A Yoneda lemma-formulation of the univalence axiom

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Abstract. Rijke has given a type-theoretic formulation of the Yoneda lemma and a proof of it from Martin-Löf's J-rule and the function extensionality axiom. Escardó has derived the J-rule from Rijke's type-theoretic formulation of the Yoneda lemma. Here we give a Yoneda lemma-formulation of Voevodsky's axiom of univalence. Based on the work of Escardó, and applying Coquand's technique of reducing the J-rule to the transport and the contractibility of singleton types, we derive the univalence axiom from its Yoneda lemma-formulation.

1 Introduction

The univalent perspective on the foundations of mathematics, which was based on the homotopic interpretation of Martin-Löf's intensional type theory[1] (ITT) by Voevodsky in [18], Awodey and Warren in [2], and was inspired by Hofmann and Streicher's groupoid interpretation of ITT in [8], reinforced the type-theoretic approach to the subject and brought mathematics and computer science even closer.

Univalent type theory (UTT) is the extension of ITT with Voevodsky's axiom of univalence (`UA`), the most important univalent concept, that reflects the standard mathematical practice of identifying isomorphic objects. According to `UA`, an equivalence between two types generates a proof of their equality. The converse, i.e., the generation of an equivalence between two types from a proof of their equality, follows easily from Martin-Löf's J-rule, the induction principle that corresponds to the inductive definition of equality between the terms of a type, as a new type[2].

[1] See e.g., [11], [12] and [13]. The canonicity property of ITT i.e., the fact that every closed term of the type of natural numbers is reduced to a numeral, makes ITT a programming language. As it is mentioned in [6], this is "a major compelling aspect of ITT compared to non-constructive foundations such as set theory".

[2] One of the key-features of ITT is the use of two kinds of equality for the terms a, b of a type A. The definitional, or judgmental equality $a \equiv b$ expresses that a and b are by definition equal, while the propositional equality $a =_A b$, or simpler $a = b$, is a new type, and every term $p : a =_A b$ is interpreted as a "proof" that a and b are propositionally equal. Through the "rough" homotopic interpretation of ITT, p is a "path" from the point a to point b in the "space" A.

It is well-known that proofs and computations in constructive mathematics rely heavily on the choice of representations. The univalence axiom allows one to identify equivalent definitions and hence makes program and data refinement possible[3]. Moreover, it implies function extensionality[4], and shapes the theories of higher inductive types (HITs), of homotopy n-types and of categories within the univalent framework[5].

Originally, `UA` was motivated by the classical model of simplicial sets, developed in [10]. The computational interpretation of `UA` in cubical sets in [3] is a milestone in the development of the univalent foundations. Types in the cubical model are interpreted as cubical sets, and terms of types, including equality proofs, as n-dimensional cubes (points, lines, squares, cubes etc.). There is also a so-called glueing operation that allows to yield paths from equivalences, and hence proves the univalence axiom. Based on this model, an extension of ITT is formed, which is called cubical type theory (CTT), as every type has a "cubical" structure[6].

The central role of `UA` in the univalent foundations of mathematics and the current position of the univalent foundations in logical studies raises naturally the following question: *how one can explain the central role of* `UA` *in categorical terms*?

Here, we try to answer this question providing a Yoneda-lemma formulation of `UA`. In [15] Rijke gave a type-theoretic formulation of Yoneda lemma and proved it from the J-rule and the function extensionality axiom. In [7] Escardó took Rijke's type-theoretic formulation of Yoneda lemma as primitive and proved the J-rule from it, so that its computation rule holds definitionally. The main results included here are the following.

1. We give a Yoneda lemma-formulation (`sY-UA`) of Voevodsky's `UA`. Although, in contrast to Voevodsky's formulation of `UA`, the computation rules of `sY-UA` hold definitionally, we could have formulated (`sY-UA`) using propositional equality instead.
2. Based on the work of Escardó, and applying Coquand's technique of reducing the J-rule to the transport and the contractibility of singleton types, we

[3] For example, as it is mentioned in the Git repository of the Haskell implementation of cubical type theory (https://github.com/mortberg/cubicaltt), computing is instant for the binary representation of natural numbers, but very time-consuming for the unary representation of a large number. Consequently, proof-checking for large unary numbers is almost impossible. By defining a doubling structure, one can transport a proof from binary to unary numbers, where checking the transported proof for unary numbers is also instant.

[4] According to it, the pointwise equality of two functions generates a proof of their equality. Function extensionality is not provable in ITT.

[5] All these theories are developed in [17]. HITs generalize inductive types, allowing constructors to produce not only elements of the type, but also paths and higher paths i.e., elements of the iterated equality types. Homotopy type theory (HoTT) is, roughly speaking, UTT extended with HITs.

[6] As shown in [9], CTT also satisfies the canonicity property.

prove from sY-UA the principle of equivalence induction with a definitional computation rule (Theorem 2).
3. From sY-UA we prove Voevodsky's UA (Proposition 1, Proposition 2 and Corollary 2).

