
<tr>
<td>(EQO3)</td>
<td>$(\forall x, y, z \in \text{ob})(x =_o y \land y =_o z \rightarrow x =_o z)$</td>
</tr>
<tr>
<td>(EQM2)</td>
<td>$(\forall f, g \in \text{mor})(f =_m g \rightarrow g =_m f)$</td>
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</tr>
<tr>
<td>(CMP2)</td>
<td>$(\forall f, g, h \in \text{mor})(\pi_1 g =_o \pi_0 f \land \pi_1 h =_o \pi_0 g$ $\rightarrow (f \circ g) \circ h =_m f \circ (g \circ h))$</td>
</tr>
<tr>
<td>(ID2)</td>
<td>$(\forall x \in \text{ob})(\forall f \in \text{mor})(\text{dom}(f) =_o x \rightarrow f \circ \text{id}(x) =_m f)$</td>
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<tr>
<td>(ID3)</td>
<td>$(\forall x \in \text{ob})(\forall f \in \text{mor})(\text{cod}(f) =_o x \rightarrow \text{id}(x) \circ f =_m f)$</td>
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The rest of the usual language of category theory (and other operations) can now be defined on top of this.

\[ \text{dom}(f) \equiv \pi_0 f \]
\[ \text{cod}(f) \equiv \pi_1 f \]
\[ f_F \equiv \pi_2 f \]
\[ f : a \xrightarrow{m} b \equiv f \in \text{mor} \land \text{dom}(f) =_\circ a \land \text{cod}(f) =_\circ b \]
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\[ f : a \rightarrow^m b : \equiv f \in \text{mor} \land \text{dom}(f) =_o a \land \text{cod}(f) =_o b \]

\[ \text{hom}^{EC}(a, b) : \equiv \{ f \mid f : a \rightarrow^m b \} \]
**Definition (Functor)**

In the following we will write $f_o$ for $(\pi_0 f)$ and $f_m$ for $(\pi_1 f)$.

We say a term $f$ is a functor between two categories $\mathcal{C}$ and $\mathcal{D}$ (Notation: $f \in \text{functor}(\mathcal{C}, \mathcal{D})$) if the conjunction of the following properties holds:

\begin{align*}
(F1) & \quad x, y \in \text{ob} \quad \land \quad x =_\circ y \rightarrow f_o(x) =_\circ f_o(y) \\
(F2) & \quad g, h \in \text{mor} \quad \land \quad g =_m h \rightarrow f_m(g) =_m f_o(h) \\
(F3) & \quad g \in \text{mor} \quad \rightarrow \quad \text{dom} \ (f_m(g)) =_\circ f_o(\text{dom}(g)) \\
(F4) & \quad g \in \text{mor} \quad \rightarrow \quad \text{cod} \ (f_m(g)) =_\circ f_o(\text{cod}(g)) \\
(F5) & \quad x \in \text{ob} \quad \rightarrow \quad f_m(id(x)) =_m id \ (f_o(x)) \\
(F6) & \quad g, h \in \text{mor} \quad \land \quad \text{dom}(g) =_\circ \text{cod}(h) \\
& \quad \rightarrow \quad f_m(g \circ h) =_m f_m(g) \circ f_m(h)
\end{align*}

All of this can be written as an elementary formula.
Definition (Functor)

In the following we will write $f_0$ for $(\pi_0 f)$ and $f_m$ for $(\pi_1 f)$.

We say a term $f$ is a functor between two categories $\mathcal{C}$ and $\mathcal{D}$ (Notation: $f \in \text{functor}(\mathcal{C}, \mathcal{D})$) if the conjunction of the following properties holds:

(F1) $x, y \in \text{ob}\mathcal{C} \land x =^\mathcal{C} y \rightarrow f_0(x) =^\mathcal{D} f_0(y)$

(F2) $g, h \in \text{mor}\mathcal{C} \land g =^\mathcal{C} m h \rightarrow f_m(g) =^\mathcal{D} m f_0(h)$

(F3) $g \in \text{mor}\mathcal{C} \rightarrow \text{dom}_\mathcal{D}(f_m(g)) =^\mathcal{D} f_0(\text{dom}(g))$

(F4) $g \in \text{mor}\mathcal{C} \rightarrow \text{cod}_\mathcal{D}(f_m(g)) =^\mathcal{D} f_0(\text{cod}(g))$

(F5) $x \in \text{ob}\mathcal{C} \rightarrow f_m(id(x)) =^\mathcal{D} m id_\mathcal{D}(f_0(x))$

