1 Separation of Variables

We consider ODEs of the following form

$$\begin{cases} x' = f(x)g(t) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

Example 1.

$$\begin{cases} x' = \frac{x}{t} \\ x(t_0) = x_0 \quad (t_0 \neq 0) \end{cases}$$

We "compute"

$$\begin{aligned} \frac{dx}{dt} = \frac{x}{t} \Rightarrow \frac{dx}{x} = \frac{dt}{t} \\ \Rightarrow \int \frac{dx}{x} = \int \frac{dt}{t} \end{aligned}$$

Hence $log|x| = log|t| + C \Leftrightarrow x = Ct$

Surely this approach can be formalized.

Theorem 1. *Let $I, J \subset \mathbb{R}$ intervals, $g \in \mathcal{C}(I)$ and $f \in \mathcal{C}(J)$.*
(i) If $f(x_0) = 0$, then $x(t) = x_0$ is the global solution to (1).
(ii) If $f(x_0) \neq 0$ then there exists an open interval $D \subset I$ and a unique solution to (1) *$x \in \mathcal{C}^1(D)$ that satisfies*

$$\int_{x_0}^{x(t)} \frac{d\xi}{f(\xi)} = \int_{t_0}^{t} g(\tau)d\tau \tag{2}$$

Proof. Assume $x : D \to U \subset D$ is a solution and $f(x(t)) \neq 0$. Then we compute

$$\frac{x'(t)}{f(x)} = g(t)$$

Hence by Integration we have

$$\int_{x_0}^{x} \frac{d\xi}{f(\xi)} = \int_{t_0}^{t} \frac{x'(\tau)}{f(x(\tau))} = \int_{t_0}^{t} g(\tau)d\tau \tag{3}$$

So let U be the maximal interval with $x_0 \in U$ and $f(x) \neq 0, x \in U$. With (3) in mind we define (for $x \in U$)

$$F(x) := \int_{x_0}^{x} \frac{d\xi}{f(\xi)} \qquad G(t) := \int_{t_0}^{t} g(\tau)d\tau$$

We immediately see that $F \in \mathcal{C}^1(U)$ and $F'(x) = \frac{1}{f(x)} \neq 0$. Hence F is either strong monotone increasing or decreasing. Thus $F : U \to V := F(U)$ is a bijection with some differentiable inverse $F^{-1} : V \to U$. Furthermore V is open and $0 = F(x_0) \in V$. We define the pre-image $W := G^{-1}[V] \subset I$, which is open and contains t_0 since $G(t_0) = 0$. So finally let D be the biggest interval around t_0 that is contained in W, then i.p. $G(J) \subset V$. We now define our candidate for a solution by

$$x(t) := F^{-1}(G(t)), t \in J \tag{4}$$

Since F^{-1} and G are $\mathcal{C}^1$ so is x. Furthermore

$$x(t_0) = F^{-1}(G(t_0)) = F^{-1}(0) = x_0$$
$$x'(t) = \frac{1}{F'(F^{-1}(G(t)))} G'(t) = f(x)g(t)$$

The uniqueness follows, since (3) holds for any solution and F^{-1} is a bijection. □

2 The easiest PDE

We consider functions $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$. We denote the derivative w.r.t the first component

$$\frac{\partial u}{\partial t}(t, x) = u_t(t, x)$$

Consider the following PDE (the transport equation).

$$(TE) \begin{cases} u_t(t,x) + b(t,x) \cdot \nabla u(t,x) = f(t,x) & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0,x) = g(x) & x \in \mathbb{R}^n \end{cases}$$

We will solve this PDE by reducing it to a system of ODEs (Note that this is a special case of the *methods of characteristics*).

Case 1 We consider the special case

$$b(t,x) = b \in \mathbb{R}^n \quad f(t,x) \equiv 0 \quad g \in \mathcal{C}^1$$

Thus the PDE now reads

$$u_t(t,x) + b \cdot \nabla u(t,x) = 0 \tag{5}$$
$$u(0,x) = g(x) \tag{6}$$

Now (5) just says that the derivative of a solution must vanish in the direction $(1, b)$. Suppose u is a solution and define

$$z(s) := u(t+s, x+sb) \qquad s \in [-t, \infty)$$

Then we can differentiate

$$\dot{z} = u_t(t+s, x+sb) + b \cdot \nabla u(t+s, x+sb) = 0$$

So z is constant. Note that this is expected since $(t+s, x+sb)$ parametrizes the lines through (t,x) in direction $(1,b)$. Furthermore we know that

$$z(-t) = u(0, x-bt) = g(x-bt) z(0) = u(t,x)$$

Hence $u(t,x) = g(x-bt)$.

Remark 1. The solution is constant along all lines $(t+s, x+sb), s \in [-t, \infty)$, so it suffices to know the value for one point on each line. This now also justifies the name "*transport equation*" since the initial condition g is transported along in direction $(1,b)$.