We work in the informal framework of UTT found in [17]. Our presentation of the Yoneda lemma-formulation of UA is done in two stages. First we introduce a weak Yoneda lemma-formulation of univalence (wY-UA), and then a strong one (sY-UA), in order to fully prove UA from sY-UA.

1.1 A weak Yoneda lemma-formulation of the univalence axiom

In [15] Rijke viewed a type family $P : A \to \mathcal{U}$ over a type $A : \mathcal{U}$ as a presheave of a locally small category $\mathcal{C}$ i.e., as an element of $\mathtt{Set}^{\mathcal{C}^{\mathrm{op}}}$, and gave a type-theoretic version of the Yoneda lemma. Recall that $\mathtt{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the category of contravariant set-valued functors on $\mathcal{C}$, and by the definition of a locally small category $\mathcal{C}$, if $A, B \in \mathcal{C}_0$, where $\mathcal{C}_0$ denotes the objects of $\mathcal{C}$ and $\mathcal{C}_1$ denotes the arrows of $\mathcal{C}$, the collection $\mathrm{Hom}_{\mathcal{C}}(A, B) \equiv \{f \in \mathcal{C}_1 \mid f : A \to B\}$ is a set. According to the Yoneda lemma (see [1], section 8.2), if $\mathcal{C}$ is a locally small category, $C \in \mathcal{C}_0$ and $F \in \mathtt{Set}^{\mathcal{C}^{\mathrm{op}}}$, then there is an isomorphism $\mathrm{Hom}_{\mathtt{Set}^{\mathcal{C}^{\mathrm{op}}}}(\mathcal{Y}(C), F) \simeq F(C)$, which is natural in both F and C, where $\mathcal{Y} : \mathcal{C} \to \mathtt{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the Yoneda embedding i.e., the functor

$$\mathcal{Y}(C) \equiv \mathrm{Hom}_{\mathcal{C}}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathtt{Set}$$

$$\mathcal{Y}(f : C \to C') \equiv \mathrm{Hom}_{\mathcal{C}}(-, f) : \mathrm{Hom}_{\mathcal{C}}(-, C) \to \mathrm{Hom}_{\mathcal{C}}(-, C')$$

defined post-compositionally. Through the Yoneda lemma, the Yoneda embedding is shown to be an embedding i.e., an injective on objects, faithful, and full functor. If one corresponds $\mathcal{U}$ to $\mathtt{Set}$, and of course, the fixed universe of types $\mathcal{U}$ is closed under function types, a type A to $\mathcal{C}^{\mathrm{op}}$, and take $\mathrm{Hom}_A(a, b) \equiv (a =_A b)$, for every $a, b : A$, and if one defines the Yoneda embedding $\mathcal{Y}_a : A \to \mathcal{U}$, where $\mathcal{Y}_a(x) \equiv (x =_A a)$, for every $x : A$, and if for every $P, Q : A \to \mathcal{U}$ one defines

$$\mathrm{Hom}(P, Q) \equiv \prod_{x:A} \big(P(x) \to Q(x)\big),$$

then

$$\mathrm{Hom}(\mathcal{Y}_a, P) \equiv \prod_{x:A} \big(\mathcal{Y}_a \to P(x)\big) \equiv \prod_{x:A} \big((x =_A a) \to P(x)\big) \equiv \prod_{x:A} \prod_{p:x=_A a} P(x).$$

Theorem 1 (Yoneda lemma (YL) in ITT + Function extensionality, Rijke, 2012). *Let $P : A \to \mathcal{U}$ and $a : A$. There is a pair of quasi-inverses (j, i) : $\mathrm{Hom}(\mathcal{Y}_a, P) \simeq P(a)$ i.e., $(j \circ i)(u) = u$, for every $u : P(a)$, and $(i \circ j)(\sigma) = \sigma$, for every $\sigma : \prod_{x:A} \prod_{p:x=_A a} P(x)$ such that*

$$i(u)(a, \mathtt{refl}_a) \equiv u, \quad u : P(a),$$

$$j(\sigma) \equiv \sigma(a, \mathtt{refl}_a), \quad \sigma : \prod_{x:A} \prod_{p:x=_A a} P(x).$$

In [7] Escardó showed that the YL in ITT, taken as primitive, implies the existence of a pair of quasi-inverses $(j, i) : \left(\prod_{x:A}\prod_{p:x=_A a} B\right) \simeq B$, where $B : \mathcal{U}$, such that $i(b)(a, \mathtt{refl}_a) \equiv b$,, for every $b : B$, and $j(\sigma) \equiv \sigma(a, \mathtt{refl}_a)$, for every $\sigma : \prod_{x:A}\prod_{p:x=_A a} B$. Moreover, if $b : B$, $x : A$, and $p : x =_A a$, then $i(b)(x, p) =_B b$. The contractibility of singleton types follows from this fact, and since the transport is an easy consequence of YL, then by Coquand's result in [5], the J-rule follows. Note that Escardó avoided the use of functional extensionality in proving the converse of Theorem 1.

