Modelle der Mengenlehre - SS13

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These notes include part of the material discussed in the Tutorium and in the Exercises that correspond to the Vorlesung “Modelle der Mengenlehre” of Prof. Dr. Hans-Dieter Donder. Of course, possible mistakes in these notes are not related to Prof. Donder at all. Many extra, or optional exercises can be found here. Exercises that provide a certain technique, found also in other proofs in the Vorlesung or connected to the given weekly exercises, start with [**T**].

Please feel free to send me your comments, or your suggestions regarding these notes.

1. Some History

1. **Bolzano**: he considered infinite sets (he also introduced the term *Menge*).

2. **Cantor**: he introduced cardinals-ordinals, topology of reals, he proved that $\mathbb{R}$ is uncountable, that the set of algebraic numbers $\mathbb{A}$ is countable, and he formulated the continuum hypothesis CH.

3. The axioms of ZF are due to Zermelo, except Replacement (**Fraenkel**, **Skolem**), and Foundation (**von Neumann**).

2. Introductory facts

1. Unlike group-axioms (first the models, the groups, and then the axioms) the set-axioms are given first and then we study their models!!!

2. The axioms of ZF are generally "accepted". The infamous axiom of choice AC is not considered that innocent.

3. There exist many set theories (e.g., in Bernays-Gödel ST we have two sorts (types) of objects: sets and classes). Note that in ZF classes are informal objects.

4. There exist many constructive set theories e.g., CZF, based on intuitionistic logic (Aczel-Rathjen).

5. The language of ZF.

3. Remarks on the Axioms

1. Note that the axioms are actually first-order ones.

2. First we show by Extensionality the uniqueness of $\emptyset$, and "then" we write $\emptyset \in V$. The same pattern is followed in every similarly written axiom.

3. The converse of Extensionality is provable by the equality axioms.

4. Cantor's full Comprehension axiom scheme is false.

5. Thus we need more axioms to describe our intuition about sets.

6. Note that the scheme of Separation (Aussonderung) is derivable by the rest of the axioms.

7. Note that the existence of an inductive set is equivalent to the existence of an infinite set, but that requires the notion of a finite set. Of course, it is intuitively expected that an inductive set is not finite (Jech p.26, Ex. 2.4).

8. Show that each set is a class.

4. On the axiom of foundation

9. Verify the axiom of foundation on specific sets. There are sets x, like ω, for which there exists only one element not intersecting x, while there are sets, like $\mathbb{R}$, such that every element doesn't intersect x.

10. It is not used in actual mathematics but it is important for the formation of the set-theoretic universe. It is also very important in the construction of models of set theory.

11. There exist no infinite $\ni$-chains

$$x_0 \ni x_1 \ni x_2 \ni \ldots \ .$$

12. There exist no cycles

$$x_0 \in x_1 \ldots x_n \in x_0.$$

13. $\nexists_x (x \in x)$.

14. $\nexists_x (\mathcal{P}(x) \subseteq x)$.

15. $\forall_{x,y}(x \notin y \ \vee \ y \notin x)$.

5. On well-orderings

A *partial ordering* (p.o) $(u, <)$ is an irreflexive and transitive relation $<$ on u. If $(u_1, <), (u_2, <)$ are p.o., a function $f : u_1 \to u_2$ is called *order-preserving*, if

$$\forall_{x,y \in u_1}(x < y \to f(x) < f(y)).$$

If $(u_1, <), (u_2, <)$ are linear p.o., an order-preserving f is also called *increasing*. If $f : u_1 \to u_2$ is $1{-}1$ and onto u_2, then f is called an *isomorphism*, if f, f^{-1} are o.p. (in this case we write $u_1 \cong u_2$). If $u_2 = u_1$ and f is an isomorphism, f is called an *automorphism*. A *well-ordering* (w.o.) is a p.o. $(w, <)$ such that

$$\forall_{u \subseteq w}(u \neq \emptyset \to u \text{ has a least element}).$$

Clearly, a w.o. is a linear p.o. (if $x \neq y$, then $\min\{x, y\}$ is in w).

1. If $(w, <)$ is a well-ordered set, then show that there exists no sequence $\alpha : \mathbb{N} \to w$ such that

$$\alpha_0 > \alpha_1 > \alpha_2 > \ldots \ .$$

   Show also that this property (the non-existence of infinitely decreasing chains) implies the existence of a minimum element for each non-empty subset of w (for that one uses actually the principle of dependent choices).

