Bachelor's Thesis

Constructive Differentiability

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Declaration

Eidesstattliche Erklärung
Hiermit erkläre ich, dass ich die vorliegende Arbeit eigenständig und ohne fremde Hilfe angefertigt habe. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Ort, Datum	Unterzeichner

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1 Introduction

In the early 20^{th} century, L. E. J. Brouwer developed the first concept of constructive mathematics. As the name suggest, constructive mathematics distinguishes from the classical counterpart by the aim of finding or rather constructing a mathematical object in order to prove its existence. In classical mathematics we work with axioms which allow the principle of indirect proofs, also called proof by contradiction. Thereby one assumes the non-existence of a mathematical object and then derives a contradiction from the assumption. However, this kind of proof is not valid in constructive mathematics. The constructive viewpoint requires a verificational interpretation not only of the existential quantifier but all the logical expressions.

In his time, Brouwer could only convince a few mathematicians. However, he was the one who laid the foundation of an accurate, structured approach to constructive mathematics. Though, his approach differs from constructive mathematics known today. The so-called intuitionism is considered to be a philosophy of mathematics, in which mathematics is assumed to be the result of precise, constructive thinking, which produces its own objects and does not presuppose them.[1]

It took quite some years until a classical mathematician achieved progress in the development of constructive mathematics. E. Bishop published a modern version of Brouwer's view in his book *Foundations of Constructive Analysis* in 1967 [1]. Unlike Brouwer, Bishop's constructive mathematics (BISH) (see also [5] and [6]) is not contradictory to classical mathematics. Instead, he developed a large part of the 20^{th} century's classical analysis further into constructive analysis. His approach is that for every constructive theorem and proof in BISH, a counterpart theorem and proof in classical analysis exists. See also Bishop's and D. Bridges's book *Constructive Analysis* from 1985 [2], which is a revision and extension of [1].

In 2006 M. Bridger published *Real Analysis: A Constructive Approach* [3]. He follows the same path as Bishop; to prove an existing theorem by providing a construction of the object in question. However, Bridger's goal was a bit different: he wants to show, that the constructive approach "makes sense - not just to math majors, but to students from all branches of the sciences" [3, Preface].

The goal of this thesis is to to study the field of differentiability in constructive analysis as this is done by Bishop and Bridger. Therefore we have to introduce all relevant definitions and notions, following Bishop's book *Constructive Analysis* [2] and Bridger's book *Real Analysis: A Constructive Approach* [3]. The thesis will be structured as follows:

According to Bishop, we find that in constructive analysis, unlike in classical analysis, we only work with uniform continuity. This concept and its basic properties are introduced in chapter two, which allows us to study uniform differentiability.

[1] See Bridges' and Palmgren's *Constructive Mathematics*, 2018 [4]

In chapter three we discuss uniform differentiability by working with Bishop's elaboration on constructive differentiability. Thereby we use Definitions and Propositions given by Bishop but structure them in a new way. Besides basic properties of differentiability, we introduce Bishop's approximation to Rolle's Theorem, to the Mean Value Theorem and to Taylor's Theorem. These approximations can be proven constructively.

After studying Bishop's constructive differentiability we continue with the study of Bridger's approach to uniform differentiability in constructive analysis. In order to define uniform differentiability, Bridger provides three different definitions of differentiability and proves their equivalence. Since Bridger's definition of differentiability demands for fewer assumptions, we complete the chapter with additional theorems on continuity.

Finally we compare two approaches to uniform differentiability and discuss their differences. Thereby we find out, that Bridger follows more closely the idea of constructive analysis. In addition, we prove that we can apply Bridger's theorems on continuity to a modeled version of Bishop's differentiability.

2 Bishop continuity

In classical analysis, one can prove that pointwise continuity on a compact interval implies uniform continuity. However, a proof of the so-called Uniform Continuity Theorem is not expected to be given in constructive analysis and not a disprove of it either. The concept of uniform continuity is given by the next Definition. We abbreviate this concept to continuity, since the classical version of continuity - that is pointwise continuity - is not used within this thesis. By the assumption of uniform continuity from the start, nothing essential is lost.

Definition 2.1 (Bishop - continuity)**.** Let $[a, b]$ be a compact interval, $a, b \in \mathbb{R}, a < b$ and $f : [a, b] \to \mathbb{R}$ and $\omega_f : \mathbb{R}^+ \to \mathbb{R}^+$ be functions. The function f is called *continuous* on $[a, b]$ if for each positive real number ε and for x, y in $[a, b]$ such that $|x - y| \leq \omega_f(\varepsilon)$ we have

$$|f(x) - f(y)| \leq \varepsilon.$$

The function $\omega_f : \varepsilon \mapsto \omega(\varepsilon)$ is called a *modulus of continuity* for f on $[a, b]$.

We also say the pair (f, ω_f) is in $\mathrm{Cont_B}\big([a, b]\big)$ with

$$\mathrm{Cont_B}\big([a, b]\big) := \Big\{(f, \omega_f) \in \mathbb{F}([a, b], \mathbb{R}) \times \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+) \;\big|\; \omega_f : \mathrm{cont_B}(f)\Big\}$$

and $\omega_f : \mathrm{cont_B}(f) :\Leftrightarrow \forall \varepsilon > 0 \; \forall x, y \in [a, b] : \Big(|x - y| \leq \omega_f(\varepsilon) \Rightarrow |f(x) - f(y)| \leq \varepsilon\Big).$

The pair $(f, \omega_f) \in \mathbb{F}(J, \mathbb{R}) \times \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+)$ on an arbitrary interval J is in $\mathrm{Cont_B}(J)$, thus f is *continuous* on J, if it is continuous on every compact subinterval of J.

Remark 2.2.

1. $\mathbb{F}([a, b], \mathbb{R})$ is the set of all functions $f : [a, b] \to \mathbb{R}$.
2. Instead of saying f is a continuous function on $[a, b]$ we may also say f is in $C([a, b])$.
3. $f \vee g = \max_{x \in I}\{f(x), g(x)\}$

Theorem 2.3. Let I be an interval in $\mathbb{R}$ and (f, ω_f) and (g, ω_g) be in $\mathrm{Cont_B}(I)$. Then there exist moduli of continuity $\omega_{f+g}, \omega_{fg}$ and $\omega_{f \vee g}$ in $\mathbb{F}(\mathbb{R}^+, \mathbb{R}^+)$ such that the pairs $(f + g, \omega_{f+g}), (fg, \omega_{fg})$ and $(f \vee g, \omega_{f \vee g})$ are in $\mathrm{Cont_B}(I)$.

If f is bounded away from 0 on every compact subinterval J of I - that is, if $|f(x)| \geq c$ for all x in J and some $c > 0$ (depending on J) - then there exists $\omega_{f^{-1}}$ such that the pair $(f^{-1}, \omega_{f^{-1}})$ is in $\mathrm{Cont_B}(I)$.

Proof. Due to the definition of continuity, it is enough to consider the case in which I is compact. (f, ω_f) and (g, ω_g) are in $\mathrm{Cont_B}(I)$. Thus ω_f and ω_g are moduli of continuity of f and g.

1. Goal: $(f + g, \omega_{f+g}) \in \mathrm{Cont_B}(I)$
   We begin by writing out, what we must prove.
   Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|x - y| \leq \omega_{f+g}(\varepsilon) = \min\big\{\omega_f\big(\frac{\varepsilon}{2}\big), \omega_g\big(\frac{\varepsilon}{2}\big)\big\}$. We must show:
   $$|(f + g)(x) - (f + g)(y)| \leq \varepsilon$$

Since we know about f and g separately, we rearrange and use the triangle inequality to obtain

$$\begin{aligned}|(f+g)(x)-(f+g)(y)| &= |f(x)+g(x)-f(y)-g(y)|\\ &\le |f(x)-f(y)|+|g(x)-g(y)|\\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

The second inequality holds, since (f,ω_f) and (g,ω_g) are in $\mathrm{Cont_B}(I)$ and $|x-y|\le \omega_{f+g}(\varepsilon)$. Therefore $(f+g,\omega_{f+g})$ is in $\mathrm{Cont_B}(I)$ with $\omega_{f+g}(\varepsilon)=\min\{\omega_f(\frac{\varepsilon}{2}),\omega_g(\frac{\varepsilon}{2})\}$ being a modulus of continuity of $f+g$.

2. Goal: $(fg,\omega_{fg})\in \mathrm{Cont_B}(I)$
We begin by writing out, what we must prove.
Let $\varepsilon>0$ be arbitrary, $x,y\in I$ s.t. $|x-y|\le \omega_{fg}(\varepsilon)=\min\{\omega_f(\frac{\varepsilon}{2m}),\omega_g(\frac{\varepsilon}{2m})\}$ with

$$m:=\max\Big\{\max\{|f(x)|:x\in I\},\max\{|g(x)|:x\in I\}\Big\}+1$$

We must show:

$$|(fg)(x)-(fg)(y)|\le\varepsilon$$

By using the triangle inequality we obtain

$$\begin{aligned}|(fg)(x)-(fg)(y)| &= |f(x)g(x)-f(y)g(y)|\\ &= |f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)|\\ &\le |f(x)g(x)-f(x)g(y)|+|f(x)g(y)-f(y)g(y)|\\ &\le |f(x)|\cdot|g(x)-g(y)|+|f(x)-f(y)|\cdot|g(y)|\\ &\le |f(x)|\frac{\varepsilon}{2m}+\frac{\varepsilon}{2m}|g(y)|\\ &= \big(|f(x)|+|g(y)|\big)\cdot\frac{\varepsilon}{2m}\\ &\le (m+m)\cdot\frac{\varepsilon}{2m}=\varepsilon\end{aligned}$$

Therefore (fg,ω_{fg}) is in $\mathrm{Cont_B}(I)$ with $\omega_{fg}(\varepsilon)=\min\{\omega_f(\frac{\varepsilon}{2m}),\omega_g(\frac{\varepsilon}{2m})\}$ being a modulus of continuity of fg.