As in the case of Martin-Löf's J-rule, Voevodsky's original formulation of `UA` does not involve the Yoneda lemma. If $A : \mathcal{U}$, and using the J-rule, one shows easily the existence of a term

$$\mathtt{IdtoEqv} : \prod_{X:\mathcal{U}}\prod_{p:X=_{\mathcal{U}}A} X \simeq_{\mathcal{U}} A$$

such that $\mathtt{IdtoEqv}(A, \mathtt{refl}_A) \equiv (\mathrm{id}_A, e_A)$, where $(\mathrm{id}_A, e_A) : A \simeq_{\mathcal{U}} A$, and $A \simeq_{\mathcal{U}} B \equiv \sum_{f:A\to B} \mathtt{isequiv}(f)$ is Voevodsky's notion of equivalence between types A and B in a universe $\mathcal{U}$ (see [17], section 2.4). According to `UA`, there is a term

$$\mathtt{ua} : \prod_{X:\mathcal{U}}\prod_{e:X\simeq_{\mathcal{U}}A} X =_{\mathcal{U}} A$$

such that

$$\mathtt{ua}(X, \mathtt{IdtoEqv}(X, p)) = p, \qquad p : X =_{\mathcal{U}} A,$$

$$[\mathtt{IdtoEqv}(X, \mathtt{ua}(X, e))]^*(x) = e^*(x),$$

for every $x : X$, where if $e : A \simeq_{\mathcal{U}} B$, we write $e \equiv (e^*, e^{**})$ with $e^* : A \to B$ and $e^{**} : \mathtt{isequiv}(e^*)$. Because of the `IdtoEqv`-computation rule we get $\mathtt{ua}(A, (\mathrm{id}_A, e_A)) = \mathtt{refl}_A$. Following the simpler writing found in book-HoTT, we write $\mathtt{ua}(\mathrm{id}_A) = \mathtt{refl}_A$, and $\mathtt{IdtoEqv}(\mathtt{ua}(f), x) = f(x)$.

Extending Rijke's categorical interpretation, we view the fixed universe $\mathcal{U}$ as a category with objects the types in $\mathcal{U}$ and $\mathrm{Hom}_{\mathcal{U}}(A, B) \equiv A \simeq_{\mathcal{U}} B$, which is locally small, since $(A \simeq_{\mathcal{U}} B) : \mathcal{U}$. Clearly, $\mathcal{U}$ can be seen as a category identical to its opposite. If we correspond the successor universe $\mathcal{U}'$ of $\mathcal{U}$ to `Set`, a contravariant functor from $\mathcal{U}$ to $\mathcal{U}'$ is a type family $P : \mathcal{U} \to \mathcal{U}'$. Next we define the corresponding Yoneda functor from $\mathcal{U}$ to ${\mathcal{U}'}^{\mathcal{U}}$.

Definition 1. *Let $A, B : \mathcal{U}$, $e : A \simeq_{\mathcal{U}} B$, $X : \mathcal{U}$, and $\mathcal{Y} : \mathcal{U} \to (\mathcal{U} \to \mathcal{U}')$ be defined by $A \mapsto \mathcal{Y}_A$ and $e \mapsto \mathcal{Y}(e)$, where $\mathcal{Y}_A : \mathcal{U} \to \mathcal{U}$ with*

$$\mathcal{Y}_A(X) \equiv X \simeq_{\mathcal{U}} A,$$

$$\mathcal{Y}(e) : \mathrm{Hom}(\mathcal{Y}_A, \mathcal{Y}_B) \equiv \prod_{X:\mathcal{U}}\prod_{e':X\simeq_{\mathcal{U}}A} X \simeq_{\mathcal{U}} B$$

$$\mathcal{Y}(e) \equiv \lambda(X : \mathcal{U}, e' : X \simeq_{\mathcal{U}} A).e \circ e'.$$

It is immediate to show that for every $\epsilon : B \simeq_{\mathcal{U}} C$ we have that $\mathcal{Y}(\epsilon \circ e) \equiv \mathcal{Y}(\epsilon) \circ \mathcal{Y}(e)$, and $\mathcal{Y}(\mathrm{id}_A) \equiv 1_{\mathcal{Y}_A}$. Next we formulate the first, and weaker version of a Yoneda lemma-formulation of `UA`. Instead of the definitional equality, in the computation rule of the term i we could have used propositional equality.

