On the Constructive Elementary Theory of the Category of Sets

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Contents

1 Introduction

The Elementary Theory of the Category of Sets (ETCS) was first introduced by William Lawvere in [4] in 1964 to give an axiomatization of sets.
The goal of this thesis is to describe the Constructive Elementary Theory of the Category of Sets (CETCS), following its presentation by Erik Palmgren in [2].
In chapter 2. we discuss basic elements of Category Theory. Category Theory was first formulated in the year 1945 by Eilenberg and Mac Lane in their paper "General theory of natural equivalences" and is the study of generalized functions, called arrows, in an abstract algebra. Hence the comparison to "archery" describes the art and the beauty of this field of mathematics fittingly. Category Theory is considered as a very abstract field in mathematics and was therefore often overlooked by many mathematicians in the past. However, already in the late 1940s many principles of Category Theory were applied in mathematics, like algebraic topology and abstract algebra. Today it is not only used in mathematical fields, but also in computer science, linguistics and philosophy. The Category Theory explained in chapter 2. is meant to prepare for the study of the CETCS.
In chapter 3. we present the constructive approach to Lawvere's ETCS. Unlike ETCS, CETCS is formulated in intuitionistic logic instead of classical logic.
All notions and results in our thesis are found in [1] and [2]. Namely, all categorical notions are found in [1] and all notions related to the CETCS are found in [2].

2 Elements of basic Category Theory

2.1 The category Set

As an introduction we want to look at a simple concept, which we have encountered several times in basic mathematics lectures or even in school.

Consider two sets A and B and a function f from A to B denoted by

$$f : A \longrightarrow B.$$

As usual, we will call A the *domain* of f, denoted by $A = dom(f)$ and B the *codomain* of f, denoted by $B = cod(f)$.

Consider the functions $g : B \longrightarrow C$ and $h : C \longrightarrow D$. Now we can define the *composition* of f and g, denoted by $g \circ f : A \longrightarrow C$, for every $x \in A$

$$(g \circ f)(x) = g(f(x)).$$

These functions can be visualized by the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

We can also observe that for every set A there exists the identity function

$$id_A : A \longrightarrow A,$$

given by

$$id_A(x) = x.$$

Finally we note, that the functions satisfy following rules:

- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$
- Unit law: $f \circ id_A = f = id_B \circ f$

Now we identify the sets as objects and the arrows as functions between sets. In category theory we denote this structure as the category of sets **Set**. The aim of categories is to generalize this concept of sets and function and view it as an abstract algebra.

2.2 Basic definitions

Definition 2.1 (Category)**.** *A category $\boldsymbol{C}$ consists of:*

- *a collection of objects C_0 denoted as $A, B, C, \dots$*
- *a collection of arrows or morphisms C_1 denoted as $f, g, h, \dots$*
- *for a given arrow $f : A \longrightarrow B$ there exist two objects*

  $$dom(f) = A \text{ and } cod(f) = B,$$

  called domain and codomain.
- *a collection of compositions C_2. An element h of C_2 can be represented by two arrows $f : A \longrightarrow B$ and $g : B \longrightarrow C$ with $cod(f) = dom(g)$ and is denoted as*

  $$h = g \circ f : A \longrightarrow C$$

  .
- *an identity arrow given by*

  $$1_A : A \longrightarrow A,$$

  for every A.

Furthermore, $\boldsymbol{C}$ satisfies following properties:

- *Unit law: for given $f : A \longrightarrow B$*

  $$f \circ 1_A = f = 1_B \circ f$$

  $$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[rr, bend right, "f"'] & A \arrow[r, "f"] \arrow[rr, bend left, "f"] & B \arrow[r, "1_B"] & B \end{tikzcd}$$

- *Associativity:*

  $$h \circ (g \circ f) = (h \circ g) \circ f$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend left=40, "(h\circ g)\circ f"] \arrow[rrr, bend right=40, "h\circ(g\circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & C \arrow[r, "h"] & D
\end{tikzcd}$$

Definition 2.2 (Functor)**.** *Consider two categories* $\boldsymbol{C}$ *and* $\boldsymbol{D}$*. A mapping*

$$F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$$

is called functor, if it satisfies following properties:

- $F(f : A \longrightarrow B) = F(f) : F(A) \longrightarrow F(B)$
- $F(1_A) = 1_{F(A)}$
- $F(f \circ g) = F(f) \circ F(g)$.

Definition 2.3 (Equalizer)**.** *Consider a category* $\boldsymbol{C}$*. An equalizer consists of the following data:*

- *two parallel arrows* $f, g : A \longrightarrow B$
- *an arrow* $e : E \longrightarrow A$ *such that*

  $$f \circ e = g \circ e$$

  and for a given arrow $z : Z \longrightarrow A$ *following universal property is satisfied: there exists a unique arrow* $u : Z \longrightarrow E$ *such that following diagram commutes:*

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"] & &
\end{tikzcd}$$

Definition 2.4 (Pullback)**.** *Consider a category* $\boldsymbol{C}$ *and two arrows* $f : A \longrightarrow C$ *and* $g : B \longrightarrow C$*. A pullback consists of following data:*

- *two arrows* $p_1 : P \longrightarrow A$ *and* $p_2 : P \longrightarrow B$.
- *given arrows* $z_1 : Z \longrightarrow A$ *and* $z_2 : Z \longrightarrow B$ *with* $fz_1 = gz_2$

Then there exists a unique arrow $u : Z \longrightarrow P$ with $z_1 = p_1 u$ and $z_2 = p_2 u$, as shown in the following diagram:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

2.3 Basic properties of Set

2.3.1 Epis and monos

Recall the definition of an injective and surjective function. We now want to discuss these properties in the context of categories:

Definition 2.5 (Epis and Monos)**.** *Consider a category **C** and an arrow $f : A \longrightarrow B$. We call f*

- *a monomorphism (or mono), if for all arrows $g, h : C \longrightarrow A$ with $fg = fh$ implies $g = h$. We will denote a mono as: $f : A \rightarrowtail B$*
- *an epimorphism (or epi), if for all arrows $i, j : B \longrightarrow D$ with $if = jf$ implies $i = j$.*

To visualize these definitions we often use diagrams to avoid losing track of domain and codomain, while working with many arrows. In the case of monomorphisms we obtain following diagram:

$$\begin{tikzcd}
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$

It is easy to view these definitions as an abstract concept of injective and surjective functions. In the category **Set** these definitions are equivalent.

Theorem 2.6. *In **Set** a function $f : A \longrightarrow B$ is injective if and only if f is a monomorphism.*

Proof. To prove that f is a monomorphism, assume f is an injection, i.e for every $x, y \in A$ and $f(x) = f(y)$ implies $x = y$. Consider two function

$$g, h : C \longrightarrow A$$

Then

$$\begin{gathered}(f \circ g)(x) = (f \circ h)(x) \\ f(g(x)) = f(h(x))\end{gathered}$$

holds. Since f is injective, it follows that $g(x) = h(x)$ and since x is arbitrary, we conclude $g = h$. Hence f is a mono.
Now suppose f is a mono. Without loss of generality $|A| > 1$, because if $|A| = 1$ it satisfies injectivity of f trivially.
Now we choose $a, a' \in A$ with $a \neq a'$. We want to show $f(a) \neq f(a')$, thus implying the injectivity of f. Consider the functions defined as follows:

$$g, h : \{x\} \longrightarrow A \text{ given by } g(x) = a \text{ and } h(x) = a',$$

where $\{x\}$ is any set containing only one element x. Since $g \neq h$ and f is a monomorphism, $fg \neq fh$. Together we get:

$$f(a) = f(g(x)) \neq f(h(x)) = f(a'),$$

which proves that f is injective. $\square$

Following Palmgren in [2] we can also generalize the notion of a monic arrows.
Suppose a sequence of mappings $r_1 : R \longrightarrow X_1, \ldots, R \longrightarrow X_n$. This sequence is called *jointly monic*, if and only if for any $f, g : A \longrightarrow R$ the following holds:

$$r_1 f = r_1 g, \ldots, r_n f = r_n g \text{ implies } f = g.$$

Note that f and g being jointly monic does not imply that f and g are monic. We will use the same notation $(r_1, \ldots, r_n) : R \rightarrowtail (X_1, \ldots, X_n)$

After proving the equivalence above it is natural to assume the following:

Theorem 2.7. *In* ***Set*** *a function $f : A \longrightarrow B$ is surjective if and only if f is an epimorphism.*

Proof. Let $f : A \longrightarrow B$ be an epimorphism. We want to show by contradiction that f is surjective. Assume that f is not surjective. We now can choose $b \in B$ such that for all $x \in A$: $f(x) \neq b$. Define the functions g, h as follows:

$$g : B \longrightarrow \{y, y'\} \text{ given by } g(x) = y \text{ for all } x \in A \text{ and}$$

and

$$h : B \longrightarrow \{y, y'\}, \text{ given by } h(x) = \begin{cases} y' & x = b \\ y & \text{otherwise} \end{cases}$$

where $\{y, y'\}$ is any given set with two elements. Obviously $g \neq h$, but $(g \circ f) = (h \circ f)$ because b is not in the range of f. This is a contradiction to f being an epimorphism, hence f is surjective.

Now suppose f is surjective, i.e for all $b \in B$ there exists an $a \in A$ such that $f(a) = b$. Suppose two functions $g, h : B \longrightarrow C$ such that $g \neq h$. Then there exists a $b \in B$ such that $g(b) \neq h(b)$. Together we get $g(f(a)) \neq h(f(a))$ and $(g \circ f)(a) \neq (h \circ f)(a)$ proving that f is an epi.

□

2.3.2 Elements as arrows

In contrast to other mathematical fields, in category theory we do not use the ϵ-relations. Instead, we try describe the ϵ-defined notion with the arrows of categories.

In **Set** we can even identify an element of an object (an element of a set in **Set**). In order to discuss this, we define following terms:

Definition 2.8 (Initial and terminal objects)**.** *Consider a category* $\boldsymbol{C}$*. An object*

- *0 is called initial, if for every object* C *there exists a unique arrow* $f : 0 \longrightarrow C$
- *1 is called terminal, if for every object* C *there exists a unique arrow* $f : C \longrightarrow 1$

In **Set** it is easy to see that any singleton set, i.e. a set with only one element, is terminal. For every set A there exists only one way to map all $x \in A$ to any singleton set $\{*\}$ given by the function f, which maps all elements to $*$.

The empty set $\{\emptyset\}$ is the initial object in **Set**. Now we want to show, how we can view an element in **Set** as an arrow.

Remark 2.9. Consider the category **Set** with a terminal object $1 = \{*\}$. Then we can identify every element of a set as a function.