3. Goal: $(f\vee g,\omega_{f\vee g})\in \mathrm{Cont_B}(I)$
Since it is for $x\in I$

$$(f\vee g)(x)=\max\{f,g\}(x)=\frac{f(x)+g(x)}{2}+\frac{|f(x)-g(x)|}{2}$$

we can prove $(\alpha f,\omega_{\alpha f})$ is in $\mathrm{Cont_B}(I), \alpha\in\mathbb{R}\setminus\{0\}$ and $(|f|,\omega_{|f|})$ is in $\mathrm{Cont_B}(I)$. Then by $(f+g,\omega_{f+g})\in \mathrm{Cont_B}(I)$ it follows that $(f\vee g,\omega_{f\vee g})$ is in $\mathrm{Cont_B}(I)$.
Let $\alpha\in\mathbb{R}\setminus\{0\}, \varepsilon>0$ be arbitrary, $x,y\in I$ s.t. $|x-y|\le\omega_{\alpha f}(\varepsilon)=\omega_f(\frac{\varepsilon}{|\alpha|})$. Then

$$|(\alpha f)(x)-(\alpha f)(y)|=|\alpha||f(x)-f(y)|\le|\alpha|\cdot\frac{\varepsilon}{|\alpha|}=\varepsilon$$

and therefore $(\alpha f,\omega_{\alpha f})$ is in $\mathrm{Cont_B}(I)$.

Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|x - y| \leq \omega_{|f|}(\varepsilon) = \omega_f(\varepsilon)$. Then by using the inverse triangle inequality we obtain

$$\big|(|f|)(x) - (|f|)(y)\big| = \big||f(x)| - |f(y)|\big| \leq \big|f(x) - f(y)\big| \leq \varepsilon$$

and therefore $(|f|, \omega_{|f|})$ is in $\mathrm{Cont}_{\mathrm{B}}(I)$.

Thus it follows that $(f \vee g, \omega_{f \vee g})$ is in $\mathrm{Cont}_{\mathrm{B}}(I)$.

4. Goal: $(f^{-1}, \omega_{f^{-1}}) \in \mathrm{Cont}_{\mathrm{B}}(I)$
f is bounded away from 0 on every compact subinterval J of I. Thus there exists a $c > 0$ s.t. for all $x \in I$ we have $|f(x)| > c$. We begin by writing out, what we must prove.
Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|x - y| \leq \omega_{f^{-1}}(\varepsilon) = \omega_f(\varepsilon c^2)$. We must show:

$$|(f)^{-1}(x) - (f)^{-1}(y)| \leq \varepsilon$$

Since f is bounded away from zero we obtain

$$\begin{aligned} \left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| &= \left| \frac{f(y) - f(x)}{f(x)f(y)} \right| \\ &= \frac{|f(x) - f(y)|}{|f(x)||f(y)|} \\ &\leq \frac{|f(x) - f(y)|}{c^2} \\ &\leq \frac{\varepsilon \cdot c^2}{c^2} = \varepsilon \end{aligned}$$

Therefore $(f^{-1}, \omega_{f^{-1}})$ is in $\mathrm{Cont}_{\mathrm{B}}(I)$ with $\omega_{f^{-1}}(\varepsilon) = \omega_f(\varepsilon m^2)$ being a modulus of continuity of f^{-1}.

$\square$

3 Bishop differentiability

Besides continuity, differentiability is a fundamental property of a function. It describes the rate at which a function changes. The following section provides the basic properties of this concept defined by Bishop. The concept introduced in the next Definition is classically called uniform differentiability. Similarly to continuity, we abbreviate this to differentiability.

3.1 Definition and basic properties

Definition 3.1 (Bishop - differentiability (Bd))**.** Let $[a,b]$ be a compact interval, $a,b \in \mathbb{R}, a < b$, let f and g be in $C([a,b])$ and let $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. The function f is called *differentiable* on $[a,b]$ with g being its *derivative*, if for each positive real number ε and for x,y in $[a,b]$ such that $|x-y| \le \delta(\varepsilon)$ we have

$$|f(y) - f(x) - g(x)(y-x)| \le \varepsilon|y-x|.$$

The function $\delta_f : \varepsilon \mapsto \delta_f(\varepsilon)$ is called a *modulus of differentiability* for f on $[a,b]$.

We also say the triplet (f, g, δ_f) is in $\mathrm{Dif}_\mathrm{B}\big([a,b]\big)$ with

$$\mathrm{Dif}_\mathrm{B}\big([a,b]\big) := \Big\{(f,g,\delta_f) \in C([a,b]) \times C([a,b]) \times \mathbb{F}(\mathbb{R}^+,\mathbb{R}^+) \ \Big|\ \delta_f : \mathrm{dif}_\mathrm{B}(f)\Big\}$$

and $\delta_f : \mathrm{dif}_\mathrm{B}(f) :\Leftrightarrow \forall \varepsilon > 0 \ \forall x,y \in [a,b] :$

$$\Big(|y-x| \le \delta(\varepsilon) \Rightarrow |f(y) - f(x) - g(x)(y-x)| \le \varepsilon|y-x|\Big)$$

The triplet $(f,g,\delta_f) \in C(J) \times C(J) \times \mathbb{F}(\mathbb{R}^+,\mathbb{R}^+)$ on an arbitrary interval J is in $\mathrm{Dif}_\mathrm{B}(J)$, thus f is *differentiable* on J with derivative g and modulus of differentiability δ_f in J, if f is differentiable on every proper compact subinterval of J.

Remark 3.2.

1. For the derivative g we also write $g = f', g = Df$, or $g(x) = \frac{df(x)}{dx}$.
2. As already mentioned, g is the rate of change of f. This interpretation comes from the *difference quotient*
$$\frac{f(y) - f(x)}{y - x}$$
that approaches $g(x)$ as y approaches x.

Theorem 3.3 (Calculation rules for the derivative)**.** Let I be an arbitrary interval in $\mathbb{R}$ and $(f_1, f_1', \delta_{f_1})$ and $(f_2, f_2', \delta_{f_2})$ be in $\mathrm{Dif}_\mathrm{B}(I)$. Then there exist moduli of differentiability $\delta_{f_1+f_2}$ and $\delta_{f_1 f_2}$ in $\mathbb{F}(\mathbb{R}^+,\mathbb{R}^+)$, such that the triplets

1. $(f_1 + f_2, f_1' + f_2', \delta_{f_1+f_2})$ and
2. $(f_1 f_2, f_1 f_2' + f_2 f_1', \delta_{f_1 f_2})$ are in $\mathrm{Dif}_\mathrm{B}(I)$.

In case f_1 is bounded away from 0 on every compact subinterval of I, there exists $\delta_{f_1^{-1}}$ in $\mathbb{F}(\mathbb{R}^+,\mathbb{R}^+)$, such that the triplet

3. $(f_1^{-1}, -f_1' f_1^{-2}, \delta_{f_1^{-1}})$ is in $\mathrm{Dif}_\mathrm{B}(I)$.

In addition, for the identity function id and a constant function h there exist moduli of differentiability such that

4. $(\mathrm{id}, 1, \delta_\mathrm{id})$ and
5. $(h, 0, \delta_h)$ are in $\mathrm{Dif}_\mathrm{B}(I)$.

Proof. Due to Definition 3.1 it is sufficient to proof the theorem for I being a compact interval. $(f_1, f_1', \delta_{f_1})$ and $(f_2, f_2', \delta_{f_2})$ are in $\mathrm{Dif}_{\mathrm{B}}(I)$, thus δ_{f_1} and δ_{f_2} are moduli of differentiability of f_1 and f_2. As $f_1 \in C(I)$ let ω_{f_1} be a corresponding modulus of continuity.