Weak Yoneda lemma-formulation of the univalence axiom (`wY-UA`): Let $P : \mathcal{U} \to \mathcal{U}'$ and $A : \mathcal{U}$. There is a pair of quasi-inverses

$$(j, i) : \mathrm{Hom}(\mathcal{Y}_A, P) \simeq P(A),$$

where

$$i : P(A) \to \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} P(X) \quad \& \quad j : \left(\prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} P(X) \right) \to P(A)$$

with the following i-computation-rule and j-computation-rule, respectively,

$$i(u)(A, (\mathrm{id}_A, e_A)) \equiv u, \quad u : P(A),$$

$$j(\sigma) \equiv \sigma(A, (\mathrm{id}_A, e_A)), \quad \sigma : \mathrm{Hom}(\mathcal{Y}_A, P).$$

Next we show that the i-term of `wY-UA` "constructs" the `ua`-term of `UA`.

Proposition 1 (`wY-UA`). *There is a term*

$$\mathtt{ua}' : \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} X =_{\mathcal{U}} A,$$

such that $\mathtt{ua}'(A, (\mathrm{id}_A, e_A)) \equiv \mathtt{refl}_A$.

Proof. Let $P : \mathcal{U} \to \mathcal{U}'$ defined by $P(X) \equiv (X =_{\mathcal{U}} A)$. Since

$$i : A =_{\mathcal{U}} A \to \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} X =_{\mathcal{U}} A,$$

we define $\mathtt{ua}' \equiv \lambda(X : \mathcal{U}, e : X \simeq_{\mathcal{U}} A).i(\mathtt{refl}_A)(X, e)$, hence $\mathtt{ua}'(A, (\mathrm{id}_A, e_A)) \equiv i(\mathtt{refl}_A)(A, (\mathrm{id}_A, e_A)) \equiv \mathtt{refl}_A$.

Proposition 2. *If* $X : \mathcal{U}$ *and* $p : X =_{\mathcal{U}} A$*, then*

$$\mathtt{ua}'(X, \mathtt{IdtoEqv}(X, p)) = p.$$

Proof. Let $C(X, p) \equiv \mathtt{ua}'(X, \mathtt{IdtoEqv}(X, p)) = p$. Since

$$\begin{aligned} C(A, \mathtt{refl}_A) &\equiv \mathtt{ua}'(A, \mathtt{IdtoEqv}(A, \mathtt{refl}_A)) = \mathtt{refl}_A \\ &\equiv \mathtt{ua}'(A, (\mathrm{id}_A, e_A)) = \mathtt{refl}_A \\ &\equiv \mathtt{refl}_A = \mathtt{refl}_A, \end{aligned}$$

what we want follows from the J-rule.

Next we show that the $\mathtt{ua}$-term of $\mathtt{UA}$ “constructs” the i-term of $\mathtt{wY\text{-}UA}$

Proposition 3 ($\mathtt{UA}$). *There is a term*

$$i' : P(A) \to \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} P(X),$$

such that $i'(u)(A, (\mathrm{id}_A, e_A)) = u$, *for every* $u : P(A)$.

Proof. Let $u : P(A)$. Since $\mathtt{ua}(X, e) : X =_{\mathcal{U}} A$, we have that $\mathtt{ua}(X, e)^{-1} : A =_{\mathcal{U}} X$, and consequently $\left[\mathtt{ua}(X, e)^{-1}\right]_*^P : P(A) \to P(X)$. We define

$$i'(u) \equiv \lambda(X : \mathcal{U}, e : X \simeq_{\mathcal{U}} A).\left[\mathtt{ua}(X, e)^{-1}\right]_*^P(u).$$

Hence,

$$\begin{aligned} i'(u)(A, (\mathrm{id}_A, e_A)) &\equiv \left[\mathtt{ua}(A, (\mathrm{id}_A, e_A))^{-1}\right]_*^P(u) \\ &= \left(\mathtt{refl}_A^{-1}\right)_*^P(u) \\ &\equiv \left(\mathtt{refl}_A\right)_*^P(u) \\ &\equiv \mathrm{id}_{P(A)}(u) \\ &\equiv u. \end{aligned}$$

As in Voevodsky’s $\mathtt{UA}$, the non-trivial term is the $\mathtt{ua}$-term, in $\mathtt{wY\text{-}UA}$ the non-trivial term is the i-term. The j-term of $\mathtt{wY\text{-}UA}$ is immediate, as the $\mathtt{IdtoEqv}$-term follows immediately from the J-rule.

1.2 A strong Yoneda lemma-formulation of the univalence axiom

Next, we add a natural computation-rule to the weak Yoneda lemma-formulation of univalence in order to get from this strong Yoneda lemma-formulation of univalence the term of J-type that corresponds to the univalence axiom (this is the J_e-term in Theorem 2). Following Escardó’s line of proof of the J-rule from the YL in ITT, and employing, as in the case of Escardó, Coquand’s equivalence of the J-rule to the transport and the contractibility of singleton types, we fully derive Voevodsky’s $\mathtt{UA}$ from this strong Yoneda lemma-formulation of univalence. Note that, as in the weak version $\mathtt{wY\text{-}UA}$, the computation-rules of the strong Yoneda lemma-formulation involve only judgmental equalities, although we could have also used propositional equality. First we define an obvious generalization of the notion of homotopy.