2. **(optional)** If $\alpha, \beta : \omega \to w$ show that there exist $i < j$ such that

$$\alpha(i) \leq \alpha(j) \ \wedge \ \beta(i) \leq \beta(j).$$

3. **(optional)** There exist quasi-orderings (q.o) $(u, \preceq)$ i.e., a reflexive and transitive relation on u, such that every sequence in u is *good*[1] but $(u, \preceq)$ is not a well-ordering.

4. **(optional)** We define
$$\omega^{(\infty)} = ({}^{\omega}\omega, \preceq_p),$$
where
$$\alpha \preceq_p \beta \leftrightarrow \forall_n (\alpha(n) \leq \beta(n)),$$
for each $\alpha, \beta \in {}^{\omega}\omega$. Show that $\omega^{(\infty)}$ has a bad sequence.

5. If $(w, <)$ is a w.o. and $f : w \to w$ is increasing, then
$$\forall_{x \in w}(f(x) \geq x).$$

6. If $(w, <)$ is a w.o. and $f : w \to w$ is an automorphism, then $f = \mathrm{id}_w$.

7. If $(w_1, <)$, $(w_2, <)$ are w.o. and $w_1 \cong w_2$, then the isomorphism between them is unique.

8. If $(w, <)$ is a w.o. and for each $x \in w$ we define
$$\hat{x} = \{y \in w \mid y < x\},$$
then there exists no isomorphism between $w, \hat{x}$.

9. **[T]** If $(w_1, <)$, $(w_2, <)$ are w.o., then
$$w_1 \cong w_2 \ \vee \ \exists_{y \in w_2}(w_1 \cong \hat{y}) \ \vee \ \exists_{x \in w_1}(w_2 \cong \hat{x}).$$
(The proof of this proposition is of the same style to the first Satz of the Vorlesung notes.)

6. On well-founded relations

If (u, r) is a structure, then r is called *well-founded* (w.f.r.), if
$$\forall_{v \subseteq u}(v \neq \emptyset \to \exists_{a \in v} \forall_{x \in v}(x \not r a))$$
i.e., if each non-empty subset v of u has an r-minimal element.

1. Show that a w.f.r. is irreflexive and asymmetric i.e.,
$$\forall_{x \in u}(x \not r x),$$
$$\forall_{x, y \in u}(x \ r \ y \to y \not r x).$$

2. Give an example of a w.f.r. which is not a transitive relation.

[1] A sequence $\alpha : \omega \to w$ is called *good*, if there exist $i < j$ such that $\alpha(i) \preceq \alpha(j)$; otherwise it is called *bad*.

3. Show that if r is a well-ordering, then r is a w.f.r.

4. **(optional)** Find a w.f.r. which is not a w.o.

5. If (u, r) is a w.f.r., there exists no sequence $\alpha : \omega \to u$ such that

$$\alpha_1 r \alpha_0, \ \alpha_2 r \alpha_1, \ \alpha_3 r \alpha_2 \ldots \ .$$

   Show also that this property implies the existence of an r-minimal element for each non-empty subset of u.

6. **(optional)** Formulate and prove the proposition on w.f.r. corresponding to item 5 of the previous section.

7. On transitivity of sets

1. If $u \in V$, show that the following are equivalent:
   (a) u is transitive.
   (b) $\bigcup u \subseteq u$.
   (c) $\bigcup u^+ = u$, where $u^+ = u \cup \{u\}$.

2. If u is a transitive set, then $\bigcup u$ is also transitive.

3. If u is a non-empty set each element of which is transitive, then $\bigcap u$ is also transitive.

4. If u is a non-empty set, then
$$u \text{ is transitive } \rightarrow \bigcap u = \emptyset.$$

5. **[T]** Show that for every set u there exists a transitive set v such that $u \subseteq v$. Describe the least such transitive set[2]; this is called the *transitive closure*, $\mathrm{TC}(u)$, of u.

8. On the definition of addition in ω

1. Using the Rekursionssatz für ω, show the following special case of it:
   If a is a non-empty set, x is a fixed element of a and $h : a \rightarrow a$, there exists a unique function $f : \omega \rightarrow a$ such that
$$f(0) = x$$
$$f(n+1) = h(f(n)).$$

2. Show that if $m \in \omega$, there exists a function $A_m : \omega \rightarrow \omega$ such that
$$A_m(0) = m$$
$$A_m(n+1) = A_m(n) + 1.$$

   Then define the addition of natural numbers $+$ as an appropriate set.