1. The modulus of differentiability $\delta_{f_1+f_2} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\varepsilon \mapsto \min\left\{\delta_{f_1}\left(\frac{\varepsilon}{2}\right), \delta_{f_2}\left(\frac{\varepsilon}{2}\right)\right\}$. We begin by writing out, what we must prove:
   Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|y - x| \leq \delta_{f_1+f_2}(\varepsilon)$. We must show
   $$|(f_1 + f_2)(y) - (f_1 + f_2)(x) - (f_1' + f_2')(x)(y - x)| \leq \varepsilon|y - x|$$
   Since we know about f_1 and f_2 separately, we rearrange and use the triangle inequality to obtain
   $$\begin{aligned} A &= |(f_1 + f_2)(y) - (f_1 + f_2)(x) - (f_1' + f_2')(x)(y - x)| \\ &\leq \underbrace{|(f_1)(y) - (f_1)(x) - (f_1')(x)(y - x)|}_{\leq \frac{\varepsilon}{2}|y-x|} + \underbrace{|(f_2)(y) - (f_2)(x) - (f_2')(x)(y - x)|}_{\leq \frac{\varepsilon}{2}|y-x|} \\ &\leq \varepsilon|y - x| \end{aligned}$$
   The second inequality holds, since $(f_1, f_1', \delta_{f_1})$ and $(f_2, f_2', \delta_{f_2})$ are in $\mathrm{Dif}_{\mathrm{B}}(I)$ and $|y - x| \leq \delta_{f_1+f_2}(\varepsilon)$. Therefore $\left(f_1 + f_2, f_1' + f_2', \delta_{f_1+f_2}\right)$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$ with $\delta_{f_1+f_2}(\varepsilon) = \min\left\{\delta_{f_1}\left(\frac{\varepsilon}{2}\right), \delta_{f_2}\left(\frac{\varepsilon}{2}\right)\right\}$ being a modulus of differentiability of $f_1 + f_2$.

2. Since f_1, f_2 and $f_2' \in C(I)$ with I being a compact interval, their suprema and infima on I exists. Therefore we can define
   $$M := \max\Big\{ \max\big\{|f_1(x)| : x \in I\big\}, \max\big\{|f_2(x)| : x \in I\big\}, \max\big\{|f_2'(x)| : x \in I\big\}\Big\} + 1$$
   The modulus of differentiability of $f_1 f_2$ is defined by $\delta_{f_1 f_2} : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varepsilon \mapsto \min\left\{\delta_{f_1}\left(\frac{\varepsilon}{3M}\right), \delta_{f_2}\left(\frac{\varepsilon}{3M}\right), \omega_{f_1}\left(\frac{\varepsilon}{3M}\right)\right\}$.
   Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|y - x| \leq (\delta_{f_1 f_2})(\varepsilon)$. We must show:
   $$|(f_1 f_2)(y) - (f_1 f_2)(x) - (f_1 f_2' + f_2 f_1')(x)(y - x)| \leq \epsilon|y - x|$$
   By using the triangle inequality, we obtain:
   $$\begin{aligned} B &= |(f_1 f_2)(y) - (f_1 f_2)(x) - (f_1 f_2' + f_2 f_1')(x)(y - x)| \\ &= |f_1(y) f_2(y) - f_1(x) f_2(x) - f_1(x) f_2'(x)(y - x) + f_2(x) f_1'(x)(y - x)| \\ &= |f_1(y) f_2(y) - f_1(y) f_2(x) + f_1(y) f_2(x) - f_1(x) f_2(x) - f_1(y) f_2'(x)(y - x) \\ &\quad - f_1(x) f_2'(x)(y - x) + f_2(x) f_1'(x)(y - x) + f_1(y) f_2'(x)(y - x)| \\ &\leq |f_1(y)||f_2(y) - f_2(x) - f_2'(x)(y - x)| \\ &\quad + |f_2(x)||f_1(y) - f_1(x) - f_1'(x)(y - x)| \\ &\quad + |f_1(y) - f_1(x)||f_2'(x)(y - x)| \\ &\leq 3M \frac{\varepsilon}{3M}|y - x| \\ &= \varepsilon|y - x| \end{aligned}$$
   The last but one inequality holds, since $(f_1, f_1', \delta_{f_1})$ and $(f_2, f_2', \delta_{f_2})$ are in $\mathrm{Dif}_{\mathrm{B}}(I)$ and $|y - x| \leq \delta_{f_1 f_2}(\varepsilon)$. Therefore $\left(f_1 f_2, f_1 f_2' + f_2 f_1', \delta_{f_1 f_2}\right)$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$ and a modulus of differentiability of $f_1 f_2$ is $\delta_{f_1 f_2}(\varepsilon) = \min\left\{\delta_{f_1}\left(\frac{\varepsilon}{3M}\right), \delta_{f_2}\left(\frac{\varepsilon}{3M}\right), \omega_{f_1}\left(\frac{\varepsilon}{3M}\right)\right\}$.

3. As I is compact and f_1' and $f_1^{-1} \in C(I)$, set

$$M := \max\Big\{\max\big\{|f_1^{-1}| : x \in I\big\}, \max\big\{|f_1'| : x \in I\big\}\Big\} + 1$$

The modulus of differentiability $\delta_{f_1^{-1}} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\varepsilon \mapsto \min\big\{\delta_{f_1}\big(\frac{\varepsilon}{2M^2}\big), \omega_{f_1}\big(\frac{\varepsilon}{2M^4}\big)\big\}$. Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|y-x| \leq \delta_{f_1^{-1}}(\varepsilon)$. We must show:

$$\left|\frac{1}{f_1(y)} - \frac{1}{f_1(x)} - \left(-\frac{f_1'(x)}{f_1(x)^2}\right)(y-x)\right| \leq \varepsilon|y-x|$$

By using the triangle inequality, we obtain:

$$\begin{aligned}
C &= \left|\frac{1}{f_1(y)} - \frac{1}{f_1(x)} - \left(-\frac{f_1'(x)}{f_1(x)^2}\right)(y-x)\right| \\
&= \frac{1}{|f_1(x)f_1(y)|}\left|f_1(x) - f_1(y) + \frac{f_1'(x)f_1(y)}{f_1(x)}(y-x)\right| \\
&= \underbrace{\frac{1}{|f_1(x)f_1(y)|}}_{\leq M^2}\left|f_1(y) - f_1(x) - f_1'(x)(y-x) + f_1'(x)(y-x) - \frac{f_1'(x)f_1(y)}{f_1(x)}(y-x)\right| \\
&\leq M^2|f_1(y) - f_1(x) - f_1'(x)(y-x)| + M^2\underbrace{\left|\frac{1}{f_1(x)}\right|}_{\leq M}|f_1'(x)f_1(x) - f_1'(x)f_1(y))||y-x| \\
&\leq M^2|f_1(y) - f_1(x) - f_1'(x)(y-x)| + M^3\underbrace{|f_1'(x)|}_{\leq M}|f_1(y) - f_1(x)||y-x| \\
&\leq M^2\frac{\varepsilon}{2M^2}|y-x| + M^4\frac{\varepsilon}{2M^4}|y-x| = \varepsilon|y-x|
\end{aligned}$$

The last but one inequality holds, since $(f_1, Df_1, \delta_{f_1})$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$ and $|y-x| \leq \delta_{f_1^{-1}}(\varepsilon)$. Therefore $\big(f_1^{-1}, Df_1^{-1}, \delta_{f_1^{-1}}\big)$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$ and the modulus of differentiability of f_1^{-1} is $\delta_{f_1^{-1}}(\varepsilon) = \min\big\{\delta_{f_1}\big(\frac{\varepsilon}{2M^2}\big), \omega_{f_1}\big(\frac{\varepsilon}{2M^4}\big)\big\}$.

4. The modulus of differentiability $\delta_{\mathrm{id}} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\varepsilon \mapsto \varepsilon$. Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|y-x| \leq \delta_{\mathrm{id}}(\varepsilon)$. Then:

$$|\mathrm{id}(y) - \mathrm{id}(x) - \mathrm{id}'(x)(y-x)| = |y - x - 1(y-x)| = 0 \leq \varepsilon|y-x|$$

Therefore $(\mathrm{id}, 1, \delta_{\mathrm{id}})$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$.

5. The modulus of differentiability $\delta_h : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\varepsilon \mapsto \varepsilon$. Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|y-x| \leq \delta_h(\varepsilon)$. Then:

$$|h(y) - h(x) - h'(x)(y-x)| = |c - c - 0(y-x)| = 0 \leq \varepsilon|y-x|$$

Therefore $(h, 0, \delta_h)$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$.

$\square$

Corollary 3.4. For all n in $\mathbb{N}$ there exits an operation $\delta_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that the triplets $\left(\mathrm{id}_{\mathbb{R}}^n, D(\mathrm{id}_{\mathbb{R}}^n), \delta_n\right)_{n\in\mathbb{N}}$ are in $\mathrm{Dif}_{\mathrm{B}}(\mathbb{R})$ with

$$D(\mathrm{id}_{\mathbb{R}}^n) = n \cdot \mathrm{id}_{\mathbb{R}}^{n-1} \tag{1}$$

Th function $\mathrm{id}_{\mathbb{R}}$ is called the identity function on $\mathbb{R}$.