Definition 2. *Let* $A, B : \mathcal{U}$ *and let* $Q : A \to B \to \mathcal{U}'$ *be a type family over* A *and* B (*or a relation on* A, B). *If*

$$F, G : \prod_{x:A} \prod_{y:B} Q(x, y),$$

we say that F, G are homotopic, and we write $F \approx B$, if there is a term

$$H : F \approx B \equiv \prod_{x:A} \prod_{y:B} F(x,y) =_{Q(x,y)} G(x,y).$$

Proposition 4. *Let $A : \mathcal{U}$ and $P : \mathcal{U} \to \mathcal{U}'$. If we fix some*

$$\sigma : \mathrm{Hom}(\mathcal{Y}_A, P) \equiv \prod_{X:\mathcal{U}} \prod_{f:X \simeq A} P(X),$$

there is a term

$$\begin{aligned} \mathtt{Happly}_{\mathcal{Y},\sigma} : & \prod_{\tau \in \mathrm{Hom}(\mathcal{Y}_A,P)} \prod_{p:\tau=\sigma} \tau \approx \sigma \\ \equiv & \prod_{\tau \in \mathrm{Hom}(\mathcal{Y}_A,P)} \prod_{p:\tau=\sigma} \left(\prod_{X:\mathcal{U}} \prod_{f:X \simeq A} \tau(X,f) =_{P(X)} \sigma(X,f) \right) \end{aligned}$$

such that

$$\mathtt{Happly}_{\mathcal{Y},\sigma}(\sigma, \mathtt{refl}_\sigma) \equiv \lambda(X : \mathcal{U}, f : X \simeq A).\mathtt{refl}_{\sigma(X,f)}.$$

Proof. If $C(\tau, p) \equiv \tau \approx \sigma$, then $C(\sigma, \mathtt{refl}_\sigma) \equiv \sigma \approx \sigma$ and $\lambda(X : \mathcal{U}, f : X \simeq A).\mathtt{refl}_{\sigma(X,f)} : C(\sigma, \mathtt{refl}_\sigma)$. What we want follows immediately from the based version of the J-rule (see [17], section 1.12.1, and [14]).

Next, we equip `wY-UA` with an explicit description of the terms of type $i \circ j \sim \mathrm{id}_{\mathrm{Hom}(\mathcal{Y}_A,P)}$ and $j \circ i \sim \mathrm{id}_{P(A)}$.

Strong Yoneda lemma-formulation of the univalence axiom (`sY-UA`): Let $A : \mathcal{U}$ and $P : \mathcal{U} \to \mathcal{U}'$. There is a pair of quasi-inverses

$$(j, i) : \mathrm{Hom}(\mathcal{Y}_A, P) \simeq P(A)$$

where the terms

$$i : P(A) \to \prod_{X:\mathcal{U}} \prod_{f:X \simeq A} P(X) \quad \& \quad j : \left(\prod_{X:\mathcal{U}} \prod_{f:X \simeq A} P(X) \right) \to P(A)$$

are equipped with the following i-computation-rule and j-computation-rule:

$$i(u)(A, \mathrm{id}_A) \equiv u, \quad u : P(A),$$

$$j(\sigma) \equiv \sigma(A, \mathrm{id}_A), \quad \sigma : \mathrm{Hom}(\mathcal{Y}_A, P).$$

Moreover, there are terms

$$G : i \circ j \sim \mathrm{id}_{\mathrm{Hom}(\mathcal{Y}_A,P)} \equiv \prod_{\sigma \in \mathrm{Hom}(\mathcal{Y}_A,P)} i(j(\sigma)) = \sigma,$$

$$H : j \circ i \sim \mathrm{id}_{P(A)} \equiv \prod_{u:P(A)} j(i(u)) = u,$$

equipped with the following G-computation-rule and H-computation-rule[7]:

$$\texttt{Happly}_{\mathcal{Y},\sigma}\big(i(j(\sigma)), G(\sigma)\big)(A, \mathrm{id}_A) \equiv \texttt{refl}_{\sigma(A,\mathrm{id}_A)},$$

$$H(u) \equiv \texttt{refl}_u,$$

for every $\sigma : \mathrm{Hom}(\mathcal{Y}_A, P)$ and for every $u : P(A)$, respectively.