Proof. Let a be an element of a set A. Define the function $f : 1 \longrightarrow A$ given by $f(*) = a$. Hence we can identify a as the function f.
Similarly, if we have the function $f : 1 \longrightarrow A$, we can identify the element a as $f(*)$. $\square$

2.3.3 Binary relations as monic arrows

Recall the term *jointly monic*. Palmgren [2] considers a jointly monic sequence of mappings $r_1 : R \longrightarrow X_1, \ldots, R \longrightarrow X_n$ as an n-ary relation between the objects $X_1, \ldots, X_n$. For simplicity we will elaborate this for binary relations. The generalization can be shown analogously.
In **Set**, a relation on a single set B_1 can be identified as a subset of B_1:
Let an injective function $f : A \longrightarrow B_1$. Then the range of f $Rng(f) = \{f(a) : a \in A\}$ is a subset of B_1. Define the relation $R_f = \big\{f(a) : a \in A\big\} \subseteq B_1$. We call this a 1-ary relation of B_1.

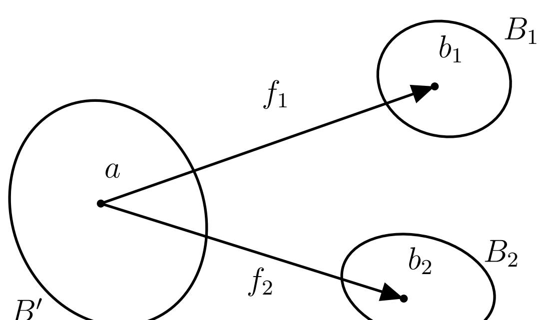
Now consider the sets B_1 and B_2. Let $b_1 \in B_1$, $b_2 \in B_2$ and $b_1 \sim b_2$, i.e b_1 is in relation to b_2. Define the set $B' = \{(b_1, b_2) \in B_1 \times B_2 : b_1 \sim b_2\}$. We want to show that $b_1 \sim b_2$ if and only if there exists an element $a \in B'$ such that $b_1 = f_1(a), b_2 = f_2(a)$ and f_1, f_2 injective.
Assume $b_1 \sim b_2$. Define f_1, f_2 as follows:

$$f_1 : B' \longrightarrow B_1 : \text{, given by } f_1(b_1, b_2) = b_1$$

and

$$f_2 : B' \longrightarrow B_2 : \text{, given by } f_2(b_1, b_2) = b_2.$$

It is easy to see that these functions are injections.



Define the set $R_{f_1,f_2} := \big\{(f_1(a), f_2(a) : a \in B'\big\}$. We want to show that there exists an element $a \in B'$ such that $b_1 = f_1(a)$ and $b_2 = f_2(a)$. To show this, we will show that the set B' of all binary relation between the sets B_1 and B_2 is equal to R_{f_1,f_2} :

$$\begin{aligned} R_{f_1,f_2} &= \{(f_1(a), f_2(a) : a \in B'\} \\ &= \{(f_1(b_1, b_2), f_2(b_1, b_2) : (b_1, b_2) \in B'\} \\ &= \{(b_1, b_2) : (b_1, b_2) \in B'\} = B' \end{aligned}$$

By this, we have shown that a binary relation $B' \subseteq B_1 \times B_2$ generates a set R_{f_1,f_2} and two injective functions f_1 and f_2, such that $B' = R_{f_1,f_2}$.
Now suppose there exists $a \in A$ and two injective functions

$$f_1 : A \longrightarrow B_1 \text{ and } f_2 : A \longrightarrow B_2$$

such that $f_1(a) = b_1$ and $f_2(a) = b_2$. Define the relation $b_1 \sim b_2$, if and only if there exists an $a \in A$ with $f_1(a) = b_1$ and $f_2(a) = b_2$. The set of all pairs (b_1, b_2) in relation is defined as

$$R_{f_1,f_2} := \Big\{(f_1(a), f_2(a)) : \exists!\, a \in A\Big\} \subseteq B_1 \times B_2.$$

So we showed that two injective functions generate a relation $R_{f_1,f_2} \subseteq B_1 \times B_2$ and together with the first part showing the equivalence.

2.3.4 Coequalizers as quotient sets

Consider a set A and a binary relation $\sim$ with following properties:

- reflexivity, i.e for all $a \in A : a \sim a$
- symmetry, i.e for all $a, b \in A$: $a \sim b$ implies $b \sim a$
- transitivity, i.e for all $a, b, c \in A$: $a \sim b$ and $b \sim c$ implies $a \sim c$

Then we call $\sim$ an *equivalence relation* on A. Now define the *equivalence class* as follows:

$$[a] = \Big\{b \in A : a \sim b\Big\}.$$

The set of all equivalence classes is called *quotient set* and is defined as:

$$A/\sim = \Big\{[a] : a \in A\Big\}.$$

Further define the canonical projection

$$\pi : A \longrightarrow A/\sim,\ a \mapsto [a]$$

and a function $f' : A \longrightarrow B$, which preserves the equivalence relation, i.e. $a \sim b$ implies $f(a) = f(b)$. Our aim is to find a function $g : A/\sim \longrightarrow B$ such that following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dr, "f"'] & A/\sim \arrow[d, dashed, "g"] \\
& B
\end{tikzcd}$$

We will now abstract this concept and introduce coequalizers:

Definition 2.10 (Coequalizer)**.** *Consider a category* $\boldsymbol{C}$*. A coequalizer consist of following data:*

- *two parallel arrows* $f, g : A \longrightarrow B$
- *an arrow* $q : B \longrightarrow Q$ *such that* $qf = qg$

If there exists an $z : B \longrightarrow Z$*, which satisfies* $zf = zg$*, there exists a unique* $u : Q \longrightarrow Z$ *such that* $uq = z$*, i.e following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\
& & Z
\end{tikzcd}$$

We now want to apply this definition to our problem above. With 2.3.3 we can identify the binary relation $\sim \subseteq A \times A$ as two monic (injective in **Set**) arrows. Define two projections r_1, r_2 from the subset $R :=\sim$

$$r_i : R \longrightarrow A \text{ for } i = 1, 2$$

and p_1, p_2 the projections

$$p_i : X \times X \longrightarrow A \text{ for } i = 1, 2$$

as shown in the following diagram:

$$
\begin{tikzcd}
& R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] \arrow[dd] & \\
A & & A \\
& A \times A \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}
$$

With the canonical projection $\pi : A \longrightarrow A/R$, the projections r_i for $i \in \{1, 2\}$ and a given function $f : A \longrightarrow B$, which preserves the equivalence relation, we can construct following coequalizer:

$$
\begin{tikzcd}
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "\pi"] \arrow[dr, "f"'] & A/R \arrow[d, dashed, "f'"] \\
& & B
\end{tikzcd}
$$

with $fr_1 = fr_2$.
This means there exists a function $f' : A/R \longrightarrow B$ such that $f'\pi(a) = f(a)$. Note that this composition is well defined, since f preserves the equivalence relation. Therefore, we conclude: given a pair of parallel mappings $f, g : R \longrightarrow A$ we can construct a quotient set by quotienting B with an equivalence relation $\sim$ given by

$$x \sim y \text{ if and only if } f(x) = g(x).$$

We will further study this concept in section 3.5.1 in the context of set- theoretic consequences of CETCS.

2.4 Membership of elements

In the subsection 2.3.2 we have seen how to view elements as arrows. Now we want explain the notion of $\Subset$. One might be tempted to read this as "elementhood", but we will use the term "membership", thus using this different notation ($\in$ means "elementhood" and $\Subset$ means "membership").
With the Remark 2.3.2 we can identify the mapping of an element $x \in A$ as a monic function $x : 1 \longrightarrow A$. We now want to study the evaluations of x under the function f. This can be seen as a special case of composition.

If we identify x as the arrow $x : 1 \longrightarrow A$ we can describe the element mapping of x under $f : A \longrightarrow B$, as the (unique) composition of two arrows:

$$y := f \circ x$$

We can now visualize $y \in B$ as the following commuting diagram:

$$\begin{tikzcd} & 1 \arrow[dl, "x"'] \arrow[dr, "y"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

Now we can define the notion of $\Subset$, which generalizes the meaning of the mapping of x under f:

Definition 2.11 ($\Subset$ - membership)**.** *Consider a category* $\boldsymbol{C}$ *with a terminal object 1,* $x \in B$ *and a monic arrow* $m : A \rightarrowtail B$*.* x *is called member of* m*, if there exists an arrow* $x' : 1 \longrightarrow A$ *such that* $x = f \circ x'$*.*

$$\begin{tikzcd} & 1 \arrow[dl, "x'"'] \arrow[dr, "x"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

We write $x \Subset m$*, if* x *is a member of* m*.*

Roughly speaking $x \Subset m$, if x is in the range of the arrow m. As usual we can generalize the notion of $\Subset -$ membership for sequence of arrows.
If for $x \in (B_1, \ldots, B_n)$ and a sequence of monic arrows $(m_1, \ldots, m_n) : A \longrightarrow (B_1, \ldots, B_n)$ there exists an arrow $x' : 1 \longrightarrow A$ such that $m_i \circ x' = x$ for every $i \in \{1, \ldots n\}$ we call $x = (x_1, \ldots, x_n)$ a member of $(B_1, \ldots, B_n)$.

With this we can reformulate the definitions of epis and monos in **Set**.

Remark 2.12. An arrow $f : A \longrightarrow B$ is called:

- onto, if for every $y \in B$ there exists $x \in A$ identified as the arrow $x : 1 \longrightarrow A$ such that $f \circ x = y$.

- mono, if for every $x \in A$ identified as the arrow $x : 1 \longrightarrow A$ and $\forall x' \in A$: $f \circ x = f \circ x'$ implies $x = x'$

Note with the use of classical logic onto and epic arrows equivalent in **Set**, but in intuitionistic logic only epi $\Rightarrow$ onto holds.
Hence in the CETCS we will usually use the term onto. We will further discuss this in chapter 3.

Here one can clearly recognize that Definition 2.5 is just the generalization of surjective and injective functions in the context of Category Theory. In Category Theory, and in order to understand [2] it is essential to view these terms as arrows.
With this we can define isomorphism, which will be very important in the CETCS.

Definition 2.13 (Isomorphism)**.** *An arrow $f : A \longrightarrow B$ is an isomorphism if f is an epi and a mono.*

This allows us to describe an element $a \in A$ as $f^{-1}(b)$ for $b \in B$ in **Set**.
In [1] Awody uses a more general definition and uses ours as a proposition, but we will use this as definition. This definition will be even clearer when we study the Constructive Elementary Theory of the Category of Sets, where we will use this even as an axiom, axiom **(G)**, for our theory.