[2]There are many propositions of that kind; e.g., see the existence of an R-closed set including a given one, if R is set-like.

9. On Ordinals

1. They were invented by Cantor to solve a problem in Fourier series.
2. What do the ordinals count, and why are they necessary? As it is noted by T. Forster in [2], "ordinals are the kind of numbers that measures the length of precisely this sort of process: transfinite and discrete". E.g., as it is asked in the exercise 3 of Blatt 3, if (u, r) is a structure and r is founded, we define

$$\begin{aligned} u_0 &:= \emptyset, \\ u_{\alpha+1} &:= \{x \in u \mid \forall_y (y \; r \; x \to y \in u_\alpha\}, \\ u_\lambda &:= \bigcup_{\alpha<\lambda} u_\alpha. \end{aligned}$$

   By Replacement one shows that there exists an ordinal ξ such that $u_{\xi+1} = u_\xi$. Also,

$$u_0 \subseteq u_1 \subseteq \ldots u_\zeta = u,$$

   where ζ is the minimum ordinal satisfying $u_{\zeta+1} = u_\zeta$.

3. The intuition behind addition and multiplication of ordinals.

10. Some basic facts

1. Show that

$$\lim(\lambda) \leftrightarrow \forall_\alpha (\alpha < \lambda \to \alpha + 1 < \lambda).$$

2. Find a non-transitive subset of some ordinal α which does not belong to α.
3. Show that for all ordinals α, β

$$\alpha < s(\beta) \leftrightarrow \alpha \leq \beta.$$

4. If $\alpha, \beta, \gamma, \delta \in \mathrm{On}$, show the following properties:
   (i) $\alpha < \beta \to \gamma + \alpha < \gamma + \beta$. Especially, $0 < \beta \to \gamma < \gamma + \beta$.
   (ii) $\alpha \leq \beta \to \alpha + \gamma \leq \beta + \gamma$. Find α, β, γ such that $\alpha < \beta$ and $\alpha + \gamma = \beta + \gamma$.
   (iii) $\alpha < \beta \to \gamma \cdot \alpha < \gamma \cdot \beta$. Especially, $1 < \beta \to \gamma < \gamma \cdot \beta$.
   (iv) $\alpha \leq \beta \to \alpha \cdot \gamma \leq \beta \cdot \gamma$. Find α, β, γ such that $\alpha < \beta$ and $\alpha \cdot \gamma = \beta \cdot \gamma$.
   (v) $\alpha + \gamma < \beta + \gamma \to \alpha < \beta$.
   (vi) $\alpha \cdot \gamma = \beta \cdot \gamma \to \gamma$ is successor ordinal $\to \alpha = \beta$.
   (vii) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. Is it true that $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$?
5. If $\alpha < \beta$ and $\gamma > 1$, then $\gamma^\alpha < \gamma^\beta$. Check that this doesn't hold for the corresponding operations on cardinals.

6. If ε_0 is the ordinal defined by[3]

$$\varepsilon_0 := \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$$

show that $\omega^{\varepsilon_0} = \varepsilon_0$ and also that ε_0 is the least ordinal α satisfying $\omega^\alpha = \alpha$. Therefore, ε_0 is the least ordinal bigger than ω which is closed under addition, multiplication and exponentiation of ordinals.

7. $\forall_{\alpha,\beta \in \mathrm{On}}(\beta \leq \alpha \to \exists!_{\gamma \in \mathrm{On}}(\alpha = \beta + \gamma))$.

8. $\forall_{\alpha > 0, \gamma} \exists!_{\beta,\rho}(\rho < \alpha \ \wedge \ \gamma = \alpha \cdot \beta + \rho)$.

9. **Cantor's normal form theorem**: Every ordinal $\alpha > 0$ can be written uniquely as

$$\alpha = \omega^{\beta_m} \cdot k_m + \ldots + \omega^{\beta_0} \cdot k_0,$$

where $m \geq 1$, $\alpha \geq \beta_m > \ldots > \beta_0$, and $k_0, \ldots, k_m \in \mathbb{N} \setminus \{0\}$. Thus the ordinal ε_0 has the following convenient representation

$$\varepsilon_0 = \omega^{\varepsilon_0} \cdot 1,$$

and that's why in general $\alpha \geq \beta_m$. The above convenience is one of the reasons we use ω as a base in the representation of ordinals in a normal form.