Proof. First we prove equation (1). To simplify the calculations, we write $\mathrm{id}_{\mathbb{R}}^n = x^n, x \in \mathbb{R}$. The proof is by induction on $n \in \mathbb{N}$:
$\boxed{n = 1}$

$$\frac{dx}{dx} = 1 \qquad \text{by Theorem 3.3 4.}$$

$\boxed{n \rightsquigarrow n+1}$ The induction hypothesis (1) holds for $n \in \mathbb{N}$. Then:

$$\begin{aligned}
\frac{dx^{n+1}}{dx} &= \frac{d(x^n x)}{dx} \\
&= x^n \cdot \frac{dx}{dx} + x \cdot \frac{d(x^n)}{dx} && \text{by Theorem 3.3 2.} \\
&= x^n + x \cdot nx^{n-1} && \text{by the induction hypothesis.} \\
&= (n+1)x^{n+1}
\end{aligned}$$

Next we have to show for $n \in \mathbb{N}$ that $\left(\mathrm{id}_{\mathbb{R}}^n, D(\mathrm{id}_{\mathbb{R}}^n), \delta_n\right) \in \mathrm{Dif}_{\mathrm{B}}(\mathbb{R})$.
We know for $x, y \in \mathbb{R}, k \in \mathbb{N}$:

$$y^n - x^n = (y - x) \cdot \sum_{k=0}^{n-1} y^k x^{n-1-k}$$

Thus we can write

$$y^n - x^n - nx^{n-1}(y - x) = (y - x) \cdot \sum_{k=0}^{n-1} y^k x^{n-1-k} - x^{n-1}$$

Define $M = |x| + 1$ and suppose $|y - x| \leq 1$. Then $|y| \leq M$ and $|x| \leq M$ and

$$\begin{aligned}
|y^k - x^k| &\leq |y - x| \cdot \sum_{p=0}^{k-1} |y|^p |x|^{k-1-p} \\
&\leq |y - x| \cdot \sum_{p=0}^{k-1} M^p M^{k-1-p} \\
&\leq |y - x| \cdot \sum_{p=0}^{k-1} M^{k-1} \\
&= |y - x| \cdot kM^{k-1}
\end{aligned}$$

Then

$$
\begin{aligned}
|y^n - x^n - nx^{n-1}(y-x)| &\le |y-x| \cdot \sum_{k=0}^{n-1} |y^k x^{n-1-k} - x^{n-1}| \\
&\le |y-x| \cdot \sum_{k=0}^{n-1} |x|^{n-1-k} |y^k - x^k| \\
&\le |y-x| \cdot \sum_{k=1}^{n-1} |x|^{n-1-k} |y-x| k M^{k-1} \\
&\le |y-x| \cdot \sum_{k=1}^{n-1} |y-x| k M^{n-2} \\
&= |y-x|^2 M^{n-2} \cdot \sum_{k=1}^{n-1} k \\
&= |y-x|^2 M^{n-2} \cdot \frac{n(n-1)}{2}
\end{aligned}
$$

For $n = 1$ we are done. Therefore consider the case $n > 1$. We can set the modulus of differentiability $\delta_n = \min\big\{1, \frac{2\varepsilon}{M^{n-2}n(n-1)}\big\}$. Then for an arbitrary $\varepsilon > 0, x, y \in \mathbb{R}$ with $|x-y| \le \delta_n$ we have

$$
\begin{aligned}
|y^n - x^n - nx^{n-1}(y-x)| &= |y-x|^2 M^{n-2} \cdot \frac{n(n-1)}{2} \\
&\le |y-x| \delta_n M^{n-2} \frac{n(n-1)}{2} \\
&\le |y-x| \varepsilon
\end{aligned}
$$

Therefore for $n \in \mathbb{N}$ holds $\big(\mathrm{id}_{\mathbb{R}}^n, D(\mathrm{id}_{\mathbb{R}}^n), \delta_n\big)$ is in $\mathrm{Dif}_{\mathrm{B}}(\mathbb{R})$.

□

With Theorem 3.3 and its Corollary we obtain the following formulas

$$
\begin{aligned}
D\big(f_1 f_2^{-1}\big) &= f_2^{-1}\big(f_2 Df_1 - f_1 Df_2\big) \\
D\Big(\sum_{k=0}^{n} a_{n-k} x^k\Big) &= \sum_{k=1}^{n} k a_{n-k} x^{k-1}
\end{aligned}
$$

for the derivatives of a quotient and a polynomial.

Theorem 3.5 (Chain Rule)**.** I and J are arbitrary intervals in $\mathbb{R}$. Let the triplet (f, f', δ_f) be in $\mathrm{Dif}_{\mathrm{B}}(I)$ and the triplet (g, g', δ_g) be in $\mathrm{Dif}_{\mathrm{B}}(J)$, s.t. f maps each compact subinterval of I into a compact subinterval of J. Then there exists $\delta_{g\circ f} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\big(g \circ f, (g' \circ f) f', \delta_{g\circ f}\big)$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$.

Proof. Due to the definition of differentiability, we may assume that I and J are compact intervals. (f, f', δ_f) is in $\mathrm{Dif}_{\mathrm{B}}(I)$, thus δ_f is a modulus of differentiability of f in I. As $f \in C(I)$ let ω_f be the corresponding modulus of continuity. δ_g is a modulus of differentiability of g in J. The modulus of differentiability $\delta_{g\circ f} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\varepsilon \mapsto \min\{\omega_f(\delta_g(\alpha)), \delta_f(\beta)\}, \alpha, \beta \in \mathbb{R}$.

We begin by showing, what we must prove:
Let $\varepsilon > 0$ be arbitrary, $x, y \in I$ s.t. $|y - x| \leq \delta_{g \circ f}(\varepsilon)$. We must show:

$$\big|g\big(f(y)\big) - g\big(f(x)\big) - g'\big(f(x)\big)f'(x)(y-x)\big| \leq \varepsilon\big|y - x\big|$$

We know from $|y - x| \leq \delta_{g \circ f}(\varepsilon)$:

1. $\big|f(y) - f(x)\big| \leq \delta_g(\alpha)$
2. $\big|g\big(f(y)\big) - g\big(f(x)\big) - g'\big(f(x)\big)\big(f(y) - f(x)\big)\big| \leq \alpha\big|f(y) - f(x)\big|$
3. $\big|f(y) - f(x)\big| \leq \big|f(y) - f(x) - f'(x)(y-x)\big| + \big|f'(x)(y-x)\big|$

Therefore we have

$$\begin{aligned}
D &= |g(f(y)) - g(f(x)) - g'(f(x))f'(x)(y-x)| \\
&\leq |g(f(y)) - g(f(x)) - g'(f(x))(f(y) - f(x))| + |g'(f(x))||(f(y) - f(x)) - f'(x)(y-x)| \\
&\leq \alpha|f(y) - f(x)| + \|g'\|_J|(f(y) - f(x)) - f'(x)(y-x)| \\
&\leq \alpha|f'(x)(y-x)| + \alpha|f(y) - f(x) - f'(x)(y-x)| + \|g'\|_J|(f(y) - f(x)) - f'(x)(y-x)| \\
&\leq \alpha\|f'\|_I|y-x| + (\alpha + \|g'\|_J) \cdot |(f(y) - f(x)) - f'(x)(y-x)| \\
&\leq \alpha\|f'\|_I|y-x| + (\alpha + \|g'\|_J)\beta|y-x|
\end{aligned}$$

As each of the summands should be less than $\varepsilon/2$, we got

$$\begin{aligned}
\big(\alpha + \|g'\|_J\big)\beta \leq \frac{\varepsilon}{2} &\Leftrightarrow \beta = \big(\alpha + \|g'\|_J\big)^{-1}\frac{\varepsilon}{2} \\
\alpha\|f'\|_I \leq \frac{\varepsilon}{2} &\Leftrightarrow \alpha = \|f'\|_I^{-1}\frac{\varepsilon}{2}
\end{aligned}$$

As $\|f'\|_I$ might be 0, we set $\alpha = \big(1 + \|f'\|_I\big)^{-1}\frac{\varepsilon}{2}$. Then:

$$\alpha\|f'\|_I|y-x| + \big(\alpha + \|g'\|_J\big)\beta|y-x| \leq \frac{\varepsilon}{2}|y-x| + \frac{\varepsilon}{2}|y-x| = \varepsilon|y-x|$$

It follows that $\big(g \circ f, (g' \circ f)f', \delta_{g \circ f}\big)$ is in $\mathrm{Dif}_{\mathrm{B}}(I)$ with $(g' \circ f)f'$ being the derivative and $\delta_{g \circ f}$ a modulus of differentiability of $g \circ f$ in I.

□

3.2 Approximate Rolle's Theorem & the Mean Value Theorem

The following Theorem is the counterpart to Rolle's Theorem in constructive analysis.

Theorem 3.6 (Bishop - approximate Rolle's Theorem)**.** Let $[a, b], a, b \in \mathbb{R}, a < b$ be an interval, (f, f', δ_f) be in $\mathrm{Dif}_{\mathrm{B}}\big([a, b]\big)$ with $f(a) = f(b)$. Then for each $\varepsilon > 0$ there exists x in $[a, b]$ with

$$|f'(x)| \leq \varepsilon$$

Proof. From (f, f', δ_f) in $\mathrm{Dif}_{\mathrm{B}}\big([a, b]\big)$ we know that the pair $(f', \omega_{f'})$ is in $\mathrm{Cont}_{\mathrm{B}}\big([a, b]\big)$ with $\omega_{f'} : \mathbb{R}^+ \to \mathbb{R}^+$ being a modulus of continuity of f'. δ_f is a modulus of differentiability for f on $[a, b]$. We choose the points $x_0, x_1, \ldots, x_n \in \mathbb{R}$ such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b \text{ and } |x_{k+1} - x_k| \leq \min\Big\{\delta_f\Big(\frac{\varepsilon}{2}\Big), \omega_{f'}\Big(\frac{\varepsilon}{2}\Big)\Big\}$$

for all k in $(0,1,\dots,n-1)$. Then we have

$$\begin{aligned}f(x_{k+1})-f(x_k) &= f'(x_k)(x_{k+1}-x_k)+f(x_{k+1})-f(x_k)-f'(x_k)(x_{k+1}-x_k)\\ &\le f'(x_k)(x_{k+1}-x_k)+\frac{\varepsilon}{2}|x_{k+1}-x_k|\\ &= (f'(x_k)+\frac{\varepsilon}{2})|x_{k+1}-x_k|\\ &< (f'(x_k)+\varepsilon)|x_{k+1}-x_k|\end{aligned}$$

for all k in $(0,1,\dots,n-1)$ and therefore it is

$$0=f(b)-f(a)=\sum_{k=0}^{n-1} f(x_{k+1})-f(x_k)\le \sum_{k=0}^{n-1}\big(f'(x_k)+\varepsilon\big)|x_{k+1}-x_k|$$