2.5 Partial and total arrows

To comprehend the the axioms of the Constructive Elementary Theory of the Category of Sets we now want to elaborate the terms partial and total functions in the context of category theory. This will especially be important to understand the **(Π)**- axiom, which we will discuss later.

Definition 2.14 (Partial and total arrow)**.** *Consider a binary relation $r = (r_1, r_2) : R \longrightarrow (X, Y)$. r is a partial arrow if r_1 is a mono. If r_1 is an iso, we call r a total function.*

In contrast to functions in **Set**, a partial arrow does not have to define $f(x)$ for all $x \in R$. It rather is defined on a subset R' of R.

In **Set** a partial arrow is called partial function. Palmgren [2] uses the term partial function in general categories. In this thesis we will use both terms depending on which category we study. A partial function can be constructed with following data:

- a subset A of X
- an inclusion map $\iota_A : A \longrightarrow X$
- two functions $f : X \longrightarrow Y$ and $g : A \longrightarrow Y$

which satisfy $f(\iota_A(a)) = g(a)$, i.e following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

We will now give a simple example of a partial function in the category **Set**.

Example 2.15. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Define an inclusion given by the identity function $id : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a function $g : \mathbb{R}^* \longrightarrow \mathbb{R}$ given by $g(x) = \frac{1}{x}$. With this we get following commuting diagram and that f is a partial function:

$$\begin{tikzcd} \mathbb{R}^* \arrow[r, hook, "id"] \arrow[dr, "\frac{1}{x}"'] & \mathbb{R} \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

We can characterize partial and total arrows as below:

Theorem 2.16. *Consider a category* $\boldsymbol{C}$ *and a relation* $r : R \longrightarrow (X, Y)$*. Then*

1. *r is a partial arrow if and only if for all $x \in X$ and $y, z \in Y$ following holds: if $(x, y) \Subset r$ and $(x, z) \Subset r$ then $y = z$.*
2. *r is a total arrow if and only if for all $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \Subset r$.*
3. *If for all $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \Subset r$, then there exists an arrow $f : X \longrightarrow Y$ such that for all $x \in X$ $(x, fx) \Subset r$.*

Proof. In this proof we will use concepts of 2.3.3.

1. Assume r is a partial arrow, that means r_1 is a mono. With the characterization of

monic arrows in 2.4 this is equivalent to:

$$\text{for all } s,t \in R, : r_1 s = r_1 t \Rightarrow s = t.$$

Now suppose $(x,y) \in r$ and $(x,z) \in r$. This means there exist arrows $a_1, a_2 : 1 \longrightarrow R$ such that $r_1 a_1 = x$ and $r_2 a_1 = y$, $r_1 a_2 = x$ and $r_2 a_2 = z$. r_1 is a mono that means $a_1 = a_2$ which implies $y = z$.

2. Assume r is a total function. That means r_1 is an iso, so r_1 is mono and onto. Consider $x \in X$. Then there exists $s \in R$ such that $r(s) = (x,*)$ because r_1 is an epi. We now can use the notion $s := r^{-1}(x,*) \in R$, since r_1 is an iso. Define $y := r_2(r_1^{-1}x)$. Of course $x = r_1(r_1^{-1}x) \in X$, which means $(x,y) \in r$, thus showing the existence. Suppose there exists another $y' \in Y$ such that $(x,y') \in r$. r is also a partial arrow and with 1) we get $y = y'$, which shows the uniqueness.
   Now assume that for all $x \in X$ there exists a unique $y \in Y$ such that $(x,y) \in r$. This means 1) is satisfied, therefore r is a partial function and r_1 is mono. We now have to show that r_1 is an onto. Since $(x,y) \in r$ for all $x \in X$ there exists a $t \in R$ such that $x = r_1 t$ and $y = r_2 t$. With the characterization of onto arrows in subsection 2.4 we can conclude that r_1 is onto, that means r_1 is a total function.

3. Suppose for all $x \in X$ there exists a unique $y \in Y$ such that $(x,y) \in r$. With 2) we can conclude that r is a total function, i.e r_1 is an iso and is invertible. Define $f := r_2 r_1^{-1}$. Then for all $x \in X$ $x = r_1 r_1^{-1} x$ and $fx = r_2 r_1^{-1} x$. With $a := r^{-1}x$ we can conclude $(x, fx) \in r$.

□

2.6 Cartesian closed categories (CCC)

In the Constructive Elementary Theory of the Category of Sets we study a special type of categories, the cartesian closed categories (CCC). This allows us to generate new objects with already existing ones, without worrying about the existence. Before defining CCCs we need some prerequisites.

2.6.1 Products of objects

For two given objects A and B we want to define a new object denoted as $A \times B$. By defining, we aim to give an abstract characterization of the structure of $A \times B$ up to isomorphism. This characterization will be given by the *universal matching property (UMP)*.

There are different versions of the UMP depending on which structure we are examining (for example for the coequalizer in Definition 2.10 we used a slightly different UMP than we will for products). This way of defining structures is standard in category theory. We now want to specifically study products of objects and give the according UMP. Defining the UMP in other structures will work similarly.

Definition 2.17 (Product)**.** *Consider a category* $\boldsymbol{C}$ *and two objects* A *and* B*. A (binary) product consists of*

- *an object* P
- *two arrows* $p_1 : P \longrightarrow A$ *and* $p_2 : P \longrightarrow B$

which satisfy the universal matching property:
For a given object X *and and two arrows* $x_1 : X \longrightarrow A$ *and* $x_2 : X \longrightarrow B$ *there exists a unique arrow* $u : X \longrightarrow P$*, such that* $x_1 = p_1 u$ *and* $x_2 = p_2 u$*.*

A product can be visualized by the following *product diagram*:

$$\begin{tikzcd}
& X \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dd, dashed, "u"] & \\
A & & B \\
& P \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

In this case Palmgren [2] uses following notation for the unique arrow u and $p := (p_1, p_2)$:

$$u \equiv \langle x_1, x_2 \rangle_p$$

Given a sequence of arrows $p = (p_1, \ldots, p_n) : P \longrightarrow (X_1, \ldots, X_n)$ this concept can be generalized for $n-$ ary products and u can be written as follows:

$$u \equiv \langle x_1, \ldots, x_n \rangle_p$$

This concept will be used in the many proofs (e.g Theorem 3.4).

As a counterpart we will define the dual of products, called coproducts.

Definition 2.18 (Coproducts)**.** *Consider two arrows $q_1 : A \longrightarrow Q$ and $q_2 : B \longrightarrow Q$. A diagram is called coproduct, if there exists an object Z and two arrows $z_1 : A \longrightarrow Z$ and $B \longleftarrow Z$, such that there exists a unique arrow $u : Q \longrightarrow Z$ with $uq_1 = z_1$ and $uq_2 = z_2$, as shown in*

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "z_1"] \arrow[dr, "q_1"'] & & B \arrow[ul, "z_2"'] \arrow[dl, "q_2"] \\ & Q \arrow[uu, dashed, "u"] & \end{tikzcd}$$

We now want to give an example of products. Therefore we use this opportunity introduce a new category, which plays a big part in the theory of CCCs.

2.6.2 Application: λ-Calculus

We now give an overview of λ-calculus, based on [1], to define the category of types, denoted as $\mathbf{C}(\lambda)$. To understand the basic concept, consider the function $x^3 + y$. If we want to express the function, in which y is the variable and x is fixed, i.e the function given by $y \mapsto x^3 + y$, we write $\lambda y.x^3 + y$. If we want to view the function with two variables, i.e the function $x \mapsto (y \mapsto x^3 + y)$, we write $\lambda x.\lambda y.x^3 + y$. Roughly speaking, we can view $\lambda-$ calculus as the theory of bound and unbound variables. It consists of:

- Types: $A, B, A \longrightarrow B, A \times B, \dots$ (there are many different types such as function types or identity types, but it suffices to view them as mere objects)
- Terms:
  - variables $x, y, z, \dots : A$, read as $x, y, z, \dots$ is a point of A
  - $a : A, b : B, \dots$, where a and b are constants
  - $\langle a, b\rangle : A \times B$, where $a : A$ and $b : B$
  - fst(c): A, where $c : A \times B$
  - $ca : A \longrightarrow B$, where $c : A \longrightarrow B$ and $a : A$
  - $\lambda y.b : A \longrightarrow B$, where $y : A$ and $b : B$

- Equations:
  - $\text{fst}(\langle a, b\rangle) = a$
  - $\text{snd}(\langle a, b\rangle) = b$
  - $\langle \text{fst}(c), \text{snd}(c)\rangle = c$
  - $(\lambda y.b)a = b[a/y]$, where $b[a/y]$ is defined as the substitution
  - $\lambda x.cx = c$, where x is a free variable in c

Further define the equivalence relation:

$$a \sim b \text{ if and only if } \lambda x.b = \lambda y.b[y/x]$$

Now we can define the category of types $\mathbf{C}(\lambda)$ as follows:
If we identify the objects as types, the arrows as the closed terms $c : A \longrightarrow B$, where $c \sim c'$, the identity arrow defined as $1_A = \lambda x.x$ and the composition defined as $c \circ a = \lambda x.c(bx)$, $\mathbf{C}(\lambda)$ is a category.

Indeed, we can verify associativity as follows:

$$\begin{aligned} c \circ (b \circ a) &= \lambda x(c((b \circ a)x)) \\ &= \lambda x(c((\lambda y.b(ay))x)) \\ &= \lambda x(c(b(ax))) \\ &= \lambda x(\lambda y(c(by)(ax)) \\ &= \lambda x((c \circ b)(ax)) = (c \circ b) \circ a \end{aligned}$$

and 1_A is the unit:

$$\begin{aligned} c \circ 1_B = \lambda x(c((\lambda y.y)x)) = \lambda x(cx) = c \\ 1_C \circ c = \lambda x((\lambda y.y)(cx)) = \lambda x(cx) = c \end{aligned}$$

Further information about λ-calculus can be found in [5].