(F6) $g, h \in \text{mor}\mathcal{C} \land \text{dom}(g) =^\mathcal{C} o \text{cod}(h)$

$\rightarrow f_m(g \circ^\mathcal{C} h) =^\mathcal{D} m f_m(g) \circ_\mathcal{D} f_m(h)$

All of this can be written as an elementary formula.
Definition (Natural transformation)

Given two functors \( f, g \in \text{functor}(\mathcal{C}, \mathcal{D}) \) between fixed categories, we call a tuple \( \eta = \langle f, g, \eta_F \rangle \) a natural transformation (Notation: \( \eta \in \text{nat}(\mathcal{C}, \mathcal{D}, f, g) \) or \( \eta : f \Rightarrow g \)) if

\[
\begin{align*}
(NAT\,1) & \quad (\forall x \in \text{ob})(\eta_F(x) : f_o(x) \xrightarrow{m} g_o(x)) \\
(NAT\,2) & \quad (\forall h \in \text{mor})(g_m(h) \circ_D \eta_F(\text{dom}(h))) \\
& \quad =_D m \eta_F(\text{cod}(h)) \circ_D f_m(h)
\end{align*}
\]
Definition (Functor category)

A (covariant) functor category from (fixed) categories $C$ to $D$ is a category $D^C$ defined as

\[ ob : \equiv \text{functor}(C, D) \]

\[ mor : \equiv \sum \sum_{f \in ob \, g \in ob} \text{nat}(C, D, f, g) \]

\[ =_o : \equiv eqo \]

\[ =_m : \equiv eqm \]

\[ id(f) : \equiv \langle f, f, \lambda x. id_D(f_\circ(x)) \rangle \]

\[ (\eta \circ \nu) : \equiv \langle \text{dom}(\nu), \text{cod}(\eta), \lambda x.(\eta_{FX} \circ_D \nu_{FX}) \rangle \]

where

\[ eqo : \equiv \{ \langle f, g \rangle \mid f, g \in ob \}
\quad \land (\forall x \in ob_C)(f_\circ(x) =_o D g_\circ(x))) \]

\[ \land (\forall h \in mor_C)(f_m(h) =_m D g_m(h))) \}\]

\[ eqm : \equiv \{ \langle \nu, \eta \rangle \mid \nu, \eta \in mor \land (\forall x \in ob_C)(\nu_{FX} =_m D \eta_{FX}) \} \]
It is possible to encode (co)cones and (co)limits as pairs $\langle c, p \rangle$ and a term $h$ which uniformly picks the unique map into (out of) the (co)limit.
The category of “Sets”

- We want something which is “close” to the usual category of sets.
- There are several different variants worth considering with different advantages and drawbacks.
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There are several different variants worth considering with different advantages and drawbacks.

1. Objects are class names, morphisms are operations.
2. Bishop Sets. (Sorry Ulrik, it’s Setoid Hell)
3. One of the above with an added choice operator.
We want something which is “close” to the usual category of sets.

There are several different variants worth considering with different advantages and drawbacks.

1. Objects are class names, morphisms are operations.
2. Bishop Sets. (Sorry Ulrik, it’s Setoid Hell)
3. One of the above with an added choice operator.
4. Set theoretic maps with choice operator. But then why even use Explicit Mathematics?
Classes built from elementary comprehension restricted to essentially $\exists$, $\lor$-free formulas. and operations as morphisms.

- built from $(x \in X)$, $x \downarrow$, $(x = y)$, and $\land$, $\rightarrow$, $\forall$
- allows $(AC)$: $(\forall x \in X)\exists y \varphi(x, y) \rightarrow \exists f (\forall x \in X) \varphi(x, fx)$
Variant 2 vs. Variant 3

- Classes built from elementary comprehension restricted to essentially $\exists, \lor$-free formulas. and operations as morphisms.
  - built from $(x \in X), x \downarrow, (x = y)$, and $\land, \rightarrow, \forall$
  - allows (AC) : $(\forall x \in X) \exists y \varphi(x, y) \rightarrow \exists f (\forall x \in X) \varphi(x, fx)$

- Bishop Sets
  - Objects are pairs of names: A carrier and an ordinary equivalence relation (A class of pairs, giving a slight trivialization compared to the usual formulation in TT.)
  - Morphisms are operations which respect relations
  - No explicit transport operation necessary.
  - can use full elementary comprehension.
The category of “Sets”