Thus for at least one k in $(0,1,\dots,n-1)$ it holds $f'(x_k)>-\varepsilon$. Furthermore, we have

$$\begin{aligned}f(x_k)-f(x_{k+1}) &= f(x_k)-f(x_{k+1})-f'(x_{k+1})(x_k-x_{k+1})+f'(x_{k+1})(x_k-x_{k+1})\\ &\le \frac{\varepsilon}{2}|x_k-x_{k+1}|+f'(x_{k+1})(x_k-x_{k+1})\\ &< \varepsilon|x_k-x_{k+1}|+f'(x_{k+1})(x_k-x_{k+1})\\ &= \varepsilon|x_k-x_{k+1}|-f'(x_{k+1})|x_k-x_{k+1}|\\ &= (\varepsilon-f'(x_{k+1}))|x_k-x_{k+1}|\end{aligned}$$

for all k in $(0,1,\dots,n-1)$ and therefore it is

$$0=f(a)-f(b)=\sum_{k=0}^{n-1} f(x_k)-f(x_{k+1})\le \sum_{k=0}^{n-1}\big(\varepsilon-f'(x_k)\big)|x_k-x_{k+1}|$$

Thus for at least one k in $(0,1,\dots,n-1)$ it holds $f'(x_k)<\varepsilon$.

Since $\frac{\varepsilon}{2}<\varepsilon$ we have for at least one k in $(0,1,\dots,n-1)$ either $|f'(x_k)|<\varepsilon$ or $|f'(x_k)|>\frac{\varepsilon}{2}$. In the first case we are done with the prove. So let's have a look at the second case: $|f'(x_k)|>\frac{\varepsilon}{2}$ for k in $(0,1,\dots,n-1)$. From $|x_{k+1}-x_k|\le\omega_{f'}(\frac{\varepsilon}{2})$ we know $|f'(x_{k+1})-f'(x_k)|\le\frac{\varepsilon}{2}$. Thus we obtain the property that $f'(x_{k+1})$ and $f'(x_k)$ are both positive or both negative. This holds obviously for all k in $(0,1,\dots,n-1)$. From above we know that for at least one value of k we have $f'(x_k)>-\varepsilon$ and for at least one value of k we have $f'(x_k)<\varepsilon$. Thus we get that $0<|f'(x_k)|<\varepsilon$ for at least one value of k .

□

In classical mathematics Rolle's theorem implies the mean value theorem, whereas in constructive mathematics the approximate Rolle's theorem implies the approximate mean value theorem. This theorem gives a basic estimate for the difference of two values of a differentiable function.

Theorem 3.7 (Bishop - approximate mean value theorem)**.** Let (f,f',δ_f) be in $\mathrm{Dif}_\mathrm{B}\big([a,b]\big)$. Then for an arbitrary $\varepsilon>0$ there exists x in $[a,b]$ with

$$|f(b)-f(a)-f'(x)(b-a)|\le\varepsilon.$$

Proof. Define the function $h(x)$ on $[a,b]$ by

$$h(x)=(x-a)\cdot\big(f(b)-f(a)\big)-f(x)(b-a),\ x\in[a,b].$$

Then $h(a)=-f(a)(b-a)=h(b)$. By Theorem 3.6 there exists $x\in[a,b]$ s.t. $|h'(x)|\le\varepsilon$ for $\varepsilon>0$. Thus

$$\varepsilon\ge|h'(x)|=|f(b)-f(a)-f'(x)(b-a)|.$$

□

3.3 Taylor's theorem

Remark 3.8. A function f on a proper interval I is *(srictly) increasing* if $f(x) \underset{(=)}{>} f(y)$ whenever $x, y \in I, x > y$. We say f is *(srictly) decreasing*, if $-f$ is *(srictly) increasing.* With theorem 3.7 we obtain that if the triple (f, f', δ_f) is in $\mathrm{Dif_B}(I)$ and $f'(x) \geq 0$ (respectively, $f'(x) \leq 0$) for all x in I, then f is increasing (resp. decreasing) on I.

Definition 3.9 (n^{th} derivative)**.** Let the triplets

$$\big(f, Df, \delta_f\big), \big(f^{(1)}, Df^{(1)}, \delta_{f^{(1)}}\big), \ldots, \big(f^{(n-1)}, Df^{(n-1)}, \delta_{f^{(n-1)}}\big)$$

be in $\mathrm{Dif_B}(I)$ such that $Df = f^{(1)}, Df^{(1)} = f^{(2)}, \ldots, Df^{(n-2)} = f^{(n-1)}$ and set $Df^{(n-1)} = f^{(n)}$. The function f is then said to be *n times differentiable on I* with $f^{(n)}$ being its n^{th} *derivative.* The class of n times differentiable functions on I is recursively defined by

$$f \in \mathrm{Dif_B}^{(n)}(I) \Leftrightarrow \big(f, f', \delta_f\big) \in \mathrm{Dif_B}(I) \text{ and } f' \in \mathrm{Dif_B}^{(n-1)}(I).$$

The function f itself may be written $f^{(0)}$ or $D^0 f$.

We now want to find a polynomial in order to approximate the function f.

Definition 3.10 (Taylor polynomial)**.** Let $f \in \mathrm{Dif_B}^{(n)}(I)$ and $a \in I$. Then we call

$$T_{a,n}f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

the n^{th} Taylor polynomial for f about a and for a given value b

$$R := f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k \tag{2}$$

is called the remainder term.

Special case: f is infinitely differentiable on an open Interval $I = (a-t, a+t)$, also written $f \in \mathrm{Dif_B}^{(\infty)}(I) := \bigcap_{n \in \mathbb{N}} \mathrm{Dif_B}^{(n)}(I)$. This means $f^{(n)}$ exists for all positive integers n. In that case, if

$$\frac{r^n f^{(n+1)}}{n!} \to 0 \qquad \text{as } n \to \infty \qquad (0 < r < t),$$

then the *Taylor series* for f about a,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

converges to f on I.

Theorem 3.11 (Taylor's theorem)**.** Let $f \in \mathrm{Dif_B}^{(n+1)}(I)$, let $\varepsilon > 0$, and $a, b \in I$. Then there exists c with $\min\{a, b\} \leq c \leq \max\{a, b\}$ such that

$$\Big| R - \frac{f^{(n+1)}(c)}{n!}(b-c)^n(b-a) \Big| \leq \varepsilon.$$

where R represents the remainder term, given by (2).

Proof. We start by defining

$$M := 1 + \max\left\{ \left| \frac{f^{(k)}(a)}{k!} \right| \;\middle|\; k \in (1, \dots, n) \right\}$$

In addition we have $f \in {\mathrm{Dif_B}}^{(n+1)}(I)$, therefore there exits a modulus of differentiability $\delta_f > 0$ s.t. $(f, f', \delta_f) \in \mathrm{Dif_B}(I)$. Set $\delta_f(\varepsilon) = \min\left\{1, \frac{\varepsilon}{2nM}, \omega_f\left(\frac{\varepsilon}{2}\right)\right\}$ where ω_f is the modulus of continuity for f on $[\min\{a,b\}, \max\{a,b\}]$. We know it holds either $|a-b| \leq \delta_f(\varepsilon)$ or $|a-b| \geq 0$.

First suppose $|a-b| \leq \delta_f(\varepsilon)$. Then

$$\begin{aligned}
|R| &= \left| f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k \right| \\
&= \left| f(b) - f(a) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k \right| \\
&\leq |f(b) - f(a)| + \sum_{k=1}^{n} \underbrace{\left| \frac{f^{(k)}(a)}{k!} \right|}_{<M} |b-a|^k \\
&\leq \frac{\varepsilon}{2} + M \sum_{k=1}^{n} \delta_f(\varepsilon)^k \\
&\leq \frac{\varepsilon}{2} + M \sum_{k=1}^{n} \frac{\varepsilon}{2nM} = \varepsilon
\end{aligned}$$

If we choose $c = b$, the theorem holds.