We now want to show that $\mathbf{C}(\lambda)$ has binary products:
Consider the types A, B and define the projections π_1, π_2 as follows:

$$\pi_1 = \lambda z.\text{fst}(z) \text{ and } \pi_2 = \lambda z.\text{snd}(z) \text{ for } z : A \times B$$

For given arrows $a : X \longrightarrow A$ and $b : X \longrightarrow B$ we get following diagram

$$\begin{tikzcd}
& X \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[dd, dashed, "{(a,b)}" description] & \\
A & & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

where $(a.b) := \lambda x.\langle ax, bx\rangle$. We now have to show the existence and the uniqueness of the universal matching property:

Existence: We have to show that $\pi_1 \circ (a, b) = a$ and $\pi_2 \circ (a, b) = b$. For this it is enough to prove $\pi_1 \circ (a, b) = a$ ($\pi_2 \circ (a, b) = b$ can be proven similarly):

$$\begin{aligned}
\pi_1 \circ (a, b) &= \lambda x(\pi_1((\lambda y.\langle ay, by\rangle)x)) \\
&= \lambda x(\pi_1\langle ax, bx\rangle) \\
&= \lambda(ax) = a
\end{aligned}$$

Uniqueness: Consider an arrow $c : X \longrightarrow A \times B$ with $\pi_1 \circ c = a$ and $\pi_2 \circ c = b$. We have to show that $(a, b) = c$. Indeed:

$$\begin{aligned}
(a, b) &= \lambda x.\langle ax, bx\rangle \\
&= \lambda x.\langle(\pi_1 \circ c)x, (\pi_2 \circ c)x\rangle \\
&= \lambda x.\langle(\lambda y(\pi_1(cy)))x, (\lambda y(\pi_2(cy)))x\rangle \\
&= \lambda x.\langle(\lambda y((\lambda z.\mathrm{fst}(z))(cy)))x, (\lambda y((\lambda z.\mathrm{snd}(z))(cy)))x\rangle \\
&= \lambda x.\langle \lambda y \mathrm{fst}(cy))x, \lambda y(\mathrm{snd}(cy))x\rangle \\
&= \lambda x.\langle \mathrm{fst}(cx), \mathrm{snd}(cx)\rangle \\
&= \lambda x.(cx) \\
&= c
\end{aligned}$$

Together we get that the UMP of products is satisfied, which means the category $\mathbf{C}(\lambda)$ has products.

2.6.3 Exponentials

Before we define the cartesian closed categories, we will discuss exponentials first. This term can be viewed as the categorical interpretation of function spaces.

Consider the category **Set** and a function

$$f : A \times B \longrightarrow C.$$

By fixing $a \in A$ we get the function

$$f_a : B \longrightarrow C$$

and if a is a variable one can define the function:

$$\tilde{f} : A \longrightarrow C^B \text{ defined as } a \mapsto f(a, y),$$

where C^B is the set of functions from B to C. With $\tilde{f}$ we can describe every function $\phi : A \longrightarrow C^B$ as $\phi = \tilde{f}$ given by $f(a, b) := \phi(a)(b)$. This means there exists the isomorphism

$$Hom(A \times B, C) \cong Hom(A, C^B).$$

In **Set** this isomorphism is given by the bijection

$$ev : C^B \times B \longrightarrow C \text{ defined by } (g, b) \mapsto g(b),$$

which satisfies following universal matching property:
For a given function $f : A \longrightarrow B$ there exists a unique function

$$\tilde{f} : A \longrightarrow C^B$$

such that

$$ev(\tilde{f}(a), b) = f(a, b)$$

and following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[r, "f \times Id_B"] \arrow[dr, "f"'] & C^B \times B \arrow[d, "ev"] \\ & C \end{tikzcd}$$

With this in mind we can give the following definition of exponentials.

Definition 2.19 (Exponential)**.** *Consider a category $\boldsymbol{C}$ with binary products. An exponential of objects B and C consists of following data:*

- *an object denoted as C^B*
- *an arrow*
$$ev : C^B \times B \longrightarrow C$$
called evaluation arrow, such that for all arrows
$$f : A \times B \longrightarrow C$$
there exists a unique arrow
$$\tilde{f} : A \longrightarrow C^B$$
such that following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[r, "\tilde{f} \times 1_B"] \arrow[dr, "f"'] & C^B \times B \arrow[d, "ev"] \\ & C \end{tikzcd}$$

Now we have all the tools to define a cartesian closed category (CCC).

Definition 2.20 (Cartesian closed category)**.** *A category $\boldsymbol{C}$ is called cartesian closed if $\boldsymbol{C}$ contains all finite products and exponentials.*

Example 2.21. We will now give some examples for cartesian closed categories.

- It is easy to see that **Set** is a cartesian closed category, since it has all finite products and exponentials. Further the evaluation $ev : C^B \times B \longrightarrow C$ is defined by $(g, b) \mapsto g(b)$.
- The category of types $\mathbf{C}(\lambda)$ is a cartesian closed category. To prove this we will use an equivalent characterization of CCCs from [1].
A category $\mathbf{C}$ is cartesian closed, if and only if following three properties are satisfied:
  - $\mathbf{C}$ contains the terminal object 1 and for every object C, there exists an arrow
$$!_C : C \longrightarrow 1$$
such that for every arrow $f : C \longrightarrow 1$,
$$f =!_C.$$

– For every object A and B, there exists the product $A \times B$ in $\mathbf{C}$ and arrows

$$p_1 : A \times B \longrightarrow A \text{ and } p_2 : A \times B \longrightarrow B,$$

such that for every arrow $f : Z \longrightarrow A$ and $g : Z \longrightarrow B$, there exists an arrow

$$\langle f, g \rangle : Z \longrightarrow A \times B$$

with following properties:

  * $p_1 \langle f, g \rangle = f$
  * $p_2 \langle f, g \rangle = g$
  * $\langle p_1 h, p_2 h \rangle = h$ for every $h : Z \longrightarrow A \times B$.

– For every object A and B, there exists the exponential object B^A and the arrow

$$ev : B^A \times A \longrightarrow B$$

and for every $f : Z \times A \longrightarrow B$, there is a given arrow

$$\tilde{f} : Z \longrightarrow B^A$$

such that

$$ev \circ (\tilde{f} \circ 1_A) = f \text{ and } ev \circ (\widetilde{f \circ 1_A}) = g$$

for every $g : Z \longrightarrow B^A$. For simplicity for given $a : X \longrightarrow A$ and $b : Y \longrightarrow B$ we write

$$a \times b = \langle a \circ p_1, b \circ p_2 \rangle : X \times Y \longrightarrow A \times B.$$

With this characterization of CCCs we will prove that $\mathbf{C}(\lambda)$ is cartesian closed. For given two objects A and B we showed, that $A \times B$ exists. Now define $B^A := A \longrightarrow B$. Further define the evaluation as

$$ev = \lambda z.fst(z)snd(z) : B^A \times A \longrightarrow B$$

for $z : Z$. Further define for given $f : Z \times A \longrightarrow B$ the transpose $\tilde{f}$ as:

$$\tilde{f} = \lambda z \lambda y.f \langle x, y \rangle : Z \longrightarrow B^A$$

for $z : Z$ and $x : A$. To show the third property of the characterization of CCCs we have to show $ev \circ (\tilde{f} \circ \times 1_A) = f$ and $ev \circ (\widetilde{f \circ 1_A}) = g$. With the rules formulated in section 2.6.2 we get:

$$\begin{aligned}
ev \circ (\tilde{f} \circ 1_A) &= (\lambda z.fst(z)snd(z) \circ [(\lambda y \lambda x.f\langle y, x\rangle) \times \lambda u.u] \\
&= \lambda v.(\lambda z.fst(z)snd(z))[\lambda y \lambda x.f\langle y, x\rangle) \times \lambda u.u]v \\
&= \lambda v.(\lambda z.fst(z)snd(z))[\lambda w.\langle(\lambda y \lambda x.f\langle y, x\rangle)fst(w), \lambda u.u)snd(w)\rangle]v \\
&= \lambda v.(\lambda z.fst(z)snd(z))[\lambda w.\langle(\lambda x.f\langle fst(w), x\rangle), snd(w)\rangle]v \\
&= \lambda v.(\lambda z.fst(z)snd(z))[\langle(\lambda x.f\langle fst(v), x\rangle), snd(v)\rangle] \\
&= \lambda v.(\lambda x.f\langle fst(v), x\rangle)snd(v) \\
&= \lambda v.f\langle fst(v), snd(v)\rangle \\
&= \lambda v.fv \\
&= f
\end{aligned}$$

$ev \circ (\widetilde{f \circ 1_A}) = g$ can be shown similarly. So we showed that $\mathbf{C}(\lambda)$ is cartesian closed.

Remark 2.22 (CCC $\sim \lambda-$ calculus)**.** Awodey [1] shows that there exists a correspondence between the logical system of λ- calculus and cartesian closed categories, which shows the equivalence of both notions:

$$\text{CCC} \sim \lambda\text{-Calculus}$$

This means that for a given cartesian closed category $\mathbf{C}$, we can construct a $\lambda-$ calculus generated by a theory $\mathcal{L}$. We denote this construction as $\mathcal{L}(\lambda)$. We will briefly describe $\mathcal{L}(\lambda)$:

- Types: define the types as the objects of $\mathbf{C}$
- Terms: define the terms as the arrows $a : A \longrightarrow B$ in $\mathbf{C}$
- Equations: a selection of equations are for example
  - $\lambda x.fst(x) = p_1$, where p_1 is a projection
  - $\lambda x.snd(x) = p_2$
  - $g(f(x) = (g \circ f)(x)$
  - $\lambda y.y = 1_A$

This sketch implies, that there exists following isomorphism:

$$\mathbf{C} \cong \mathbf{C(C(\mathcal{L}))}.$$

Before we study the CETCS in the next chapter, we want to define bicartesian categories.

Definition 2.23 (Cartesian and cocartesian)**.** *A category $\boldsymbol{C}$ is called cartesian, if following properties are satisfied:*

1. *The terminal object 1 is in $\boldsymbol{C}$.*
2. *Binary products exist.*
3. *Equalizers exists.*

A category is called cocartesian, if following properties are satisfied:

1. *The initial object 0 is in $\boldsymbol{C}$.*
2. *Binary sums exist.*
3. *Coequalizer exist.*

If $\boldsymbol{C}$ is cartesian and cocartesian, then we call $\boldsymbol{C}$ a bicartesian category.

In the CETCS we often use the existence of binary products in proofs (e.g in (3.5.3)). Since we work with bicartesian categories, we know of the existence of these structures, which simplifies proofs and makes the theory clearer.

3 Constructive Elementary Theory of the Category of Sets (CETCS)

In this section we will discuss the constructive version of the Elementary Theory of the Category of Sets, based on paper [2]. From this we will derive some basic set-theoretic consequences and compare them to standard categorical formulations. All the information, theorems and proofs used are based on [2].