11. On the cumulative hierarchy

1. Show that $\forall_\alpha(V_\alpha \in V)$.

2. Find a transitive set u having a non-transitive element x.

3. If the rank of a set x is defined as in the Vorlesung by[4]

$$\mathrm{rn}(x) := \min\{\alpha \in \mathrm{On} \mid x \in V_\alpha\},$$

show that $y \in x \to \mathrm{rn}(y) < \mathrm{rn}(x)$. Does the converse hold?

4. Show that the following are equivalent:
(i) Foundation Axiom.
(ii) $V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$.

5. Show the following:
(i) $\forall_{\alpha \in \mathrm{On}}(\alpha \in V_{\alpha+1} \setminus V_\alpha)$.
(ii) $\forall_{\alpha \in \mathrm{On}}(\mathrm{rn}(\alpha) = \alpha + 1)$.

[3] This is a very important ordinal in proof theory as the infamous theorem of Gentzen on the consistency of arithmetic shows. For that see [6], Chapter 10.

[4] If one defines the rank of x as the least α such that $x \in V_{\alpha+1}$, one gets $\mathrm{rn}(\alpha) = \alpha$.

6. Show that
$$A \in V \leftrightarrow \{\mathrm{rn}(x) \mid x \in A\} \text{ is bounded in On}.$$

7. The rank of a set, as defined above, is always a successor ordinal i.e.,
$$\forall_{x \in V} \exists_{\alpha \in \mathrm{On}} (\mathrm{rn}(x) = \alpha + 1),$$
where
$$\alpha = \sup\{\mathrm{rn}(y) \mid y \in x\}.$$

8. Using the standard set-theoretic constructions of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ (see for example [1], Chapter 5), show that $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, {}^{\mathbb{R}}\mathbb{R} \in V_{\omega+\omega}$.

Suppose that $v \in V$ and ϕ is a formula of the language of ZFC. The *relativization* ϕ^u of ϕ to u is generated by replacing all occurrences $x \in V$ in ϕ by $x \in u$. Then we define
$$u \models \phi := \mathrm{ZFC} \vdash \phi^u.$$
Then show the following:

1. $\forall_{\alpha \in \mathrm{On}} (V_\alpha \models$ Extensionality Axiom).
2. $\forall_{\alpha \in \mathrm{On}} (\alpha \neq 0 \to V_\alpha \models$ Empty set Axiom).
3. $\forall_{\alpha \in \mathrm{On}} (\mathrm{limitordinal}(\alpha) \to V_\alpha \models$ Pair Axiom).
4. $\forall_{\alpha \in \mathrm{On}} (V_\alpha \models$ Union axiom).
5. $\forall_{\alpha \in \mathrm{On}} (\mathrm{limitordinal}(\alpha) \to V_\alpha \models$ Power set Axiom).
6. $\forall_{\alpha \in \mathrm{On}} (V_\alpha \models$ Separation Scheme).
7. $\forall_{\alpha \in \mathrm{On}} (\alpha > \omega \to V_\alpha \models$ Infinity Axiom).
8. What about the converse to 2, 3, 5, 7?
9. Consider the following formulation of the Axiom of Choice: if R is a relation, then there exists a function F such that $\mathrm{dom}(F) = \mathrm{dom}(R)$, in order to show:
$$\forall_{\alpha \in \mathrm{On}} (V_\alpha \models \text{ Axiom of Choice}).$$

In other words, if λ is any limit ordinal number $> \omega$, then
$$V_\lambda \models \mathrm{ZF}_0,$$
where
$$\mathrm{ZF}_0 = \mathrm{ZF} \setminus \{\text{Replacement Axiom}\},$$

and ZF is considered here to contain the Separation Scheme[5]. One could ask if the Replacement Scheme is derivable by the rest axioms. Next results show the *independence* of the Replacement Scheme.

(i) Find a well-ordering structure $(w, <)$ such that $< \in V_{\omega+\omega}$, and $\mathrm{otp}(w, <) \notin V_{\omega+\omega}$.
(ii) Conclude by (i) that

$$V_{\omega+\omega} \not\models \text{Replacement Axiom}.$$

The above result is expected from Gödel's second incompleteness theorem. If there was a limit ordinal λ such that $V_\lambda \models$ Replacement Axiom too, then we would have that $V_\lambda \models$ ZF. But then ZF could prove its own consistency, and that contradicts Gödel's second incompleteness theorem.