The last two computation-rules, which make the difference between the two Yoneda lemma-formulations of univalence, are justified as follows. Since $G(\sigma) : i(j(\sigma)) = \sigma$, we have that $\texttt{Happly}_{\mathcal{Y},\sigma}\big(i(j(\sigma)), G(\sigma)\big)$ is a term of type

$$\prod_{X:\mathcal{U}} \prod_{f:X \simeq A} i(j(\sigma))(X, f) =_{P(X)} \sigma(X, f),$$

hence $\texttt{Happly}_{\mathcal{Y},\sigma}\big(i(j(\sigma)), G(\sigma)\big)(A, \mathrm{id}_A)$ is a term of type $i(j(\sigma))(A, \mathrm{id}_A) =_{P(A)} \sigma(A, \mathrm{id}_A)$. By the j, i-computation-rules we have that

$$i(j(\sigma))(A, \mathrm{id}_A) \equiv i(\sigma(A, \mathrm{id}_A))(A, \mathrm{id}_A) \equiv \sigma(A, \mathrm{id}_A),$$

therefore $\texttt{Happly}_{\mathcal{Y},\sigma}\big(i(j(\sigma)), G(\sigma)\big)(A, \mathrm{id}_A)$ is a term of type $\sigma(A, \mathrm{id}_A) =_{P(A)} \sigma(A, \mathrm{id}_A)$. Similarly, if $u : P(A)$, we have that $H(u) : j(i(u)) = u$, and since $j(i(u)) \equiv i(u)(A, \mathrm{id}_A) \equiv u$, we get $H(u) : u =_{P(A)} u$. It is natural to demand, as is the case in all axioms of this kind, like the J-rule, that the terms associated to these computation-rules are the most expected ones. Note though, that the H-computation-rule is not significant, while the G-computation-rule makes the whole difference between the two Yoneda lemma-formulations of univalence. Next follows the (strong) analogue to a lemma of Escardó in [7].

Lemma 1 (**sY-UA**). *If $B : \mathcal{U}'$, there are terms*

$$i_B : B \to \prod_{X:\mathcal{U}} \prod_{f:X \simeq A} B \quad \& \quad j_B : \left(\prod_{X:\mathcal{U}} \prod_{f:X \simeq A} B \right) \to B,$$

such that

$$i_B(b)(A, \mathrm{id}_A) \equiv b, \quad b : B,$$

$$j_B(\sigma) \equiv \sigma(A, \mathrm{id}_A), \quad \sigma : \prod_{X:\mathcal{U}} \prod_{f:X \simeq A} B,$$

and terms

$$G_B : \prod_{\sigma \in \mathrm{Hom}(\mathcal{E}_A, B)} i_B(j_B(\sigma)) = \sigma \quad \& \quad H_B : \prod_{b:B} j(i(b)) = b,$$

[7] If we had used propositional equality in the formulation of the i-rule, we could have also written write the G-computation-rule using propositional equality.

such that

$$\texttt{Happly}_{\mathcal{Y},\sigma}\Big(i_B(j_B(\sigma)), G_B(\sigma)\Big)(A, \mathrm{id}_A) \equiv \texttt{refl}_{\sigma(A,\mathrm{id}_A)}, \qquad \sigma : \mathrm{Hom}(\mathcal{Y}_A, B)$$

$$H_B(b) \equiv \texttt{refl}_b, \qquad b : B.$$

Moreover, if $b : B$*,* $X : \mathcal{U}$*,* $f : X \simeq A$*, and*

$$\big[\sigma_b \equiv \lambda(X : \mathcal{U}, f : X \simeq A).b\big] : \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} B,$$

then

$$\Big\{\texttt{Happly}_{\mathcal{Y},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)(X, f)\Big\} \;:\; \big[i_B(b)(X, f) =_B b\big]$$

such that

$$\texttt{Happly}_{\mathcal{Y},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)(A, \mathrm{id}_A) \equiv \texttt{refl}_b.$$

Proof. We apply `sY-UA` on the constant type family $P : \mathcal{U} \to \mathcal{U}'$, defined by $P(X) \equiv B$, for every $X : \mathcal{U}$. To show $i_B(b)(X, f) =_B b$ we work as follows. Since $i_B(j_B(\sigma_b)) \equiv i_B(\sigma_b(A, \mathrm{id}_A)) \equiv i_B(b)$, we get $G_B(\sigma_b) : i_B(b) = \sigma_b$. Since $\texttt{Happly}_{\mathcal{Y},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)$ is of type

$$\prod_{X:\mathcal{U}} \prod_{f:X\simeq A} i_B(j_B(\sigma_b))(X, f) =_B \sigma_b(X, f),$$

the term $\texttt{Happly}_{\mathcal{Y},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)(X, f)$ is of type

$$i_B(b)(X, f) =_B \sigma_b(X, f) \equiv i_B(b)(X, f) =_B b.$$

Consequently, the term

$$\texttt{Happly}_{\mathcal{Y},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)(A, \mathrm{id}_A)$$

is of type $i_B(b)(A, \mathrm{id}_A) =_B \sigma_b(A, \mathrm{id}_A) \equiv b =_B b$, and by the G-computation-rule of `sY-UA` we get

$$\texttt{Happly}_{\mathcal{Y},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)(A, \mathrm{id}_A) \equiv \texttt{refl}_{\sigma_b(A,\mathrm{id}_A)} \equiv \texttt{refl}_b.$$

The next corllary explains why we need to use the successor universe $\mathcal{U}'$ of $\mathcal{U}$ in the formulation of `sY-UA`.