3.1 Constructivism

Since we will present the constructive version of the Elementary Theory of the Category of Sets, we will therefore outline the basic principles of constructivism.
Unlike in most fields of mathematics, where one generally uses classical logic, constructive mathematics is formulated using intuitionistic logic. That means for a formula φ the *stability rule*

$$\neg\neg\varphi \Rightarrow \varphi \tag{1}$$

is not necessarily satisfied in intuitionistic logic. Since (1) does not hold for every formula in intuitionistic logic, we can not accept proofs by contradiction in the constructive setting, because following tautology would need to hold:

$$(\neg\varphi \Rightarrow \bot) \Rightarrow \varphi.$$

Additionally a proof in the constructive setting does not use the principle of excluded middle, i.e. the statement

$$\varphi \vee \neg\varphi \tag{2}$$

is not necessarily satisfied for every formula φ.
According to constructive mathematics, every part of a proof has to be constructed. The axiom of choice (AC) is often utilized in classical logic to assert the existence of objects, without providing a method to construct them. Therefore this axiom can not be utilized in the constructive setting.
Naturally the set of axioms in the CETCS is bigger and more "specific" than in ETCS. For example the axiom **(G)**, that an onto and monic arrow $f : A \longrightarrow B$ is an isomorphism, is an axiom in CETCS, but is provable in ETCS.

3.2 Axioms of ETCS

Before we study the constructive version of the Elementary Theory of the Category of Sets, we will first discuss the ETCS introduced by Lawvere. ETCS has eight axioms:

- Finite roots exist:
  This means that all products $A \times B$, coproducts $A + B$, equalizers, coequalizer and a terminal object 1 exists.
- For all pairs of objects the exponential exists:
  This means that for every object A and B the object B^A exists.
- There exists the Dedekind-Peano object:
  This means that there exists a *natural numbers object* (NNO) in a category $\mathbf{C}$, i.e there exists a sequence of mappings $1 \xrightarrow{0} N \xrightarrow{s} N$, which satisfies following universal matching property:
  For every other sequence of mappings $1 \xrightarrow{b} A \xrightarrow{h} A$ there exists a unique mapping $f : N \longrightarrow A$ such that
  $$f0 \equiv b \text{ and } fS \equiv hf.$$
  This can be visualised as follows:
  $$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "b"'] & N \arrow[r, "S"] \arrow[d, "f"] & N \arrow[d, "f"] \\ & A \arrow[r, "h"'] & A \end{tikzcd}$$
- The terminal object 1 is separating:
  1 is separating if and only if for every arrow $f : A \longrightarrow B$ and $g : A \longrightarrow B$ with $f \neq g$ there exists $a \in A$, such that $f \circ a \neq g \circ a$. In essence this gives us a way to check, if two arrows arrows are the same.
- Axiom of Choice holds:
  This means for every arrow $f : A \longrightarrow B$ there exists a $g : B \longrightarrow A$ such that $f \circ g \circ f = f$.
- For all objects $A \not\cong 0$, i.e A is not empty.
- For all elements a of a sum, a is a member of one of its injections
- There is an object A with more than one element

3.3 Axioms of CETCS

As stated above, the constructive version of the Elementary Theory of the Category of Sets requires more axioms than the ETCS. We will first list all the axioms needed and introduce some abbreviations:

- **(C)** Every category **C** is bicartesian, i.e cartesian and cocartesian.
- **(Π)**: Dependent product exists
  For given arrows $g : Y \longrightarrow X$ and $f : X \longrightarrow I$ there exists an evaluation arrow $ev : P \longrightarrow Y$, an arrow $\varphi : F \longrightarrow I$ and two projections $\pi_1 : P \longrightarrow F$ and $\pi_2 : P \longrightarrow X$ such that following diagram commutes and the square is a pullback:

$$\begin{tikzcd} Y & P \arrow[l, "ev"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \arrow[d, "\varphi"] \\ & X \arrow[r, "f"] & I \arrow[from=1-1, to=2-2, "g"'] \end{tikzcd} \tag{3}$$

  The pullback satisfies following properties: if for all partial functions $\psi : (\xi, \vartheta) : R \longrightarrow (X, Y)$ and $i \in I$:

  1. for all $(x, y) \in (X, Y) : (x, y) \Subset \psi$ implies $gy \equiv x$ and $fx \equiv i$
  2. $fx = i$ holds, then there exists $y \in Y$ such that $(x, y) \Subset \psi$

  are satisfied, then there exists a unique $s \in F$, such that $\varphi s = i$ and for all $(x, y) \in (X, Y)$ following equivalence holds for $\alpha = (\pi_1, \pi_2, \mathrm{ev}) : P \rightarrowtail (F, X, Y)$:

$$(s, x, y) \in \alpha \text{ if and only if } (x, y) \in \psi \tag{4}$$

  If diagram (3) satisfies these properties, we call the diagram a *universal* Π-*diagram.*

- **(G)**: Every arrow $f : A \longrightarrow B$ that is an onto and a mono is an isomorphism.
- **(PA)**: For every object A there exists an onto arrow $f : A \longrightarrow P$ such that P is a choice object. P is a choice object, if for every onto arrow $f : A \longrightarrow P$ there exists a $g : P \longrightarrow A$ such that $fg = 1_P$.
- **(I)**: The intial object 0 has no elements.
- **(DP)**: In a sum (the dual of a product) diagram $A \xrightarrow{f} S \xleftarrow{g} B$ following holds:

$$\forall z \in S : z \Subset f \text{ or } z \Subset g$$

- **(NT)**: For every sum diagram $1 \xrightarrow{x} S \xleftarrow{y} 1$: $x \neq y$.
- **(FCT)**: Every arrow f can be factorized as a monic arrow m and an onto arrow e: $f = m \circ e$
- **(EFF)**: All equivalence relations $r = (r_1, r_2)$ are effective, i.e for $r : R \rightarrowtail (X, X)$ there exists a mapping $e : X \longrightarrow E$ such that following equivalence holds for every $(x_1, x_2) \in (X_1, X_2)$:

$$(x_1, x_2) \Subset (r_1, r_2) \text{ if and only if } ex_1 \equiv ex_2$$

Note that it is still uncertain, whether this set of axioms is optimal or if all the axioms are independent from one another.

3.4 Π-Axiom

To understand the (Π)-axiom we now will discuss this in the category **Set**. The aim of this subsection is rather to give an intuition for the (Π)- axiom than to give a formal introduction to this topic.

Consider a function $\lambda_0 : I \longrightarrow Set$ in **Set**, which satisfies following property:

$$\text{if } i = j \text{ then } \lambda_0(i) \simeq \lambda_0(j) \tag{5}$$

Further define the function

$$\lambda_1 : I \times I \longrightarrow \mathbb{F}(Set, Set) \text{ given by } (i, j) \mapsto \big(\lambda_0(i) \longrightarrow \lambda_0(j)\big). \tag{6}$$

We can interpret λ_1 as a "transport map", which can be visualized as follows:

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_1(i,j)"] \arrow[dr, "\lambda_1(i,k)"'] & \lambda_0(j) \arrow[d, "\lambda_1(j,k)"] \\
& \lambda_0(k)
\end{tikzcd}$$

We will now study the elements $\phi = (\phi)_{i \in I} \in \prod_{i \in I} \lambda_0(i)$. If $\phi \in \prod_i \lambda_0(i)$ then for every $i \in I : \phi_i \in \lambda_0(i)$ where ϕ can be interpreted as an n-tupel. We can visualize ϕ as a path which intersects every set $\lambda_0(i)$ at exactly one point for every $i \in I$.

Now suppose for every $i \in I$ $\lambda_0(i) = X$, where X is a fixed set. Then the transport map $\lambda_{i,j} := \lambda_1(i,j)$ is given by the identity function:

$$\lambda_{i,j} : X \longrightarrow X, x \mapsto x$$

and ϕ is a function, because of property (5), which is given by:

$$\phi : I \longrightarrow X.$$

The (Π)- axiom is the categorical way to express that there exists an object in the category **C**, which behaves like $\prod\limits_{i \in I} \lambda_0(i)$ in **Set**.

We will now present a characterization of the Π- diagram, where $F = 1$ in the diagram.

Lemma 3.1. *Consider a cartesian category* ***C****, which satisfies the axiom* ***(G)*** *and two arrows* $g : Y \longrightarrow X$ *and* $f : X \longrightarrow I$*. Then for a given arrow* $\phi = (r_1, r_2) : R \longrightarrow (X, Y)$ *the diagram*

$$\begin{tikzcd}
Y \arrow[drr, "g"'] & R \arrow[l, "r_2"'] \arrow[r, "a"] \arrow[d, "r_1"] & 1 \arrow[d, "i"] \\
& X \arrow[r, "f"'] & I
\end{tikzcd} \tag{7}$$

is a (Π)- *diagram, if and only if following properties are satisfied:*

1. ϕ *is a partial arrow*
2. *For all* $x \in X$ *with* $fx = i$ *implies* $\exists y \in Y$ *such that* $(x, y) \Subset \phi$
3. *For all* $x \in X$ *and* $y \in Y$ *with* $(x, y) \Subset \phi$ *implies* $fx = i$ *and* $gy = x$

Proof. Suppose (7) is a (Π)- diagram. Since 1 is the initial object, the arrow $i : 1 \longrightarrow I$ is monic. Since (7) is a Π-diagram, we conclude that r_1 is monic, which means that ϕ is a partial arrow and 1) is satisfied. To show property 2) suppose $i = fx$ for any $i \in I$. We have to show that there exists $y \in Y$, such that $(x, y) \Subset \phi$. Since $x \in X$ and $i \in I$ there exists a unique arrow $u : 1 \longrightarrow R$ such that $x = r_1 u$ and $i = au$, as shown in the following diagram:

$$\begin{tikzcd}
1 \arrow[drr, bend left, "x"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "i"'] & & \\
& R \arrow[r, "a"] \arrow[d, "r_1"] & 1 \arrow[d, "i"] \\
& X \arrow[r, "f"'] & I
\end{tikzcd}$$

Define $y : 1 \longrightarrow Y$ as $y = r_2 u$. Then $r_1 u = x$ and $r_2 u = y$, which implies $(x, y) \Subset \phi$. That means 2) is satisfied. 3) is satisfied trivially because the (Π)- diagram (7) commutes.