- Example: Construction of the image of a morphism in Bishop Sets.

\[ \begin{array}{ccc}
  X & \xrightarrow{g} & Y \\
  & \downarrow{ig} & \\
  im(g) & \xrightarrow{j} & z \\
  & \downarrow{n} & \\
  y & \xleftarrow{\tilde{g}} & y. \\
\end{array} \]

- This does not work for classes & operations without added choice.
Image construction

\[ \text{im}(g) \equiv \langle \text{dom}(g), (x \sim_{\text{im}(g)} y) \leftrightarrow (g_F x \sim_{\text{cod}(g)} g_F y) \rangle \]

\[ i_g \equiv \langle \text{dom}(g), \text{im}(g), \lambda x. x \rangle \]

\[ \tilde{g} \equiv \langle \text{im}(g), \text{cod}(g), g_F \rangle \]

\[ j \equiv \langle \text{im}(g), z, h_F \rangle \]
\[
\text{im}(g) \equiv \langle \text{dom}(g), (x \sim_{\text{im}(g)} y) \leftrightarrow (g_F x \sim_{\text{cod}(g)} g_F y) \rangle
\]
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\[
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\]
\[
j \equiv \langle \text{im}(g), z, h_F \rangle
\]

- \(i_g\) is a regular epi.
- the pullback \(k^*i_g\) along any \(k : z \to \text{cod}(g)\) is the coequalizer of its kernel pair.
- Coequalizers of kernel pairs are stable under pullback.
- So in Bishop Sets every morphism has a factorization into some (regular-)epi and a monomorphism.
- coequalizers in general are \textit{not} stable under pullback.
- Bishop Sets are a regular category.
Other properties:

- For all variants we have: LCCC
- Exactness (= Every congruence is a kernel pair) is only possible with choice: Let \( r \xrightarrow{(\partial_0, \partial_1)} x \times x \) be a congruence.

\[ \{ \langle \partial_0 s, \partial_1 s \rangle \mid s \in r \} \]

\[ \pi_1 \]

\[ ? \]

\[ \pi_0 \]

\[ r \xrightarrow{\partial_1} x \]

\[ \xrightarrow{\partial_0} \]

\[ x \xrightarrow{f} y \]

Congruences are, upto renaming, equivalence relations, but we can’t construct an inverse morphism to \( r_0 \xleftarrow{} \langle \partial_0 r_0, \partial_1 r_0 \rangle \) w/o some way to choose an element from the preimage.
Other properties:

- For all variants we have: LCCC
- Exactness (= Every congruence is a kernel pair) is only possible with choice: Let \( (\partial_0, \partial_1) \hookrightarrow x \times x \) be a congruence

\[
\{\langle \partial_0 s, \partial_1 s \rangle \mid s \in r \}
\]

\[
\begin{array}{ccc}
\pi_0 & \pi_1 \\
\downarrow & \downarrow \\
? & r \\
\downarrow & \downarrow \\
\partial_0 & \partial_1 \\
\downarrow & \downarrow \\
x & x \\
\downarrow & \downarrow \\
f & f \\
\downarrow & \downarrow \\
y & y
\end{array}
\]

Congruences are, up to renaming, equivalence relations, but we can’t construct an inverse morphism to \( r_0 \mapsto \langle \partial_0 r_0, \partial_1 r_0 \rangle \) w/o some way to choose an element from the preimage.

- The equivalent notion in MLTT does not suffer from this: Moerdijk, Palmgren (2002). Theorem 12.7: The category \textbf{Sets} [Bishop Sets] is a stratified pseudotopos [...]

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Category Theory and Universes in Explicit Mathematics.
There have been several definitions of universes in categories. (Without any claim to completeness)

Let $C$ be a locally cartesian closed category, $el$ be some morphism in $C$ and $S[x]$ be a formula. We call $S$ a universe in $C$ if the following axioms hold.