12. Extensions of Exercise 2, Blatt 3

1. Show that if $\alpha \in \mathrm{On}$, then $\alpha + \omega$ is the least limit ordinal above α. Hence, if λ is a limit ordinal, $\lambda + \omega$ is the immediate limit ordinal above λ. In this case $\lambda + \omega$ can be called the *immediate next* limit ordinal of λ.

2. Show that if λ is a limit ordinal, there is no, generally, an *immediate previous* limit ordinal of λ. Find a limit ordinal with no immediate previous limit ordinal.

3. If α though, is an infinite successor ordinal, then there exists the maximum limit ordinal below α. Moreover, if α is an infinite ordinal, the following are equivalent:

   (i) α is a successor ordinal.
   (ii) α can be written uniquely as

   $$\alpha = \lambda + n,$$

   for some limit ordinal λ and some natural number n.

[5] As you already know the Separation scheme is proved by the Replacement scheme, but in many textbooks it is included in the list of axioms of ZF and it is noted or proved later that it is derivable (see e.g., [3] or [4]). In that way it is clear that ZF_0 contains the required Separation Scheme.

13. Normal functions

A function $F : \mathrm{On} \to \mathrm{On}$ is called *normal*, if it satisfies the following:

(i) F is (strictly) increasing i.e.,

$$\forall_{\alpha,\beta}(\alpha < \beta \to F(\alpha) < F(\beta)).$$

(ii) F is continuous (with respect to the order topology) i.e.,

$$\forall_{\lambda \in \mathrm{LOn}}(F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha)),$$

where LOn denotes the class of limit ordinals.

1. Show that $F_1(\alpha) = \beta + \alpha$, $F_2(\alpha) = \beta \cdot \alpha$ and $F_3(\alpha) = \beta^\alpha$ are normal functions.

2. Give an example of an increasing function which is not continuous (consider e.g., the successor function $\alpha \mapsto \alpha + 1$.)

3. Give an example of a continuous function which is not increasing (consider e.g., a constant function).

4. Show that the function $F : \mathrm{On} \to \mathrm{On}$ defined by

   $$\alpha \mapsto \alpha^2$$

   is not continuous (consider e.g., $n \to \omega$ while $n^2 \to \omega \neq \omega^2$).

5. Suppose that F is continuous. Then the following are equivalent:

   (i) F is normal.
   (ii) $\forall_\alpha(F(\alpha) < F(\alpha + 1))$.
   Hint: Fix an ordinal α satisfying $\alpha < \beta$ and $F(\alpha) \geq F(\beta)$. Take β to be the least ordinal satisfying these properties. If $\beta = \gamma + 1$, then $\alpha \leq \gamma$ and we have by the least property of β that:

   $$F(\gamma + 1) \leq F(\alpha) < F(\gamma) < F(\gamma + 1),$$

   which is a contradiction. Note that $\alpha \neq \gamma$, since otherwise hypothesis (ii) is violated. If $\beta \in \mathrm{LOn}$, then by continuity

   $$F(\beta) = \bigcup_{\delta<\beta} F(\delta) \leq F(\alpha).$$

   But since $\alpha < \beta$ already, we get

   $$F(\alpha) < F(\alpha + 1) \leq F(\beta) \leq F(\alpha),$$

   which is again a contradiction.

6. Suppose that $F : \mathrm{On} \to \mathrm{On}$ is increasing. Then the following hold:

   (i) $\forall_\alpha(F(\alpha) \geq \alpha)$.
   (ii) $\forall_{\alpha,\beta}(F(\alpha) < F(\beta) \to \alpha < \beta)$.
   Hint: (i) is a special case of Exercise 5 of paragraph 5.
   (ii) Suppose that $F(\alpha) < F(\beta)$ and $\alpha \geq \beta$. Use the increasing property of F to reach a contradiction.