Now suppose the three properties are satisfied. We have to show that (7) is a (Π)- diagram. Since the third property is satisfied the diagram (7) is commutes. We now have to show that the square in (7) satisfies the pullback properties.
Since the second property is satisfied, for $i = fx$ and $(x, y) \Subset \phi$ for $y \in Y$, there exists $a : 1 \longrightarrow R$ such that $r_1 a = x$ and $r_2 a = y$. This shows the existence of the arrow $a : 1 \longrightarrow R$ from the pullback property. We now will show that a is unique. Suppose there exists another arrow $a' : 1 \longrightarrow R$ such that $r_1 a' = x$. Since the first property is satisfied, r_1 is a mono and we can conclude that $a = a'$, which shows that $a : 1 \longrightarrow R$ is unique, thus showing that (7) satisfies the pullback properties and together with the first part, that it is a (Π)- diagram. □

Remark 3.2. Now we want to briefly discuss the universal (Π)- diagram of the arrows $g : Y \longrightarrow X$ and $f : X \longrightarrow I$:

$$\begin{tikzcd}
Y \arrow[dr, "p"'] & P \arrow[l, "ev"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \arrow[d, "\varphi"] \\
& X \arrow[r, "f"'] & I
\end{tikzcd} \tag{8}$$

If a (Π)-diagram is universal, which means that the properties of the (Π)- axiom are satisfied, then for any other (Π)- diagram

$$
\begin{tikzcd}
Y & P' \arrow[l, "ev'"'] \arrow[r, "\pi_1'"] \arrow[d, "\pi_2'"] & F' \arrow[d, "\varphi'"] \\
 & X \arrow[r, "f"'] \arrow[from=ul, "p"'] & I
\end{tikzcd}
\tag{9}
$$

there exists a unique mapping $n : F' \longrightarrow F$ such that $\varphi' \equiv \varphi n$. Additionally for the unique arrow $m : P' \longrightarrow P$ with $n\pi_1' \equiv \pi_1 m$ and $\pi_2' \equiv (ev)m$, $ev' = (ev)m$ holds.

In [2] (see Lemma 7.3 in [2]) Palmgren shows that m exists and $ev' = (ev)m$ is satisfied, if and only if for all $v \in F, x \in X$ and $y \in Y$ following equivalence holds for two given (Π)- diagrams:

$$(v, x, y) \Subset (\pi_1', \pi_2', ev') \Longleftrightarrow (nv, x, y) \Subset (\pi_1, \pi_2, ev). \tag{10}$$

3.5 Set-theoretic consequences

3.5.1 Quotient Sets

Recall that equalizers can be identified as quotient sets in the category **Set**. We will now study this in the context of CETCS:

Proposition 3.3 (Quotient Sets)**.** *Consider a bicartesian category **C** and an equivalence relation $r = (r_1, r_2) : R \rightarrowtail (X, X)$. Then there exists an arrow $q : X \longrightarrow Q$ such that for all $(x_1, x_2) \in (X, X)$ the following holds:*

$$(x_1, x_2) \Subset r \Longrightarrow qx_1 = qx_2 \tag{11}$$

Additionally, if there exists an arrow $f : X \longrightarrow Y$ with

$$(x_1, x_2) \Subset r \Longrightarrow fx_1 = fx_2, \tag{12}$$

*there exists a unique arrow $h : Q \longrightarrow Y$ with $hq = f$. If **(EFF)** is satisfied (11) is an equivalence.*

Proof. Since **C** is a bicartesian, especially a cocartesian, category, we know coequalizer exist and we can define following coequalizer with $qr_1 = qr_2$:

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

We will now show the first part:
Consider $(x_1, x_2) \in (X, X)$ with $(x_1, x_2) \Subset r$. We have to show that $qx_1 = qx_2$. Since $(x_1, x_2) \Subset r$ there exists an arrow $a : 1 \longrightarrow R$ such that $x_1 = r_1 a$ and $x_2 = r_2 a$. Together we get:

$$qr_1 = qr_2 \iff qr_1 a = qr_2 a \iff qx_1 = qx_2.$$

So we showed (11).
Now consider $f : X \longrightarrow Y$, which satisfies (12). Since $fx_1 = fx_2$ for all $x_1, x_2 \in X$ and for all $t \in R : fr_1 t = fr_2 t$ ($r_i t \in X$ for $i = \{1, 2\}$). Because all mapped elements are the same the arrows $f \circ r_i : R \longrightarrow Y$ for $i = \{1, 2\}$ are identical. Since q is a coequalizer and $fr_1 = fr_2$ there exists a unique arrow $h : Q \longrightarrow Y$ with $hq = f$, shown in the following diagram:

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "q"] \arrow[dr, "f"] & Q \arrow[d, dashed, "h"] \\ & & Y \end{tikzcd}$$

Suppose the category $\mathbf{C}$ satisfies the axiom **(EFF)**, i.e there exists an arrow $e : X \longrightarrow E$ such that for all $(x_1, x_2) \in (X, X)$:

$$(x_1, x_2) \Subset r \Longleftrightarrow ex_1 = ex_2 \tag{13}$$

Similar to the proof of (12) we can conclude that $er_1 = er_2$. Since q is a coequalizer there exists a unique arrow $e' : Q \longrightarrow E$ such that $e = e'q$.

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "q"] \arrow[dr, "e"] & Q \arrow[d, dashed, "e'"] \\ & & E \end{tikzcd}$$

Suppose $qx_1 = qx_2$ holds. Then:

$$qx_1 = qx_2 \iff e'qx_1 = e'qx_2 \iff ex_1 = ex_2$$

With (13) we can conclude $(x_1, x_2) \Subset r$. $\square$

3.5.2 Induction

Recall the induction principle:
Consider a formula $\varphi(n)$ over the natural numbers n. Suppose the formula φ is satisfied for 0, i.e $\varphi(0)$. Further suppose the successor function $S : \mathbb{N} \longrightarrow \mathbb{N}$ and for any given $n \in \mathbb{N}$ satisfying $\varphi(n)$, we can conclude $\varphi(S(n))$:

$$\varphi(0) \wedge \forall n \in \mathbb{N} : (\varphi(n) \Rightarrow \varphi(S(n))$$

If this holds, then by the Peano-axioms we can conclude that φ is satisfied for all $n \in \mathbb{N}$:

$$\forall n \in \mathbb{N} : \varphi(n)$$

We will now study this concept as a set theoretic consequence of the constructive Elementary Theory of the Category of Sets:

Proposition 3.4 (Induction)**.** *Consider a category* $\boldsymbol{C}$*, which satisfies the axiom* ***(G)*** *and has the natural numbers object (NNO). Furthermore suppose a monic function* $r : R \rightarrowtail N$ $0 \isinE r$ *and for given* $n \in N$ *with* $n \isinE r$ *implies* $Sn \isinE r$*. Thus, for all* $n \in N : n \isinE r$*.*

Proof. Since $0 \isinE r$, there exists an arrow $z : 1 \longrightarrow R$ with $0 = rz$. We can now construct following pullback diagram:

$$\begin{tikzcd}
1 \arrow[dr, dashed, "x"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& P \arrow[r, "q"] \arrow[d, "p"] & R \arrow[d, "r"] \\
& R \arrow[r, "S \circ r"'] & N
\end{tikzcd}$$

with the pullback properties $px = u$ and $qx = v$ for given arrows $u, v : 1 \longrightarrow R$. We want to show that p is an iso, so we have to show p is an onto and a mono, as axiom**(G)** holds. First we will show that p is a mono. Consider two arrows $a, b : 1 \longrightarrow P$ such that $pa = pb$. Since r is monic, we get together with the pullback properties following equality:

$$pa = pb \iff (Sr)pa = (Sr)pb \iff (qa)r = (qb)r \iff qa = qb$$

We now have to show that $a = b$. From the pullback property we get a unique arrow $x : 1 \longrightarrow P$ such that $px = u$ and $qx = v$. Since x is unique we can conclude that

$x = a = b$, which shows that p is a mono.
Since $ru \Subset r$ we get $Sru \Subset r$ by assumption. Since u was arbitrary we can conclude with the pullback properties and the remark in 2.4 that p is onto. Together with axiom **(G)** we get that p is an isomorphism. We can now define the inverse $p^{-1} : R \longrightarrow R$, thus $qp^{-1} : R \longrightarrow R$. Since (NNO) holds we get following commuting diagram:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "z"'] & N \arrow[r, "S"] \arrow[d, "f"] & N \arrow[d, "f"] \\
& R \arrow[r, "qp^{-1}"'] & R
\end{tikzcd}$$

that means $f0 = z$ and $fS = qp^{-1}f$. Now we get $rf = 0$ and

$$(rf)S = rqp^{-1}f = S(rf). \tag{14}$$

Since the identity arrow 1_N also satisfies (14) we get $rf = 1_N$. Hence for every $n \in N$ $rfn = n$ holds and $n \Subset r$, proving the proposition. $\square$

If we identify the membership of an element $n \in N$ in r as satisfying the formula φ and interpret the arrow $S : N \longrightarrow N$ as the successor function $S : \mathbb{N} \longrightarrow \mathbb{N}$, we can clearly see the similarities between both approaches to the induction principle.

3.5.3 Constructing new relations with logical operations

In this subsection we want to combine two sequences of jointly monic functions $r = (r_1, \dots, r_n) : R \rightarrowtail (X_1, \dots, X_N)$ and $s = (s_1, \dots, s_n) : S \rightarrowtail (X_1, \dots, X_N)$ by using logical operators $\wedge, \vee, \rightarrow$ and the quantifiers $\forall$ and $\exists$.

For a given object X and the identity arrow $1_X : X \longrightarrow X$ of a category $\mathbf{C}$ which satisfies the axioms **(G)**, (Π), **(DP)**, **(FCT)** and **(I)** we can formulate following universally true relation:

$$x \Subset 1_X, \text{ since } 1_X x = x, \forall x \in X$$

Now consider the arrow $f_A : 0 \longrightarrow X$. Then a universal false statement is given by the relation

$$\neg(x \Subset f_x), x \in X.$$

Theorem 3.5. *Consider a bicartesian category* $\boldsymbol{C}$*, which satisfies the axioms* ***(G)****,* (Π)*,* ***(DP)****,* ***(FCT)*** *and* ***(I)****. For given jointly monic sequences* $r = (r_1, \ldots, r_n) : R \rightarrowtail (X_1, \ldots, X_n)$ *and* $s = (s_1, \ldots, s_n) : S \rightarrowtail (X_1, \ldots, X_n)$ *there exists arrows* $(r \wedge s), (r \vee s), (r \rightarrow s) : Q \longrightarrow (X_1, \ldots, X_n)$ *such that for all* $x \in X$ *following holds:*

1. $x \Subset (r \wedge s)$ *if and only if* $x \Subset r$ *and* $x \Subset s$
2. $x \Subset (r \vee s)$ *if and only if* $x \Subset r$ *or* $x \Subset s$
3. $x \Subset (r \rightarrow s)$ *if and only if* $x \Subset r$ *implies* $x \Subset s$

If $m : M \rightarrowtail (X_1, \ldots, X_n)$ *is an extended relation there exists* $\forall(m) : A \rightarrowtail (X_1, \ldots, X_n)$ *and* $\exists(m) : E \rightarrowtail (X_1, \ldots, X_n)$ *such that for all* $(x_1, \ldots, x_n) \in (X_1, \ldots, X_n)$ *following holds:*

4. $x \Subset \exists(m)$ *if and only if there exists* $y \in Y$*,* $(x_1, \ldots, x_n, y) \Subset m$
5. $x \Subset \forall(m)$ *if and only if for all* $y \in Y$*,* $(x_1, \ldots, x_n, y) \Subset m$

Proof. It suffices to prove this for $n = 1$ and denote $X_1 = X$, $r_1 = r$ and $s_1 = s$.