Corollary 1 (`sY-UA`)**.** *If* $E_A \equiv \sum_{X:\mathcal{U}} X \simeq_{\mathcal{U}} A$*, there is a term*

$$M_e : \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} (X,f) =_{E_A} (A, \mathrm{id}_A),$$

such that $M_e(A, \mathrm{id}_A) \equiv \mathtt{refl}_{(A,\mathrm{id}_A)}$.

Proof. By Lemma 1 there is a pair of quasi-inverses

$$(j_{E_A}, i_{E_A}) : \left(\prod_{X:\mathcal{U}} \prod_{f:X\simeq A} E_A\right) \simeq E_A$$

with the associated terms and computation rules. Let $\tau : \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} E_A$, where $\tau \equiv \lambda(X:\mathcal{U}, f:X\simeq A).(X,f)$. We have that

$$\begin{aligned}(X,f) &\equiv \tau(X,f)\\ &=_{E_A} \left[i_{E_A}(j_{E_A}(\tau))\right](X,f)\\ &\equiv \left[i_{E_A}(\tau(A,\mathrm{id}_A))\right](X,f)\\ &\equiv \left[i_{E_A}((A,\mathrm{id}_A))\right](X,f)\\ &=_{E_A} (A,\mathrm{id}_A).\end{aligned}$$

Since the term

$$\mathtt{Happly}_{\mathcal{Y},\tau}\Big(i_{E_A}(j_{E_A}(\tau)), G_{E_A}(\tau)\Big)(X,f)$$

is of type $\left[i_{E_A}(j_{E_A}(\tau))\right](X,f) = \tau(X,f)$, we have that the term

$$\left[\mathtt{Happly}_{\mathcal{Y},\tau}\Big(i_{E_A}(j_{E_A}(\tau)), G_{E_A}(\tau)\Big)(X,f)\right]^{-1}$$

is of type $\tau(X,f) = \left[i_{E_A}(j_{E_A}(\tau))\right](X,f)$. Since the term

$$\mathtt{Happly}_{\mathcal{Y},\sigma_{(A,\mathrm{id}_A)}}\Big(i_{E_A}(j_{E_A}(\sigma_{(A,\mathrm{id}_A)})), G_{E_A}(\sigma_{(A,\mathrm{id}_A)})\Big)(X,f)$$

is of type

$$\left[i_{E_A}((A,\mathrm{id}_A))\right](X,f) =_{E_A} (A,\mathrm{id}_A),$$

we can define the required term of type $(X,f) =_{E_A} (A,\mathrm{id}_A)$ by

$$M_e(X,f) \equiv \left[\mathtt{Happly}_{\mathcal{Y},\tau}\Big(i_{E_A}(j_{E_A}(\tau)), G_{E_A}(\tau)\Big)(X,f)\right]^{-1} *$$

$$\mathtt{Happly}_{\mathcal{Y},\sigma_{(A,\mathrm{id}_A)}}\Big(i_{E_A}(j_{E_A}(\sigma_{(A,\mathrm{id}_A)})), G_{E_A}(\sigma_{(A,\mathrm{id}_A)})\Big)(X,f).$$

Consequently,

$$M_e(A,\mathrm{id}_A) \equiv \left[\mathtt{Happly}_{\mathcal{Y},\tau}\Big(i_{E_A}(j_{E_A}(\tau)), G_{E_A}(\tau)\Big)(A,\mathrm{id}_A)\right]^{-1} *$$

$$\texttt{Happly}_{\mathcal{Y},\sigma_{(A,\mathrm{id}_A)}}\Big(i_{E_A}(j_{E_A}(\sigma_{(A,\mathrm{id}_A)})), G_{E_A}(\sigma_{(A,\mathrm{id}_A)})\Big)(A,\mathrm{id}_A)$$

$$\begin{aligned}
&\equiv \left[\texttt{refl}_{\tau(A,\mathrm{id}_A)}\right]^{-1} * \texttt{refl}_{(A,\mathrm{id}_A)}\\
&\equiv \left[\texttt{refl}_{(A,\mathrm{id}_A)}\right]^{-1} * \texttt{refl}_{(A,\mathrm{id}_A)}\\
&\equiv \texttt{refl}_{(A,\mathrm{id}_A)} * \texttt{refl}_{(A,\mathrm{id}_A)}\\
&\equiv \texttt{refl}_{(A,\mathrm{id}_A)}.
\end{aligned}$$

Next we describe the J-rule, actually its based version, that corresponds to `sY-UA`. Note that `wY-UA` also implies the corresponding term, but we need `sY-UA` to get its computation-rule.