7. If F is normal, then F has an unbounded class of *fixed points*. Actually, show that

$$\forall_\alpha \exists_\beta(\alpha \leq \beta \ \wedge \ F(\beta) = \beta \ \wedge \ \forall_{\alpha<\gamma<\beta}(F(\gamma) > \gamma)),$$

   i.e., β is the least fixed point of F above α.
   Hint: Define

$$\beta := \bigcup_{n\in\omega} \beta_n,$$

   where

$$\beta_0 := \alpha,$$
$$\beta_{n+1} := F(\beta_n).$$

   Then,

$$F(\beta) = F(\bigcup_{n\in\omega} \beta_n) = \bigcup_{n\in\omega} F(\beta_n) = \bigcup_{n\in\omega} \beta_{n+1} = \beta.$$

   Note that if α itself is a fixed point, then $\beta = \alpha$, and trivially α satisfies the least property. If α is not a fixed point, consider an ordinal γ such that $\alpha < \gamma < \beta$ and $F(\gamma) = \gamma$. Then there exists n such that $\gamma < \beta_n$ and we can take n to be the least natural having this property. Since there is an m such that $n = m + 1$ we have that

$$\beta_m \leq \gamma < \beta_{m+1}.$$

   Hence

$$\gamma < \beta_{m+1} = F(\beta_m) \leq F(\gamma) = \gamma,$$

   which is of course a contradiction.

8. **[T]** If F is normal, then the *derivative* F' of F is the function $F' : \mathrm{On} \to \mathrm{On}$ defined by:

$$F'(0) := \mu\beta(F(\beta) = \beta)$$
$$F'(\alpha + 1) := \mu\beta(F'(\alpha) < \beta \ \wedge \ F(\beta) = \beta)$$
$$F'(\lambda) := \mu\beta(\forall_{\alpha<\lambda}(F'(\alpha) < \beta) \ \wedge \ F(\beta) = \beta),$$

   where λ is a limit ordinal and $\mu\beta(\phi(\beta))$ denotes the minimum ordinal satisfying the formula ϕ. Then the following hold:

(i) F' enumerates the fixed points of F, in other words

$$\mathrm{rng}(F') = \mathrm{Fix}(F),$$

where $\mathrm{Fix}(F)$ denotes the class of the fixed points of F.
(ii) Show that F' is also normal.

Hint: (i) Clearly, $\mathrm{rng}(F') \subseteq \mathrm{Fix}(F)$, and we show equality by assuming that $\mathrm{Fix}(F) \setminus \mathrm{rng}(F') \neq \emptyset$. Then we consider the least ordinal in $\mathrm{Fix}(F) \setminus \mathrm{rng}(F')$, which cannot be 0, and we reach a contradiction supposing that α is a successor ordinal, or α is a limit ordinal.
(ii) To show that F is normal we first see that by the definition of F'

$$F'(\alpha) < F'(\alpha + 1),$$

for each α. To show that F is continuous it suffices to show that $\bigcup_{\alpha<\lambda} F'(\alpha)$ satisfies the definition of $F'(\lambda)$, if $\lambda \in \mathrm{LOn}$. Clearly $\bigcup_{\alpha<\lambda} F'(\alpha) \in \mathrm{Fix}(F)$, since by the continuity of F

$$F(\bigcup_{\alpha<\lambda} F'(\alpha)) = \bigcup_{\alpha<\lambda} F(F'(\alpha)) = \bigcup_{\alpha<\lambda} F'(\alpha).$$

Also,

$$\forall_{\alpha<\lambda}(F'(\alpha) < \bigcup_{\alpha<\lambda} F'(\alpha)),$$

since

$$F'(\alpha) < F'(\alpha + 1) \leq \bigcup_{\alpha<\lambda} F'(\alpha).$$

To show that $\bigcup_{\alpha<\lambda} F'(\alpha)$ is the least ordinal satisfying the above properties we consider β, an arbitrary fixed point of F, such that $\forall_{\alpha<\lambda}(F'(\alpha) < \beta)$. But then we have immediately that

$$\bigcup_{\alpha<\lambda} F'(\alpha) \leq \beta.$$

By Exercise 5 we conclude that F' is normal.

9. **[T]** A closed and unbounded class of ordinals is called a *club*. Show that if $F : \mathrm{On} \to \mathrm{On}$ then

$$F \text{ is normal} \to \mathrm{rng}(F) \text{ is a club}.$$

Corollary: Using the normality of the derivative F' of some normal function F and the fact $\mathrm{rng}(F') = \mathrm{Fix}(F)$, we conclude that the class of fixed points of a normal function is a club.