1. Consider following pullback square:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"] & S \arrow[d, "s"] \\ R \arrow[r, "r"] & X \end{tikzcd}$$

   Define the monic arrow $(r \wedge s) : P \longrightarrow X$. Assume $x \Subset r$ and $x \Subset s$, i.e there exists $u : 1 \longrightarrow S$ and $v : 1 \longrightarrow R$ such that $su = x$ and $rv = x$. By the pullback property there exist a unique arrow $z : 1 \longrightarrow P$, such that $pz = v$ and $qz = u$. Especially $(r \wedge s)u = x$ holds, hence $x \Subset (r \wedge s)$.

   Now assume $x \Subset (r \wedge s)$, i.e there exists $z : 1 \longrightarrow P$ such that $(r \wedge s)z = x$. Then $rpz = x$ and $sqz = x$ which implies $x \Subset r$ and $x \Subset s$.

2. Consider the sum diagram (the dual of a product) $R \xrightarrow{i} U \xleftarrow{j} S$ and the unique arrow $f : U \longrightarrow X$ with properties $r = fi$ and $s = fj$, i.e following sum diagram commutes:

$$
\begin{tikzcd}
R \arrow[r, "i"] \arrow[dr, "r"'] & U \arrow[d, dashed, "f"] & S \arrow[l, "j"'] \arrow[dl, "s"] \\
& X &
\end{tikzcd}
$$

With the axiom **(FCT)** we can factorize f as $f = em$ where $e : U \longrightarrow I$ is an onto and $m : I \longrightarrow X$ is a mono. We want to show that $(r \vee s) := m$ satisfies 2.

Suppose $x \in X$ satisfies $x \Subset r$. Then there exists $t \in R$ such that $x = rt = fit = meit$. As $eit : 1 \longrightarrow I$, thus $x \Subset m$ is satisfied. Similarly we can show $x \Subset m$ if we assume $x \Subset s$.

Now assume $x \Subset m$. We have to show that $x \Subset r$ or $x \Subset s$. Since e is an onto there exists $u \in U$ such that $x = meu = fu$. Since $R \xrightarrow{i} U \xleftarrow{j} S$ is a sum and **(DP)** is satisfied without loss of generality we conclude $u \Subset i$, which means there exists $t \in R$ such that $u = it$ which implies $x \Subset r$, since

$$x = fu = fit = rt.$$

3. Consider the monic mappings $Q \xrightarrow{p} R \xrightarrow{r} X$ and construct following pullback diagram:

$$
\begin{tikzcd}
Q \arrow[r, tail, "q"] \arrow[d, tail, "p"] & S \arrow[d, tail, "s"] \\
R \arrow[r, tail, "r"] & X
\end{tikzcd}
$$

Since the axiom **(Π)** is satisfied we can can construct following (Π)− diagram:

$$
\begin{tikzcd}
Q & P \arrow[l, "ev"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \arrow[d, "\varphi"] \\
& R \arrow[r, "r"'] & X
\arrow[from=1-1, to=2-2, "p"']
\end{tikzcd} \tag{15}
$$

We want to show that $(r \to s) := \varphi$ satisfies 3. Assume $x \Subset (r \to s)$. We want to show: $x \Subset r$ implies $x \Subset s$. Assume $x \Subset \varphi$, that means there exists $u : 1 \longrightarrow F$ such that $x = \varphi u$. Additionally suppose $x \Subset r$, that means there exists $v : 1 \longrightarrow R$ such that $x = rv$. We have to show $x \Subset s$. Since the square in (15) is a pullback and u and v exist, there exists (by the pullback property) a unique arrow $w : 1 \longrightarrow P$

such that $u = \pi_1 w$ and $v = \pi_2 w$. Together with the pullback diagram (15) we get following equality:

$$x = rv = r\pi_2 w = rp(ev)w = sq(ev)w$$

Since $sq(ev)w : 1 \longrightarrow S$ we conclude $x \Subset s$, what we wanted to show.
Now suppose for all $x \in X$ $x \Subset r$ implies $x \Subset s$. Construct following pullback square:

$$\begin{tikzcd}
T \arrow[r, tail, "t"] \arrow[d, tail] & Q \arrow[d, tail, "rp"] \\
1 \arrow[r, tail, "x"] & X
\end{tikzcd}$$

We want to show that the arrow $(r \to s) := \varphi : F \longrightarrow X$ from the (Π) -diagram (15) satisfies $x \Subset \varphi$. Define the arrow

$$\psi = (pt, t) : T \rightarrowtail (R, Q),$$

where $p : Q \rightarrowtail R$ is the arrow in (15). Since p and t are monic arrows, pt is monic and φ is a partial arrow. Assume $\varphi(u) \equiv v$ for $u \in R$ and $v \in Q$. Then there exists $w : 1 \longrightarrow T$ such that $u = ptw$ and $v = tw$. We now want to prove both properties of the (Π)- axiom.

(a) We have to show for all $(u, v) \in (R, Q)$ with $(u, v) \Subset \varphi$ implies $pv = u$ and $ru = x$. Since $u = ptw = pv$ holds, $u = ptw$ and $ru = rpv = x$, which means (a) in the (Π) -axiom is satisfied.

(b) To verify the second condition of (Π) we have to show that if $ru = x$ holds, then there exists $v \in Q$ with $(u, v) \Subset \psi$. Assume $ru = x$ for $u : 1 \longrightarrow R$ and $x \in X$. That means $x \Subset r$ and by assumption we get $x \Subset s$, i.e there exists $w : 1 \longrightarrow S$ such that $sw = x$. Since (3) is a pullback we get following commuting pullback diagram and unique arrow $v : 1 \longrightarrow Q$ such that $u = pv$ and $w = qv$:

$$\begin{tikzcd}
1 \arrow[dr, dashed, "v"] \arrow[drr, bend left, "w"] \arrow[ddr, bend right, "u"'] & & \\
& Q \arrow[r, "q"] \arrow[d, "p"] & S \arrow[d, "s"] \\
& R \arrow[r, "r"'] & X
\end{tikzcd}$$

Together we get $rpv = x$. Since (3) is a pullback there exists a unique $z : 1 \longrightarrow T$ with $tz = v$ such that following pullback diagram commutes:

$$\begin{tikzcd}
1 \arrow[dr, dashed, "z"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "id"'] & & \\
& T \arrow[r, "t"] \arrow[d] & Q \arrow[d, "rp"] \\
& 1 \arrow[r, "x"'] & X
\end{tikzcd}$$

This means $ptz = pv = u$ and $tz = v$, i.e $(u, v) \Subset \varphi$. Therefore the second condition of (Π)-diagram is satisfied.

Since both conditions are satisfied and the (Π)-axiom holds in the universal (Π)-diagram (15), there exists $k \in F$ such that $\varphi k = x$ which shows $x \Subset \varphi$.

4. Suppose $m = (m_1, m_2) : M \rightarrowtail (X, Y)$ are two jointly monic arrows. Since **(FCT)** holds, we can factorize m_1 as an onto arrow $e : M \longrightarrow I$ and a monic arrow $i : I \longrightarrow X$. Note that **(FCT)** has to be satisfied, since m_1 and m_2 are jointly monic does not imply that m_1 is monic. Define $\exists(m) := i$. We have to show that $\exists(m)$ satisfies the equivalence in 4. Suppose $x \Subset \exists(m)$.. Since $x \Subset \exists(m)$ there exist $t \in I$ such that $x = it$ and since e is an onto, there exists $s \in M$ such that $es = t$. Together we get following equivalence:

$$x \Subset \exists(m) \Leftrightarrow x = it \Leftrightarrow x = ies \Leftrightarrow x = m_1 s. \tag{16}$$

With (16) we can show the equivalence in 4.:
Assume $x \Subset \exists(m)$. Since $m_2 s \in Y$ and $x \Subset m_1$ with (16) we can conclude $(x, m_2 s) \Subset$

(m_1, m_2).
Now suppose there exists $y \in Y$ such that $(x, y) \Subset m = (m_1, m_2)$ is satisfied. That means there exist an arrow $s : 1 \longrightarrow M$ such that $x = m_1 s$ and $y = m_2 s$. Since $m_1 = ie$ we get $x = ies$ and since the arrow $es : 1 \longrightarrow I$ exists, $x \Subset i = \exists(m)$ is satisfied, thus showing the equivalence in 4.

5. We now want to show the equivalence in 5. Suppose a jointly monic arrow $m = (m_1, m_2) : M \rightarrowtail (X, Y)$ and two arrows $p : U \longrightarrow X$ and $q : U \longrightarrow Y$. Recall the term and notation of products in section 2.6.1. Since the category $\mathbf{C}$ is a bicartesian category, we can especially construct products. Define $m' := \langle m_1, m_2 \rangle$ as the unique arrow given by the universal mapping property for products, i.e following diagram commutes:

$$\begin{tikzcd}
& M \arrow[dl, "m_1"'] \arrow[dr, "m_2"] \arrow[dd, dashed, "m'"] & \\
X & & Y \\
& U \arrow[ul, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

Since axiom (Π) we can construct following (Π)- diagram:

$$\begin{tikzcd}
M \arrow[dr, "m'"'] & P \arrow[l, "ev"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \arrow[d, "\varphi"] \\
& U \arrow[r, "p"'] & X
\end{tikzcd} \tag{17}$$

We now want to show that $\forall(m) := \varphi$ satisfies the equivalence in 4. Suppose $x \Subset \varphi$ and consider $y \in Y$. We have to show $(x, y) \Subset m$. Since $x \Subset \varphi$ there exists $f : 1 \longrightarrow F$ such that $x = \varphi f$. Further there exists $u \in U$ such that $x = pu$ and $y = qu$ that means $(x, y) \Subset (p, q)$. Since (17) is a (Π)- diagram, the square is a pullback, that means there exists a unique arrow $w : 1 \longrightarrow P$ such that $f = p_1 w$ and$u = \pi_2 w$, i.e following diagram commutes:

$$
\begin{tikzcd}
1 \arrow[drr, bend left, "f"] \arrow[dr, dashed, "w"] \arrow[ddr, bend right, "u"'] & & \\
& P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \arrow[d, "\varphi"] \\
& U \arrow[r, "p"'] & X
\end{tikzcd}
$$

Together we conclude that $m'(ev)w = \pi_2 w = u$, which means $u \Subset m'$ implying $(x, y) \Subset \varphi$ since $x = pu$ and $y = qu$.