\begin{align*}
(U1) & \quad h \in \text{mor} \land S[h] \land f \in \text{mor} \rightarrow (PB[h, f, g, q] \rightarrow S[g]) \\
\begin{tikzcd}
ullet & ullet \\
\downarrow{g} & \downarrow{h} \\
\bullet & \bullet \\
\end{tikzcd} & \quad S[h] \Rightarrow S[g] \\
(U2) & \quad a \in \text{mor} \land \text{MONO}[a] \rightarrow S[a] \\
(U3) & \quad f : b \rightarrow c \land g : a \rightarrow b \land S[f] \land S[g] \rightarrow S[\Sigma f g] \\
(U4) & \quad f : a \rightarrow i \land g : b \rightarrow a \land S[f] \land S[g] \rightarrow S[\Pi f g] \\
(U5) & \quad a \in \text{mor} \land S[a] \rightarrow \exists f, pr_1(f : \text{cod}(a) \rightarrow \text{cod}(el) \land PB[f, el, a, pr_1]) \\
\begin{tikzcd}
ullet & e \\
\downarrow{a} & \downarrow{el} \\
\bullet & u \\
\end{tikzcd}
\end{align*}
Taking $S[x] :\equiv x \in S$, and assuming the existence of $S$ is inconsistent with elementary comprehension.
Taking $S[x] :≡ x ∈ S$, and assuming the existence of $S$ is inconsistent with elementary comprehension.

The problem is ($U2$). Closure under all monos is a very strong condition.

Weakening:

$(U2-W) \quad a ∈ ob → S[Δ(a)]$

We only require diagonals $Δ(a) : a \to a × a$ to be small.
Universe Condition

Definition (Universe Condition)

Given some universe of Explicit Mathematics $u$ and a morphism $f$, we say that $f$ is in $\mathcal{U}$ iff there exist $h, h^{-1}$ and $g$ such that

For all $x \in \operatorname{cod}(f)$

- $g(x) \in u$
- $h(x) : f^{-1}\{x\} \xrightarrow{m} gx$
- $h^{-1}(x) : gx \xrightarrow{m} f^{-1}\{x\}$
- $\operatorname{iso}(h(x), h^{-1}(x))$

“For all preimages of $f$ there is some isomorphic class in $u$.”
Theorem

\[ \mathcal{U} \text{ is closed under } (U1), (U2-W), (U3), (U4), (U5). \]

Where we have for arbitrary \( \mathcal{U} \{ f : a \rightarrow m b, u \} \)

\[
\begin{array}{c}
\begin{array}{c}
a \xrightarrow{\langle \text{cod}(h(f_F x)), h(f_F x) \rangle} \\
\downarrow \quad \downarrow
\end{array} \\
\begin{array}{c}
\sum_{x \in u} x \\
\downarrow \\
g(x) \xrightarrow{el : \equiv pr_0}
\end{array}
\end{array}
\]
Strength

- We would like a direct comparison with universes in Explicit Mathematics.
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This definition has a defect.

Closure under the join axiom on the EM side is not required.
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This definition has a defect.

Closure under the join axiom on the EM side is *not* required.

Possible Fix: (CA) as defined by Joyal, Moerdijk (1995), & Moerdijk, Palmgren (2002)
A commutative square

\[
\begin{array}{ccc}
C & \rightarrow & B \\
\downarrow f & & \downarrow g \\
A & \rightarrow & X
\end{array}
\]

is called a quasi-pullback whenever the canonical map
\[ C \rightarrow A \times_X B \] is an epi.
For any small\* morphism $a \xrightarrow{m} x$ and any epi $c \xrightarrow{m} a$ there exists a quasi-pullback of the form

\[
\begin{array}{ccc}
b & \rightarrow & c & \rightarrow & a \\
\downarrow & & \downarrow & & \downarrow \\
y & \rightarrow & x
\end{array}
\]

where $y \xrightarrow{m} x$ is epi and $b \xrightarrow{m} y$ is small.

(*) : part of $S$
Some Intuition. Let $c \notin S$

Existence of small subcovers.
May or may not hold for $\mathcal{U}$

Might give closure under join only indirectly by interpreting some other system in the internal logic.
Thank You
Pullback stability

\[ h \in \text{mor} \land S[h] \land f \in \text{mor} \rightarrow (PB[h, f, g, q] \rightarrow S[g]) \]

\[
\begin{array}{c}
\bullet \quad \rightarrow \quad \bullet \\
\downarrow{g} \quad \downarrow{h} \\
\bullet \quad \rightarrow \quad \bullet
\end{array}
\]

Descent: In the pullback square above, if \( f \) is epi then

\[ S[h] \leftrightarrow S[g] \]

\[ S[f : a \rightarrow b] \land S[g : a' \rightarrow b'] \rightarrow S[f + g : a + a' \rightarrow b + b'] \]

\[ S[f : b \rightarrow c] \land g : a \rightarrow b \rightarrow (S[f \circ g] \leftrightarrow S[g]) \]