Hint: Suppose that $\mathrm{rng}(F)$ is bounded, therefore it is a set, and let $\gamma = \bigcup \mathrm{rng}(F)$. Then by Exercise 6(i) we have that

$$\gamma \leq F(\gamma) < F(\gamma + 1) \in \mathrm{rng}(F),$$

which is absurd. To show that $\mathrm{rng}(F)$ is closed we suppose that $u \neq \emptyset$ and $u \subseteq \mathrm{rng}(F)$, and we show that $\sup u \in \mathrm{rng}(F)$ too. Of course, $\sup u \neq 0$, since $u \neq \emptyset$. If $\sup u = \alpha + 1$, then $\alpha < \sup u$, therefore there exists some $\beta \in u$ such that $\alpha < \beta \leq \alpha + 1$, hence $\beta = \alpha + 1$ and $\sup u \in \mathrm{rng}(F)$. Suppose now that $\sup u = \lambda$, for some $\lambda \in \mathrm{LOn}$. Then the class

$$v = \{\beta \in \mathrm{On} \mid \exists_{\alpha \in u}(F(\beta) = \alpha)\}$$

is a set, since F is an $1{-}1$ function. But then

$$F(\bigcup v) = \bigcup F(v) = \sup u,$$

i.e., $\sup u \in \mathrm{rng}(F)$.

Remark: The converse implication doesn't hold: Find a function $F : \mathrm{On} \to \mathrm{On}$ such that $\mathrm{rng}(F)$ is a club but F is not normal!

10. Show that every club C is the range of a unique normal function F.

    Hint: Define the function F which "enumerates in order" the elements of C and show that it is normal.

11. (i) Show that a club is a proper class.
    (ii) Find a closed class of ordinals which is not unbounded, and also an unbounded class of ordinals which is not closed.

12. **[T]** If C_1, C_2 are clubs, then their intersection $C_1 \cap C_2$ is also a club. Hence, if F_1, F_2 are normal functions, the class of their common fixed points is a club. This, of course, applies to the normal functions F, F'.

    Hint: First we show that $C_1 \cap C_2$ is closed: we suppose that $u \neq \emptyset$ and $u \subseteq C_1 \cap C_2$ and we show that $\sup u \in C_1 \cap C_2$. But since C_1, C_2 are closed, $\sup u \in C_1$, and $\sup u \in C_2$. To show that $C_1 \cap C_2$ is unbounded we fix an ordinal α and we form a sequence of ordinals

    $$\alpha < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots,$$

    such that $\alpha_i \in C_1$ and $\beta_i \in C_2$, for each $i \in \omega$. Clearly,

    $$\alpha < \sup_n \alpha_n = \sup_n \beta_n \in C_1 \cap C_2.$$

13. Suppose that $F, G : \mathrm{On} \to \mathrm{On}$ are normal functions. Then the following hold:

    (i) Their composition $F \circ G$ is a normal function.
    (ii) $\mathrm{Fix}(F \circ G) = \mathrm{Fix}(F) \cap \mathrm{Fix}(G)$.

14. If F is normal, then

    $$\lambda \text{ is a limit ordinal} \to F(\lambda) \text{ is a limit ordinal.}$$

15. If F is normal, then

$$\forall_\beta \exists_\alpha (F(\alpha) \leq \beta \ \wedge \ \forall_\gamma (F(\gamma) \leq \beta \to \gamma \leq \alpha)),$$

i.e., given an ordinal β and a normal function F, then $F(\alpha)$ is the best approximation to β below that one can give using F.

16. Using 7 and 15, try to sketch a graph modeling the graph of a normal function.

17. Show that the function $\omega : \mathrm{On} \to \mathrm{On}$ defined by

$$\alpha \mapsto \omega_\alpha$$

is normal and also that $\mathrm{rng}(\omega) = \mathrm{Kn} \setminus \omega$, where Kn denotes the class of cardinals.

18. Show that the class of limit ordinals LOn is a club (use paragraph 12 for that).

Solutions to selected exercises

Blatt 6, Aufgabe 1: The following is a simple consequence of the Collection principle (Beschränkung): if $\phi(x, y, \vec{z})$ is a ZF-formula,

$$(*) \quad \forall_{x \in u} \exists_y (\phi(x, y, \vec{z})) \to \exists_v \forall_{x \in u} \exists_{y \in v} (\phi(x, y, \vec{z})).$$

In order to show that $u \times v \in V$ within ZF$^-$ we fix $x \in u$ and by the pairing in ZF$^-$ we get

$$\forall_{y \in v} \exists_w (w =< x, y >).$$

Applying $(*)$ we get

$$\exists_a \forall_{y \in v} \exists_{w \in a} (w =< x, y >).$$

Since x was an arbitrary element of u, we have actually shown that

$$\forall_{x \in u} \exists_a \forall_{y \in v} \exists_{w \in a} (w =< x, y >).$$