Theorem 2 **(`sY-UA`).** *If $A:\mathcal{U}$, there is a term*

$$J_e : \prod_{C:\prod_{X:\mathcal{U}}\prod_{f:X\simeq A}\mathcal{U}}\ \prod_{c:C(A,\mathrm{id}_A)}\Big(\prod_{X:\mathcal{U}}\prod_{f:X\simeq A} C(X,f)\Big)$$

such that $J_e(C,c,A,\mathrm{id}_A)\equiv c$, for every $C:\prod_{X:\mathcal{U}}\prod_{f:X\simeq A}\mathcal{U}$ and $c:C(A,\mathrm{id}_A)$.

Proof. We fix $C:\prod_{X:\mathcal{U}}\prod_{f:X\simeq A}\mathcal{U}$ and $c\in C(A,\mathrm{id}_A)$. Let $E_A\equiv\sum_{X:\mathcal{U}} X\simeq A$, and $P:E_A\to\mathcal{U}$, defined by

$$P((X,f))\equiv C(X,f),$$

for every $X:\mathcal{U}$ and $f:X\simeq A$. By Corollary 1

$$M_e(X,f):(X,f)=_{E_A}(A,\mathrm{id}_A),$$

hence

$$M_e(X,f)^{-1}:(A,\mathrm{id}_A)=_{E_A}(X,f).$$

Consequently

$$\left[M_e(X,f)^{-1}\right]_*^P : P((A,\mathrm{id}_A))\to P((X,f))$$

i.e.,

$$\left[M_e(X,f)^{-1}\right]_*^P : C(A,\mathrm{id}_A)\to C(X,f).$$

We define

$$J_e(C,c,X,f)\equiv\left[M_e(X,f)^{-1}\right]_*^P(c).$$

By Corollary 1 we get

$$\begin{aligned}
J_e(C,c,A,\mathrm{id}_A) &\equiv \left[M_e(A,\mathrm{id}_A)^{-1}\right]_*^P(c)\\
&\equiv \left[(\texttt{refl}_{(A,\mathrm{id}_A)})^{-1}\right]_*^P(c)\\
&\equiv \left[\texttt{refl}_{(A,\mathrm{id}_A)}\right]_*^P(c)\\
&\equiv \mathrm{id}_{P((A,\mathrm{id}_A))}(c)\\
&\equiv \mathrm{id}_{C(A,\mathrm{id}_A)}(c)\\
&\equiv c.
\end{aligned}$$

The term J_e with its computation-rule is the inductive, or "type-theoretic" version of UA, while sY-UA is the "categorical" version of UA. In the book-HoTT (Corollary 5.8.5) the term J_e is constructed from UA, but its computation-rule involves propositional equality. Next, we show that the univalence function

$$\mathtt{ua}' \equiv \lambda(X : \mathcal{U}, f : X \simeq A).i(\mathtt{refl}_A)(X, f)$$

defined in the proof of Proposition 1, and for which we know that $\mathtt{ua}'(A, \mathrm{id}_A) \equiv \mathtt{refl}_A$, satisfies, in the context of sY-UA, also the second computation-rule of Voevodsky's UA.

Corollary 2. *If* $f : X \simeq A$*, then* $\mathtt{IdtoEqv}(X, \mathtt{ua}'(X, f)) = f$.

Proof. We define $C(X, f) \equiv \mathtt{IdtoEqv}(X, \mathtt{ua}'(X, f)) = f$. Since

$$\begin{aligned} C(A, \mathrm{id}_A) &\equiv \mathtt{IdtoEqv}(A, \mathtt{ua}'(A, \mathrm{id}_A)) = \mathrm{id}_A \\ &\equiv \mathtt{IdtoEqv}(A, \mathtt{refl}_A) = \mathrm{id}_A \\ &\equiv \mathrm{id}_A = \mathrm{id}_A, \end{aligned}$$

we have that $\mathtt{refl}_{\mathrm{id}_A} : C(A, \mathrm{id}_A)$, and we use Theorem 2.

2 Concluding remarks

As we have already stressed, although our strong Yoneda lemma-formulation of univalence supports the definitional approach to the computation-rules associated to the judgements of type theory, we could have formulated sY-UA using propositional equality instead[8].

Here we showed the proximity of UA to the J-rule also from a categorical point of view. Both admit a Yoneda lemma-formulation. The centrality of the Yoneda lemma in category theory, together with the deep connections between Martin-Löf type theory and category theory, and between homotopy type theory and $(\infty, 1)$-category theory[9], maybe sheds some light to the centrality of UA in univalent foundations. It is an open question, and hopefully future work, if this connection between the Yoneda lemma and the univalence axiom can be used in order to translate some non-trivial applications of the Yoneda lemma[10] into homotopy type theory.

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[8] It is an open question whether there is a model of univalent type theory where the computation-rule of UA holds definitionally. In cubical type theory the computation-rule of UA involves propositional equality (see [3]), and this is why in [4] the computation-rule of the J-rule involves also propositional equality.

[9] See e.g., [16] for a general discussion, and [17] for many concrete examples of these connections.

[10] Such non-trivial applications are e.g., the Tannaka duality and the Isbell conjugation.

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