Now suppose that $|a-b| \geq 0$. Consider the function

$$\begin{aligned}
g(x) = f(b) - f(x) - \frac{f'(x)}{1!}(b-x) - \frac{f''(x)}{2!}(b-x)^2 - \dots \\
- \frac{f^{(n)}(x)}{n!}(b-x)^n - R(b-x)(b-a)^{-1}
\end{aligned}$$

It holds:

$$\left.\begin{array}{c}
g(a) = f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k - R(b-a)(b-a)^{-1} = 0 \\
g(b) = f(b) - f(b) - 0 - \dots - 0 = 0
\end{array}\right\} \Rightarrow g(a) = g(b)$$

The function g is differentiable on I as a composition of differentiable functions on I. For the derivative we have:

$$\begin{aligned}
g'(x) &= -f'(X) + f'(x) - f''(x)(b-x) - \dots \\
&\quad + \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} - \frac{f^{(n+1)}(x)}{n!}(b-x)^n + R(b-a)^{-1} \\
&= -\frac{f^{(n+1)}(x)}{n!}(b-x)^n + R(b-a)^{-1}
\end{aligned}$$

By the approximate Rolle's theorem (3.6), we know that for an arbitrary $\varepsilon > 0$ there exists $c \in [\min\{a,b\}, \max\{a,b\}]$ such that $|g'(c)| \leq \varepsilon$. Let's apply the approximate

Rolles's theorem:

$$
\begin{aligned}
|g'(x)| = \left| -\frac{f^{(n+1)}(x)}{n!}(b-x)^n + R(b-a)^{-1} \right| &\leq \varepsilon |b-a|^{-1} \\
\Leftrightarrow \left| R - \frac{f^{(n+1)}(x)}{n!}(b-x)^n(b-a) \right| &\leq \varepsilon
\end{aligned}
$$

and the proof is complete.

□

4 Bridger differentiability

The following chapter also features differentiability in constructive analysis. This approach is due to Marc Bridger. Similarly to Bishop, he only uses the uniform version of continuity, as this is the stronger and more important form of continuity within constructive analysis. For the sake of completeness, we provide Bridger's definition of uniform continuity and abbreviate it to continuity.

Definition 4.1 (Bridger - continuity (Brc))**.** Let E be a subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be functions in E. Let $\omega_f : \mathbb{R}^+ \to \mathbb{R}^+$ be an operation, called *modulus of continuity.* The function f is called *continuous* on E if for each positive real number ε and for x, y in E such that $0 < |y - x| < \omega_f(\varepsilon)$ we have

$$|f(y) - f(x)| \leq \varepsilon.$$

Note that Bridger, unlike Bishop, does not require E to be a compact interval. He takes a similar approach to differentiability. Instead of dealing with the derivative of a function at a point, he engages himself in the derivative function at an interval. As with continuity, this notion of uniform differentiability is the one that is most important in later theory and applications. In order to define uniform differentiability, he provides three different definitions of differentiability and proves their equivalence.

4.1 Equivalent definitions

Definition 4.2 (Bridger - differentiability 1 (Brd1))**.** Let E be a subset of $\mathbb{R}$ and let $f, g : E \to \mathbb{R}$ be functions on E and let $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ be an operation. The function f is called *differentiable* on E with g being its *derivative* on E, if for each positive real number ε and for $x, x + h$ in E such that $0 < |h| < \delta_f(\varepsilon)$ we have

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right| \leq \varepsilon.$$

The operation $\delta_f > 0$ is called a *modulus of differentiability.*

Note that Bridger, unlike Bishop, does not require the functions f and g to be continuous and E to be a compact interval. The next definition of differentiability comes from a slightly change in notation. Let's write $y = x + h$, such that $h = y - x$.

Definition 4.3 (Bridger - differentiability 2 (Brd2))**.** Let E be a subset of $\mathbb{R}$ and let $f, g : E \to \mathbb{R}$ be functions on E and let $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ be a modulus of differentiability. The function f is called *differentiable* on E with g being its *derivative* on E, if for each positive real number ε and for x, y in E such that $0 < |y - x| < \delta_f(\varepsilon)$ we have

$$\left| \frac{f(y) - f(x)}{y - x} - g(x) \right| \leq \varepsilon \tag{3}$$

We also say the triplet $\left(f, g, \delta_f\right)$ is in $\mathrm{Dif}_{\mathrm{Br}}(E)$ with

$$\mathrm{Dif}_{\mathrm{Br}}(E) := \left\{ (f, g, \delta_f) \in \mathbb{F}(E, \mathbb{R}) \times \mathbb{F}(E, \mathbb{R}) \times \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+) \;\middle|\; \delta_f : \mathrm{dif}_{\mathrm{Br}}(f) \right\}$$

and $\delta_f : \mathrm{dif}_{\mathrm{Br}}(f) :\Leftrightarrow \forall \varepsilon {>} 0 \ \forall x, y {\in} E : \left(0 < |y - x| < \delta_f(\varepsilon) \Rightarrow \left| \frac{f(y) - f(x)}{y - x} - g(x) \right| \leq \varepsilon \right).$

Remark 4.4. The quotient (3) in definition 4.3 is called the *difference quotient* of f and may be written as $D(x, y)$.

Since the *difference quotient* must be close to $g(x)$ for $x \in E$, let's take a look at their difference and call it $r(x, y)$. Before introducing the third definition, we have to define the following property:

Remark 4.5. $r(x, y) \to 0$ as $y \to x$ on $E \subset \mathbb{R}$ means that for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $|r(x, y)| \leq \varepsilon$ whenever $x, y \in E$ and $0 < |y - x| < \delta(\varepsilon)$.

Note that $r(x, y)$ does not have to exist when $y = x$. Finally we receive the third definition:

Definition 4.6 (Bridger - differentiability 3 (Brd3))**.** Let E be a subset of $\mathbb{R}$ and let $f, g : E \to \mathbb{R}$ be functions on E and let $r(x, y)$ be a function on E defined for $x, y \in E$. The function f is called *differentiable* on E with g being its *derivative* on E, if for $x, y \in E$ with $|y - x| > 0$ and for $r(x, y) \to 0$ as $y \to x$ we have

$$f(y) - f(x) = g(x) \cdot (y - x) + r(x, y) \cdot (y - x)$$

We now show that the three definitions of g being the derivative of f are logically equivalent.

Theorem 4.7 (Equivalence between Brd1, Brd2 and Brd3)**.** For a subset E of $\mathbb{R}$ and the functions $f, g : E \to \mathbb{R}$ on E, the following statements are equivalent.

1. There is an operation $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ with the property that for each positive real number ε and for $x, x + h$ in E such that $0 < |h| < \delta_f(\varepsilon)$ we have
$$\left| \frac{f(x + h) - f(x)}{h} - g(x) \right| \leq \varepsilon.$$
2. There is an operation $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ with the property that for each positive real number ε and for x, y in E such that $0 < |y - x| < \delta_f(\varepsilon)$ we have
$$\left| \frac{f(y) - f(x)}{y - x} - g(x) \right| \leq \varepsilon.$$
3. There is a function $r(x, y)$ with $x, y \in E$ such that
$$f(y) - f(x) = g(x) \cdot (y - x) + r(x, y) \cdot (y - x)$$
if $|y - x| > 0$ and $r(x, y) \to 0$ as $y \to x$.

Proof. To prove the equivalence of the three statements, it is sufficient to prove the following implications.

1. $(2 \Rightarrow 3)$
Let E be a subset of $\mathbb{R}$ and let the triplet (f, g, δ_f) be in $\mathrm{Dif}_{\mathrm{Br}}(E)$, thus for an arbitrary $\varepsilon > 0$, $x, y \in E$ with $0 < |y - x| < \delta_f(\varepsilon)$ it is
$$\left| \frac{f(y) - f(x)}{y - x} - g(x) \right| \leq \varepsilon$$

Now set $r(x,y) = \frac{f(y)-f(x)}{y-x} - g(x)$. Thus $r(x,y)$ is defined on E for $x, y \in E$ and $0 < |y-x|$. We have

$$\begin{aligned}
& r(x,y) = \frac{f(y)-f(x)}{y-x} - g(x) && \mid +g(x) \\
\Leftrightarrow\ & r(x,y) + g(x) = \frac{f(y)-f(x)}{y-x} && \mid \cdot(y-x) \\
\Leftrightarrow\ & r(x,y)\cdot(y-x) + g(x)\cdot(y-x) = f(y) - f(x)
\end{aligned}$$

whenever $|r(x,y)| \leq \varepsilon$ with $x, y \in E$ and $0 < |y-x| < \delta_f(\varepsilon)$.

2. $(3 \Rightarrow 1)$
   There is a function $r(x,y)$, defined for $x, y \in E$ and $|y-x| > 0$, with the property that
   $$f(y) - f(x) = g(x)\cdot(y-x) + r(x,y)\cdot(y-x)$$
   where $r(x,y) \to 0$ as $y \to x$. Now set $y = x + h$. Then
   $$\begin{aligned}
   |r(x,y)| &= \left|\frac{f(y)-f(x)}{y-x} - g(x)\right| \\
   &= \left|\frac{f(x+h)-f(x)}{h} - g(x)\right|
   \end{aligned}$$
   whenever $|r(x,y)| \leq \varepsilon$ with $x, y \in E$ and $0 < |y-x| < \delta_f(\varepsilon)$ or rather $x, x+h \in E$ and $0 < |h| < \delta_f(\varepsilon)$.

3. $(1 \Rightarrow 2)$
   For each positive real number $\varepsilon > 0$, let $x, x+h \in E$ such that $0 < |h| < \delta_f(\varepsilon)$. Then
   $$\left|\frac{f(x+h)-f(x)}{h} - g(x)\right| \leq \varepsilon$$
   Now set $h = y - x$. Thus
   $$\begin{aligned}
   \varepsilon &\geq \left|\frac{f(x+h)-f(x)}{h} - g(x)\right| = \left|\frac{f(x+(y-x))-f(x)}{y-x} - g(x)\right| \\
   &= \left|\frac{f(y)-f(x)}{y-x} - g(x)\right|
   \end{aligned}$$
   whenever $x, y \in E$ and $0 < |y-x| < \delta_f(\varepsilon)$.