Now suppose for a fixed $x \in X$ and for all $y \in Y$, $(x, y) \Subset m$ is satisfied. We have to show that $x \Subset \varphi$. Construct following pullback:

$$
\begin{tikzcd}
N \arrow[r, "n"] \arrow[d] & M \arrow[d, "m_1"] \\
1 \arrow[r, "x"'] & X
\end{tikzcd}
\tag{18}
$$

Define the partial arrow

$$(m'n, n) : N \rightarrowtail (U, M).$$

$(m'n.n)$ is indeed a partial arrow, since m' and n are both monomorphisms. We now want to show that the properties of the (Π)- axiom is satisfied:

(a) Assume $(u, v) \Subset (m'n, n)$. Then there exists an arrow $t : 1 \longrightarrow N$ such that $u = m'nt$ and $v = nt$. Together we get $m'v = m'nt = u$ and $pu = m'nt = m_1 nt = x$, which shows condition (a).

(b) Assume $u \in U$ satisfies $pu = x$. We have to show that there exists $s \in M$ such that $(u, s) \Subset (m'n, n)$. Suppose $y = qu$. Because $y \in Y$ the assumption implies $(x, y) \Subset m$, that means there exists $s \in S$ such that $x = m_1 s$ and $y = m_2 s$. Since $m's = u$ and $x = m_1 s$, with the pullback (18) there exists a unique $t : 1 \longrightarrow N$ such that $s = nt$. That means $(u, s) \Subset (m'n, n)$, hence condition (b) holds.

Since both conditions of the (Π)- axiom is satisfied there exists $f \in F$ such that $\varphi f = x$, which implies $x \Subset \varphi$, thus showing the equivalence.

$\square$

With this theorem we want two define a decidable relation:
Suppose $f, t : 1 \longrightarrow 2$ and consider a decidable relation $r : P \rightarrowtail X$. We can now construct an arrow $\chi_r : X \longrightarrow 2$ such that

$$x \Subset r \wedge \chi_r(x) = t \text{ or } (\neg x \Subset r) \wedge \chi_r(x) = f.$$

Since $\neg x \Subset r$ is a universal false statement $x \Subset r$ if and only if $\chi_r(x) = t$.

3.6 Correspondence to standard categorical formulations

In this section we will show how the CETCS fits in the standard categorical formulations. More specifically, we want to show that a category $\mathbf{C}$ is locally cartesian closed, if and only if it satisfies the (Π)- axiom.

Definition 3.6 (Locally cartesian closed)**.** *Consider a cartesian category* $\boldsymbol{C}$*.* $\boldsymbol{C}$ *is called locally cartesian closed if it satisfies the generalized exponential axiom or the* (Π)*- axiom.*

Note that Awodey [1] uses a different definition for locally cartesian closed categories, using functors defined on the objects of the *slice category.* However, this definition implies the use of the axiom of choice to construct specific objects, which we can not use in the constructive setting.

We will now formulate a theorem, which we will later use to proof that a category is cartesian closed, if and only if it satisfies the (Π)- axiom. The proof needs simple, but lengthy prerequisites (e.g image factorization and covers), which is not essential to understand this equivalence, which can be found in [2], section 7.

Theorem 3.7. *Consider a category* $\boldsymbol{C}$*, which satisfies axiom* ***(G)*** *and two arrow* $g : Y \longrightarrow X$ *and* $f : X \longrightarrow I$*. Further suppose the diagram*

$$\begin{tikzcd} Y \arrow[dr, "p"'] & P \arrow[l, "ev"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \arrow[d, "\varphi"] \\ & X \arrow[r, "f"'] & I \end{tikzcd} \tag{19}$$

is a universal (Π)*- diagram. Then for every* $i \in I$ *and every* $\psi = (r_1, r_2) : R \longrightarrow (X, Y)$*, which satisfies the properties from Lemma 3.1, there exists a unique* $v : 1 \longrightarrow F$ *such that for all* $x \in X$ *and* $y \in Y$ *following equivalence holds:*

$$(x, y) \in \psi \Leftrightarrow (v, x, y) \in \alpha,$$

where $\alpha = (\pi_1, \pi_2, ev) : P \longrightarrow (F, X, Y)$.

Now we will proof the main theorem of this section, which we will later use to proof that locally cartesian categories satisfy the (Π)- axiom.

Theorem 3.8. *Consider a category* $\boldsymbol{C}$*, which satisfies axiom* ***(G)*** *and two arrow* $g : Y \longrightarrow X$ *and* $f : X \longrightarrow I$*. Further consider the* (Π)*- diagram*

$$\begin{tikzcd} Y \arrow[drr, "g"'] & & P \arrow[ll, "ev"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \arrow[d, "\varphi"] \\ & & X \arrow[r, "f"'] & I \end{tikzcd} \tag{20}$$

Suppose for every $i \in I$ *and every* $\psi = (r_1, r_2) : R \longrightarrow (X, Y)$*, which satisfies Lemma 3.1, there exists a unique arrow* $v : 1 \longrightarrow F$ *with* $\phi v = i$*, such that for all* $x \in X$ *and* $y \in Y$ *following equivalence holds:*

$$(x, y) \in \psi \Leftrightarrow (c, x, y) \in \alpha,$$

where $\alpha = (\pi_1, \pi_2, ev) : P \longrightarrow (F, X, Y)$*. Then (20) is a universal* (Π)*- diagram.*

Proof. Consider another (Π)- diagram

$$\begin{tikzcd} Y \arrow[drr, "g"'] & & P' \arrow[ll, "ev'"'] \arrow[r, "\pi_1'"] \arrow[d, "\pi_2'"] & F' \arrow[d, "\varphi'"] \\ & & X \arrow[r, "f"'] & I \end{tikzcd} \tag{21}$$

and form following pullback diagram:

$$\begin{tikzcd} Q \arrow[r, "!"] \arrow[d, "q"] & 1 \arrow[d, "v'"] \\ P' \arrow[r, "\pi_1"'] & F' \end{tikzcd} \tag{22}$$

Further define the composed (Π)- diagram, given by:

$$\begin{tikzcd} Y \arrow[dr, "g"'] & P' \arrow[l, "ev'q"'] \arrow[r, "!"] \arrow[d, "\pi_2'"] & F' \arrow[d, "\varphi' v'"] \\ & X \arrow[r, "f"'] & I \end{tikzcd} \tag{23}$$

We now want to proof following equivalence for $x \in X$ and $y \in Y$:

$$(x, y) \in (\pi_2' q, (ev)'q) \Longleftrightarrow (v', x, y) \in (\pi_1', \pi_2', ev')$$

Suppose $(x, y) \in (\pi_2' q, (ev)'q)$ holds. Then there exists $u : 1 \longrightarrow Q$ such that $x = \pi_2' qu$ and $(ev)'qu$. Since (22) is a pullback, the diagram commutes and $v' = v'!u = \pi_1' qu$ holds. This means for the arrow $qu : 1 \longrightarrow P'$, $(v', x, y) \in (\pi_1', \pi_2', (ev)')$ is satisfied.
Now suppose $(v', x, y) \in (\pi_1', \pi_2', ev')$ holds. Then there exists $t : 1 \longrightarrow P'$ such that $v' = \pi_1' t, x = \pi_2' t$ and $y = (ev)'t$. Since $v' = \pi_1 t$, there exists by the pullback property of (22) a unique arrow $s : 1 \longrightarrow Q$ such that $t = qs$. That means that for the arrow $s : 1 \longrightarrow q'$ $(x, y) \in (\pi_2' q, ev'q)$ is satisfied, proving the equivalence.

Since (23) is a (Π)- diagram, the pair of arrows $\psi = (\pi_2 q, (ev)'q)$ satisfies the three properties from Lemma 3.1 for $i = \varphi' v'$. By the assumption, there exists a unique arrow $v : 1 \longrightarrow F$ such that $\varphi v = \varphi' v' = i$ and for all $x \in X$, $y \in Y$ following equivalence holds:

$$(v, x, y) \in (\pi_1, \pi_2, ev) \Longleftrightarrow (x, y) \in \psi. \tag{24}$$

This implies for every $x \in X$ and $y \in Y$ following equivalence is satisfied:

$$(v, x, y) \in (\pi_1, \pi_2, ev) \Longleftrightarrow (v', x, y) \in (\pi_1', \pi_2', ev') \tag{25}$$

By Theorem 3.5 there exists a unique $\chi : F' \longrightarrow F$ such that for all $v' : 1 \longrightarrow F$, $\varphi\chi v' = \varphi v'$ is satisfied and for every $x \in X$ and $y \in Y$ following equivalence holds:

$$(\chi v, x, y) \in (\pi_1, \pi_2, ev) \Longleftrightarrow (v', x, y) \in (\pi_1', \pi_2', ev') \tag{26}$$

To show that (20) is a universal (Π)- diagram by Remark 3.1, we have to show that there exists a (unique) arrow $\kappa : P' \longrightarrow P$ such that $\pi_1 \kappa = \chi \pi_1$ and $\pi_2 \kappa = \pi_2$.

The existence of the arrow κ follows immediately from Remark 3.1, since equivalence (26) holds. The uniqueness of κ can be concluded from the uniqueness of (25). This shows that (20) is a universal (Π)- diagram, which we wanted to proof. $\square$

Now the main result of this section is a simple conclusion from Theorem 3.7 and 3.8:

Corollary 3.9. *Consider a category **C**, which satisfies axiom **(G)**. **C** is a locally cartesian closed category, if and only if **C** satisfies the* (Π)- *axiom.*

Proof. Suppose **C** is a locally cartesian closed category. Theorem 3.7 states that **C** satisfies the (Π)-axiom.
Now suppose **C** satisfies the (Π)- axiom. Then Theorem 3.8 implies that for every arrow $g : Y \longrightarrow X$ and $f : X \longrightarrow I$, there exists a universal (Π)- diagram. $\square$

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Eigenständigkeitserklärung

Ich versichere, dass ich die vorgelegte Bachelorarbeit eigenständig und ohne fremde Hilfe verfasst, keine anderen als die angegebenen Quellen verwendet und die den benutzten Quellen entnommenen Stellen als solche kenntlich gemacht habe.

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