Theorem 2. *(TE) with $f \equiv 0, b(t,x) = b \in \mathbb{R}^n, g \in \mathcal{C}^1$ has a unique solution $u \in \mathcal{C}^1((0,\infty) \times \mathbb{R}^n) \cap \mathcal{C}^0([0,\infty) \times \mathbb{R}^n)$ given by*

$$u(t,x) := g(x-bt)$$

Proof. Existence. Check that $g(x-bt)$ is a solution.
Uniqueness. Assume that u_1, u_2 are solutions and define

$$w := u_1 - u_2$$

w solves the PDE

$$\begin{cases} w_t - b \cdot \nabla u = 0 \\ w(0,x) = 0 \end{cases}$$

Define $z(s) := w(t+s, x+sb)$. As above

$$\dot{z}(s) = 0$$

and

$$\begin{aligned} z(-t) &= w(0, x-bt) = 0 \\ z(0) &= w(t,x) \end{aligned}$$

Hence $w \equiv 0$. □

Case 2 $b(t,\cdot) \in \mathcal{C}^1(\mathbb{R}^n)$ for all $t \in [0,\infty)$. We try to do the same thing as in the first case and look for curves along which the solution is constant. Such a curve is of the following form

$$\Gamma(s) := (s, \gamma(s)) \qquad \gamma : \mathbb{R} \to \mathbb{R}^n$$

For a curve γ and a solution u we have

$$\begin{aligned} &\frac{d}{ds}(u(s,\gamma(s))) \\ &u_t(s,\gamma(s)) + \dot{\gamma}(s) \times \nabla u(s,\gamma(s)) \end{aligned}$$

Hence if γ is a curve through $(t,x) \in (0,\infty) \times \mathbb{R}^n$ then u is constant along γ if and only if

$$\begin{cases} \dot{\gamma}(s) = b(s,\gamma(s)) \\ \gamma(t) = x \end{cases}$$

(This is an ODE!) In that case

$$u(t,x) = u(t,\gamma(t)) = u(0,\gamma(0)) = g(\gamma(0))$$

Theorem 3. *Assume u is a solution of (TE) with $f \equiv 0$ and that for $(t,x) \in (0,\infty)$ γ solves*

$$\begin{cases} \dot{\gamma}(s) = b(s,\gamma(s)) \\ \gamma(t) = x \end{cases}$$

Then it holds that

$$\begin{aligned} &u(t,x) = g(\gamma(0)) \\ &u(t,x) = u(s,\gamma(s)), \forall s \in [0,t] \end{aligned}$$

Remark 2. Suppose Γ is the set of all these solutions γ. Then, as in the first case, the initial condition is transported along Γ. Note that if γ is such a solution and $(t,x) \in \Gamma(s)$ then there might be no s such that $\Gamma(s) \cap \{t = 0\} \neq \emptyset$. That means that the set of curves that intersect the "zero-plane" might not cover all of $[0,\infty) \times \mathbb{R}^n$.

Example 2. (i) $b(t,x) = b \in \mathbb{R}^n$. For $(t,x) \in (0,\infty) \times \mathbb{R}^n$ solve

$$\begin{cases} \dot{\gamma}(s) = b \\ \gamma(t) = x \end{cases}$$

The solution is $\gamma(s) = (x - bt) + s$ so that by Theorem 3

$$u(t,x) = g(\gamma(0)) = g(x - bt)$$

(ii) $b(t,x) = x$. We have to solve

$$\begin{cases} \dot{\gamma}(s) = \gamma(s)\gamma(t) = x \end{cases}$$

The solution is $\gamma(s) = xe^{s-t}$, so by Theorem 3

$$u(t,x) = g(\gamma(0)) = g(xe^{-t})$$

Case 3 We consider the General Transport equation, i.e. $f \in \mathcal{C}^0$.

Theorem 4. *Under the assuptions of the last theorem with $f \in \mathcal{C}^0$ instead of $f \equiv 0$ we have that*

$$u(t,x) = g(\gamma(0)) + \int_0^t f(s,\gamma(s))ds$$

Proof. By an easy computation (since u is a solution):

$$\frac{d}{ds}u(s,\gamma(s)) = u_t(s,\gamma(s)) + \dot{\gamma}(s)\cdot u(s,\gamma(s)) = f(s,\gamma(s))$$

Hence by the fundamental theorem of calculus

$$u(t,x) = u(t,\gamma(t)) = \underbrace{u(0,\gamma(0))}_{=g(\gamma(0))} + \int_0^t f(s,\gamma(s))ds$$

□

Remark 3. Note that we can not directly say the solution is transported along the curves Γ but it is still uniquely determined by them.

Example 3. Consider

$$\begin{cases} u_t + x\cdot\nabla u = t \\ u(0,x) = g(x) \end{cases}$$

By the last example $\gamma(s) = xe^{s-t}$ so by theorem 4

$$u(t,x) = g(xe^{-t}) + \underbrace{\int_0^t s ds}_{=\frac{t^2}{2}}$$