Therefore the three definitions of differentiability are equivalent.

$\square$

4.2 Some basic properties

The modulus of differentiability $\delta_f(\varepsilon)$ is not unique; if you have one, any positive smaller number is also a modulus of differentiability. In contrast, the derivative of a function is unique as we prove with the following theorem.

Theorem 4.8 (Uniqueness of the derivative)**.** Suppose E is a subset of $\mathbb{R}$ and the triplets $\big(f, g_1, \delta_{1_f}\big)$ and $\big(f, g_2, \delta_{2_f}\big)$ are in $\mathrm{Dif}_{\mathrm{Br}}(E)$. Then $g_1 = g_2$.

Proof. Let E be a subset of $\mathbb{R}$ and let the triplets $\big(f, g_1, \delta_{1_f}\big)$ and $\big(f, g_2, \delta_{2_f}\big)$ be in $\mathrm{Dif}_{\mathrm{Br}}(E)$. Then we have

$$\begin{aligned}|g_1(x) - g_2(x)| &= \left|g_1(x) - \frac{f(y)-f(x)}{y-x} + \frac{f(y)-f(x)}{y-x} - g_2(x)\right| \\ &\leq \left|g_1(x) - \frac{f(y)-f(x)}{y-x}\right| + \left|\frac{f(y)-f(x)}{y-x} - g_2(x)\right|\end{aligned}$$

Both of these last quantities can be made arbitrarily small if $0 < |y - x|$ is sufficiently small. As g_1 and g_2 are derivatives of f, we can choose another modulus of differentiability for f such that $|y - x| < \delta(\varepsilon) = \min\big\{\delta_{1_f}\big(\frac{\varepsilon}{2}\big), \delta_{2_f}\big(\frac{\varepsilon}{2}\big)\big\}$. Thus

$$\left|g_1(x) - \frac{f(y)-f(x)}{y-x}\right| + \left|\frac{f(y)-f(x)}{y-x} - g_2(x)\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

It follows that $g_1(x) = g_2(x)$ for x in E.

□

Theorem 4.9. If E is a subset of $\mathbb{R}$ and the triplet $\big(f, f', \delta_f\big)$ is in $\mathrm{Dif}_{\mathrm{Br}}(E)$, then f' is continuous on E.

Proof. Let $E \subseteq \mathbb{R}$ and the triplet $\big(f, f', \delta_f\big) \in \mathrm{Dif}_{\mathrm{Br}}(E)$. The difference quotient is symmetric in x and y:

$$D(x, y) = \frac{f(y)-f(x)}{y-x} = \frac{-(f(y)-f(x))}{-(y-x)} = \frac{f(x)-f(y)}{x-y} = D(y, x).$$

Set the modulus of differentiability $\delta_f(\varepsilon) = \frac{\varepsilon}{2}$. Then if $|y - x| \leq \delta_f(\varepsilon)$, we have both $|D(x, y) - f'(x)| \leq \frac{\varepsilon}{2}$ and $|D(y, x) - f'(y)| \leq \frac{\varepsilon}{2}$ and thus

$$\begin{aligned}|f'(y) - f'(x)| &= |f'(y) - D(y, x) + D(x, y) - f'(x)| \\ &\leq |f'(y) - D(y, x)| + |D(x, y) - f'(x)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Therefore f' is continuous on E.

□

Corollary 4.10. Let E be a subset of $\mathbb{R}$ and the triplet $\big(f, f', \delta_f\big)$ be in $\mathrm{Dif}_{\mathrm{Br}}(E)$. If E is a compact interval, then f' is bounded on E.

Proof. Let E be a compact interval in $\mathbb{R}$. By Theorem 4.9, f' is continuous on E. By Bishop [2, Proposition 4.6], we know that for continuous functions on compact intervals the supremum and the infimum exist. If a supremum and a infimum exist for a function, this function is bounded. Thus f' is bounded in E.

□

Theorem 4.11. Let E be a subset of $\mathbb{R}$ and the triplet $\left(f, f', \delta_f\right)$ be in $\mathrm{Dif}_{\mathrm{Br}}(E)$. If f' is bounded on E, then f is continuous in E.

Proof. Let E be a compact interval in $\mathbb{R}$. Then by Theorem 4.10, f' is bounded on E. Let B be the bound of f', hence $\forall x \in E : |f'(x)| \leq B$. Set $|y-x| \leq \min\left\{\delta_f(1), \frac{\varepsilon}{B+1}\right\}$. By definition of differentiability, we receive

$$\begin{aligned}\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| \leq 1 \Rightarrow \left|\frac{f(y)-f(x)}{y-x}\right| &\leq B+1 \\ \Rightarrow |f(y)-f(x)| \leq (B+1)\underbrace{|y-x|}_{\leq \frac{\varepsilon}{B+1}} &\leq \varepsilon\end{aligned}$$

Therefore f is continuous in E.

$\square$

4.3 Examples

Theorem 4.12. The triplet $\left(e^x, e^x, \delta_{exp}\right)$ is in $\mathrm{Dif}_{\mathrm{Br}}\big((-\infty, c]\big)$ for $c \in \mathbb{R}$.

Proof. To prove theorem 4.12, we have to introduce some properties of the exponential function: $\forall x > 0:$

1. $\frac{e^{-x}-1}{-x} \leq 1 \Leftrightarrow e^x - 1 \leq e^x \cdot x$
2. $1 \leq \frac{e^x-1}{x} \Leftrightarrow x \leq e^x - 1$

Let $\varepsilon > 0$ be arbitrary, x, y in $(-\infty, c]$ with $x < y$ and $0 < |y-x| \leq \delta_{exp}(\varepsilon) = \frac{\varepsilon}{e^c}$. By using the properties of the exponential function and with $y - x > 0$, we can follow

$$y - x \leq e^{y-x} - 1 \leq e^{y-x} \cdot (y-x)$$

or rather

$$e^x \cdot (y-x) \leq e^y - e^x \leq e^y \cdot (y-x) \tag{4}$$

Then

$$\left|\frac{e^y - e^x}{y-x} - e^x\right| \leq |e^y - e^x| \leq e^y \cdot |y-x| \leq e^c \cdot \frac{\varepsilon}{e^c} = \varepsilon$$

Therefore the triplet $\left(e^x, e^x, \delta_{exp}\right)$ is in $\mathrm{Dif}_{\mathrm{Br}}\big((-\infty, c]\big)$ for $c \in \mathbb{R}$.

$\square$

Theorem 4.13. The triplet $\left(\ln(x), \frac{1}{x}, \delta_{\ln}\right)$ is in $\mathrm{Dif}_{\mathrm{Br}}\big([R, \infty)\big)$ for $R \in \mathbb{R}^+$.

Proof. Using inequality (4) above, with $x \mapsto \ln(x)$ and $y \mapsto \ln(y)$, we get

$$\begin{aligned}x(\ln(y) - \ln(x)) \leq y - x &\leq y(\ln(y) - \ln(x)) \\ \Leftrightarrow \frac{1}{y}(y-x) \leq \ln(y) - \ln(x) &\leq \frac{1}{x}(y-x) \\ \Leftrightarrow \frac{1}{y} \leq \frac{\ln(y)-\ln(x)}{y-x} &\leq \frac{1}{x}\end{aligned}$$

Then

$$\left|\frac{\ln(y)-\ln(x)}{y-x} - \frac{1}{x}\right| \leq \left|\frac{1}{x} - \frac{1}{x}\right| = 0 \leq \varepsilon$$

Therefore the triplet $\left(\ln(x), \frac{1}{x}, \delta_{\ln}\right)$ is in $\mathrm{Dif}_{\mathrm{Br}}\big([R, \infty)\big)$ for $R \in \mathbb{R}^+$.

$\square$

5 A comparison between Bishop and Bridger differentiability

The last chapters outline Bishop's and Bridger's definitions of differentiability in constructive analysis. Before comparing their approaches the next paragraphs shortly recap the definitions.

Definition 5.1 (Bd)**.** Let E be a compact interval in $\mathbb{R}$, $f, g \in C(E)$ be continuous functions in E and $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ be a modulus of differentiability of f on E. The function f is called *differentiable* on E if the triplet (f, g, δ_f) is in $\mathrm{Dif_B}(E)$, with

$$\mathrm{Dif_B}(E) := \Big\{(f, g, \delta_f) \in C(E) \times C(E) \times \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+) \bigm| \delta_f : \mathrm{dif_B}(f)\Big\}$$

and $\delta_f : \mathrm{dif_B}(f) :\Leftrightarrow \forall \varepsilon > 0\ \forall x, y \in E :$

$$\Big(|y - x| \leq \delta(\varepsilon) \Rightarrow |f(y) - f(x) - g(x)(y - x)| \leq \varepsilon |y - x|\Big)$$

Definition 5.2 (Brd2)**.** Let $E \subset \mathbb{R}$, $f, g \in \mathbb{F}(E, \mathbb{R})$ be functions in E and $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ be a modulus of differentiability of f on E. The function f is called *differentiable* on E if the triplet (f, g, δ_f) is in $\mathrm{Dif_{Br}}(E)$, with

$$\mathrm{Dif_{Br}}(E) := \Big\{(f, g, \delta_f) \in \mathbb{F}(E, \mathbb{R}) \times \mathbb{F}(E, \mathbb{R}) \times \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+) \bigm| \delta_f : \mathrm{dif_{Br}}(f)\Big\}$$

and $\delta_f : \mathrm{dif_{Br}}(f) :\Leftrightarrow \forall \varepsilon > 0\ \forall x, y \in E : \Big(0 < |y - x| < \delta_f(\varepsilon) \Rightarrow \Big|\frac{f(y) - f(x)}{y - x} - g(x)\Big| \leq \varepsilon\Big).$

The differences are easy to see. Bishop requires E to be a compact interval in $\mathbb{R}$. f and its derivative are continuous functions. In contrast, Bridger only requires E to be a set in $\mathbb{R}$ and f and its derivative not to be continuous. However, he proves the continuity of f' and of f if E is a compact interval. See Theorem 4.9 and 4.11.