Applying $(*)$ once more we get

$$\exists_c (\forall_{x \in u} \exists_{w' \in c} \forall_{y \in v} \exists_{w \in w'} (w =< x, y >)).$$

Next we define

$$d := \bigcup c.$$

Since $w =< x, y >$ and $w \in w' \in c$ we get

$$\forall_{x \in u} \forall_{y \in v} (< x, y > \in d).$$

Hence

$$u \times v = \{e \in d \mid \exists_{x \in u} \exists_{y \in v} (e =< x, y >),$$

which is in V by Σ_0-comprehension (Aussonderung). Note that all along this proof we used only Σ_0-collection i.e., the collection principle, and the corresponding $(*)$, for Σ_0-formulas $\phi(x, y, \vec{z})$.

Remark on Aufgabe 1, Blatt 7: For the proof of the direction

$$\exists_F (F : \mathrm{On} \overset{\text{onto}}{\longrightarrow} V) \to V = \mathrm{OD}$$

we suppose that F is a class defined as

$$F = \{(x, y) \mid \phi(x, y)\},$$

where ϕ is free of parameters.

Blatt 8, Aufgabe 2: By the definition of TC(x) we have that

$$y = \mathrm{TC}(x) \leftrightarrow \ \text{transitive}(y) \ \wedge \forall_t (\text{transitive}(t) \wedge x \subseteq t \to y \subseteq t),$$

which is a Π_1-formula. Also, by the construction of $\mathrm{TC}(x)$ we have that

$$y = \mathrm{TC}(x) \leftrightarrow \exists_f(\mathrm{function}(f) \ \wedge \ \mathrm{dom}(f) = \omega \ \wedge \ y = \bigcup \mathrm{rng}(f) \ \wedge$$

$$\wedge \ (0, x) \in f \ \wedge \ \forall_{n\in\omega}\forall_{w\in\mathrm{rng}(f)}((n, w) \in f \to (n+1, \bigcup w) \in f)),$$

which is a Σ_1-formula, therefore $y = \mathrm{TC}(x)$ is a Δ_1-formula.

A fact related to Blatt 8, Aufgabe 3: Suppose that $y = G(x)$ is a Σ_1-formula and

$$F(\alpha) = G(F \restriction \alpha),$$

for each $\alpha \in \mathrm{On}$. Then the formula $y = F(x)$ is Δ_1.

Proof: F is a class-function and each of its restrictions $F \restriction \alpha$ is a set-function. Moreover,

$$\forall_{\alpha,\beta\in\mathrm{On}}(\alpha < \beta \to F \restriction \alpha \subseteq F \restriction \beta).$$

If we call such an initial segment of F a *good* function, we describe it as follows:

$$\mathrm{good}(f) \leftrightarrow \mathrm{ordinal}(\mathrm{dom}(f)) \ \wedge \ \forall_{\alpha\in\mathrm{dom}(f)}(f(\alpha) = G(F \restriction \alpha)).$$

Since $y = G(x)$ is a Σ_1-formula, it follows that $\mathrm{good}(f)$ is also Σ_1. Since two good functions with the same domain are identical we have the following equivalences:

$$x = F(\alpha) \leftrightarrow \exists_f(\mathrm{good}(f) \ \wedge \ \alpha \in \mathrm{dom}(f) \ \wedge \ f(\alpha) = x),$$

$$x = F(\alpha) \leftrightarrow \forall_f(\mathrm{good}(f) \ \wedge \ \alpha \in \mathrm{dom}(f) \ \to \ f(\alpha) = x).$$

By the first equivalence we get that $x = F(\alpha)$ is equivalent to a Σ_1-formula (if we add an existential quantifier to a Σ_1-formula we get one which is again equivalent to a Σ_1-formula). By the second equivalence we get that $x = F(\alpha)$ is equivalent to a Π_1-formula (since the formula $\mathrm{good}(f) \ \wedge \ \alpha \in \mathrm{dom}(f) \ \to \ f(\alpha) = x$) is already equivalent to a Π_1-formula; $\phi \to \psi \leftrightarrow \neg\phi \vee \psi$ and the negation of a Σ_1-formula is equivalent to a Π_1-formula), and if we add a universal quantifier to a Π_1-formula we get one which is equivalent to a Π_1-formula.

Hence, the formula $x = F(\alpha)$ is Δ_1.

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