The task of constructive analysis is to avoid non relevant definitions. Using a less strict approach, Bridger's definition of differentiability follows more closely this idea of constructive analysis. This chapter checks if the Theorems 4.9 and 4.11 also apply to Bishop's differentiability, if we assume that f and f' are not continuous.

Therefore we present an appropriate Definition of differentiability.

Definition 5.3 (Bd*)**.** Let E be a compact interval in $\mathbb{R}$, $f, g \in \mathbb{F}(E, \mathbb{R})$ be functions and $\delta_f : \mathbb{R}^+ \to \mathbb{R}^+$ be a modulus of differentiability of f on E. The function f is called *differentiable* in E if the triplet (f, g, δ_f) is in ${\mathrm{Dif_B}}^*(E)$, where

$${\mathrm{Dif_B}}^*(E) := \Big\{(f, g, \delta_f) \in \mathbb{F}(E, \mathbb{R}) \times \mathbb{F}(E, \mathbb{R}) \times \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+) \bigm| \delta_f : \mathrm{dif_B}(f)\Big\}$$

We start by assuming that E is an arbitrary interval in $\mathbb{R}$.

Theorem 5.4. Let E be an interval in $\mathbb{R}$, $f, f' \in \mathbb{F}(E, \mathbb{R})$ and $\delta_f \in \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+)$ be functions, such that the triplet (f, f', δ_f) is in ${\mathrm{Dif_B}}^*(E)$. Then f' is continuous on E or rather f' is in $C(E)$.

Proof. Due to Bishops Definition of continuity, it is sufficient to proof the Theorem for E being a compact interval in $\mathbb{R}$. So let E be a compact interval and let the triplet (f, f', δ_f) be in $\mathrm{Dif_B}^*(E)$. We have to show, that the pair $(f', \omega_{f'})$ is in $\mathrm{Cont_B}(E)$.
Let $\varepsilon > 0$ and $x, y \in E$ such that $|y - x| \leq \omega_{f'}(\varepsilon) = \delta_f(\varepsilon) = \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
|f'(y) - f'(x)| &= \left| f'(y) - \frac{f(y) - f(x)}{y - x} + \frac{f(y) - f(x)}{y - x} - f'(x) \right| \\
&\leq \left| f'(y) - \frac{f(y) - f(x)}{y - x} \right| + \left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| \\
&= \frac{1}{|x - y|} \cdot |f(x) - f(y) - f'(y)(x - y)| \\
&\quad + \frac{1}{|y - x|} \cdot |f(y) - f(x) - f'(x)(y - x)| \\
&\leq \frac{1}{|x - y|} \frac{\varepsilon}{2} |x - y| + \frac{1}{|y - x|} \frac{\varepsilon}{2} |y - x| \\
&= \varepsilon
\end{aligned}$$

Therefore $(f', \omega_{f'})$ is in $\mathrm{Cont_B}(E)$.

□

Now we assume that E is a compact interval in $\mathbb{R}$. By Corollary 4.10 is f' bounded on E. We continue by checking Theorem 4.11 for Bd*.

Theorem 5.5. Let E be an subset of $\mathbb{R}$, $f, f' \in \mathbb{F}(E, \mathbb{R})$ and $\delta_f \in \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+)$ be functions, such that the triplet (f, f', δ_f) is in $\mathrm{Dif_B}^*(E)$. If f' is bounded in E, then f is continuous on E.

Proof. Let E be a compact interval of $\mathbb{R}$ and let the triplet (f, f', δ_f) be in $\mathrm{Dif_B}^*(E)$. By Lemma 5.4 is f' continuous in E. As E is a compact interval, the supremum and infimum of f' exists in E and f' is bounded in E. Let B be the bound of f', hence $\forall x \in E : |f'(x)| \leq B$. We have to show, that the pair (f, ω_f) is in $\mathrm{Cont_B}(E)$.
Let x, y be in E such that $|y - x| \leq \min\{\delta_f(1), \frac{\varepsilon}{1+B}\}$. Then

$$\begin{aligned}
&|f(y) - f(x) - f'(x)(y - x)| \leq 1 \cdot |y - x| \\
&\Rightarrow |f(y) - f(x)| \leq |y - x| + |f'(x)||y - x| = \underbrace{|y - x|}_{\leq \frac{\varepsilon}{B+1}} \cdot (1 + B) \leq \varepsilon
\end{aligned}$$

Therefore (f, ω_f) is in $\mathrm{Cont_B}(E)$.

□

By assuming that E is a compact interval and $(f, f', \delta_f) \in \mathrm{Dif_B}^*(E)$, we have now shown that the Theorems 4.9 and 4.11 also apply to Bishop's differentiability. With Theorem 5.4 and 5.5 we obtain the following Corollary:

Corollary 5.6. Let E be a compact interval in $\mathbb{R}$. If (f, f', δ_f) is in $\mathrm{Dif_B}^*(E)$, then (f, f', δ_f) is in $\mathrm{Dif_B}(E)$.

Proof. Let E be a compact interval in $\mathbb{R}$ and let the triplet (f, f', δ_f) be in $\mathrm{Dif_B}^*(E)$. By Lemma 5.4 and 5.5 are f' and f continuous in E. Thus the triplet (f, f', δ_f) is in $\mathrm{Dif_B}(E)$.
□

In addition, if E is a compact interval, Bridger's definition of differentiability Brd2 is equivalent to Bd*.

Corollary 5.7. Let E be a compact interval in $\mathbb{R}$. (f, f', δ_f) is in $\mathrm{Dif}_{\mathrm{Br}}(E)$ if, and only if, (f, f', δ_f) is in $\mathrm{Dif}_{\mathrm{B}}{}^*(E)$.

Proof. Let E be a compact interval in $\mathbb{R}$ and let the triplet (f, f', δ_f) be in $\mathrm{Dif}_{\mathrm{Br}}(E)$. Then for an arbitrary $\varepsilon > 0, x, y \in E$ such that $0 < |y - x| < \delta_f(\varepsilon)$ we have

$$\Big|\frac{f(y) - f(x)}{y - x} - f'(x)\Big| \leq \varepsilon \Leftrightarrow |f(y) - f(x) - f'(x)(y - x)| \leq \varepsilon |y - x|$$

□

Last but not least we can conclude that Bridger's definition of differentiability (Brd2) implies Bishop's definition of differentiability (Bd).

Corollary 5.8. Let E be a subset of $\mathbb{R}$. If the triplet (f, f', δ_f) is in $\mathrm{Dif}_{\mathrm{Br}}(E)$, then it is in $\mathrm{Dif}_{\mathrm{B}}(E)$.

Proof. Let E be a subset of $\mathbb{R}$ and let the triplet (f, f', δ_f) be in $\mathrm{Dif}_{\mathrm{Br}}(E)$. We know:

$$(f, f', \delta_f) \in \mathrm{Dif}_{\mathrm{Br}}(E) \Leftrightarrow \underset{I \subset E \text{ compact}}{\forall} \Big((f, f', \delta_f) \in \mathrm{Dif}_{\mathrm{Br}}(I)\Big)$$

Then by Corollary 5.7 and 5.6 it holds

$$\begin{aligned} \underset{I \subset E \text{ compact}}{\forall} \Big((f, f', \delta_f) \in \mathrm{Dif}_{\mathrm{Br}}(I)\Big) &\overset{5.7}{\Rightarrow} \underset{I \subset E \text{ compact}}{\forall} \Big((f, f', \delta_f) \in \mathrm{Dif}_{\mathrm{B}}{}^*(I)\Big) \\ &\overset{5.6}{\Rightarrow} \underset{I \subset E \text{ compact}}{\forall} \Big((f, f', \delta_f) \in \mathrm{Dif}_{\mathrm{B}}(I)\Big) \end{aligned}$$

It holds

$$\underset{I \subset E \text{ compact}}{\forall} \Big((f, f', \delta_f) \in \mathrm{Dif}_{\mathrm{B}}(I)\Big) \Leftrightarrow (f, f', \delta_f) \in \mathrm{Dif}_{\mathrm{B}}(E)$$

□

With Corollary 5.8 we have shown, that Bridger's Definition of differentiability (Brd2) implies Bishop's Definition of differentiability (Bd). Hence Brd2 is the stronger Definition.

6 Litarature

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