Bachelor's Thesis

Membership-with-evidence in constructive analysis

Lion Reichl
Matrikelnummer: 11177138

June 8, 2020

supervised by
Dr. Iosif Petrakis

Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit eigenständig und ohne fremde Hilfe angefertigt habe. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Ort,Datum Unterschrifft

Contents

1 Introduction

Many mathematicians accept classical mathematics (CLASS) as it is taught in universities. The statements one can prove in CLASS are highly abstract and therefore captivate with its "simplicity" and clarity. But one should be careful to confuse clarity with insight. For example lets think about the theorem T in CLASS that for every bounded sequence of real numbers there exists a least upper bound, which itself is a real number. Even if the assertion of this theorem seems to be intuitively clear, we should take a more detailed look at the information this theorem provides. Therefore lets define a concrete case.
Let $(x_n)_{n\in\mathbb{N}^+}$ be a sequence of real numbers, such that

$$G(n) :\Leftrightarrow 2n+2 \text{ is a sum of two primes}$$
$$x_n := \begin{cases} 0, & \text{G(n) holds} \\ 1, & \neg\text{(G(n) holds)} \end{cases}$$

for every $n \in \mathbb{N}^+$.
This sequence is well-defined, as we can decide for every natural number if $G(n)$ holds or not, simply by checking all prime numbers smaller then $2n+2$.
Now theorem T predicts a least upper bound x of the sequence $(x_n)_n$. Clearly this least upper bound has to be 0 or 1, but no one expects that we can decide which one of the two cases holds, as this would solve the Goldbach conjecture. So one could say that in CLASS the assertion of existence of an object only asserts the existence of a box with a special property, but we can't open the box and take a look at the (value of the) asserted object. Hence the assertion of existence in CLASS is "incomplete", in the sense that it has no computational meaning. This means most properties we deduce reflect the general structure of a system and thus won't provide an understanding of how to construct the predicted object in a concrete case.
Moreover this notion of existence shifts the attention of practising mathematicians from the observed system to the emergent properties and structures. In particular it changes the point of view, as we try to understand a system observing it not from the inside, but from the outside.

Constructive mathematics (CM) tackles this problem by using different definitions of the logical connectives. These constitute the Brouwer-Heyting-Kolmogorov interpretation (BHK-interpretation). In this thesis we'll discuss CM based on the style of Bishop (BISH). So the following formulation of the BHK-interpretation is in short what Bishop says about the logical connectives in his book [3], the statements about negation and falsum are from [6] but express what seems to be generally accepted in the literature:

$P \wedge Q$	A proof of $P \wedge Q$, is a proof of P and a proof of Q
$P \vee Q$	A proof of $P \vee Q$, is either a proof of P or a proof of Q
$P \Rightarrow Q$	A proof of $P \Rightarrow Q$, is a rule that converts any given proof of P into a proof of Q
$\forall x \in A\ P(x)$	A proof of $\forall x \in A\ P(x)$ is an algorithm which for any $x \in A$ returns a proof for $P(x)$
$\exists x\ P(x)$	A proof of $\exists x\ P(x)$, is a specific element x and a proof that $P(x)$ holds
$\perp$	There is no proof of $\perp$
$\neg Q$	A proof of $\neg Q$, is a proof of $Q \Rightarrow \perp$

Of course there are some obvious questions arising from these informal definitions:

What is a proof?

What is a rule or algorithm?

What can we say about the collection of all proofs?

To answer these questions, and ultimately answer the question: "What is constructive mathematics", we will promote a more exact treatment of Bishop-style constructive mathematics (BCM). Since

> an answer provided from a formal treatment of BCM, that cannot be "captured" by BISH itself, is not necessarily the "right" answer

(Petrakis in [6] p. 156), we work within BISH. In the first part of this thesis we will formulate precisely some notions Bishop used and we will define some necessary extensions, which we need for a more formal treatment of CM. Then we will present an exact, but informal, BHK-interpretation with the use of Bishop set theory (BST) hoping that this brings more clarity about what CM is within BISH. We will finish the thesis by discussing how such a more precise analysis looks like. Except for the interpretation of the integral in section 5, that is done for the first time here, this thesis is based on [6].

2 Basics of BST

The informal approach to constructive analysis of Bishop and Bridges in [3] was perfect for showing that constructive analysis can be done, since the resulting analysis is very similar to read to classical mathematics. But in this thesis we focus on a more formal level. Therefore we use the semi-formal, Bishop set theory (BST) as presented in [4], where it is called constructive set and function theory (CSFT), and [6]. This theory formalizes the basic notions of BISH like set, membership to a set and function (while the last one grounds on the informal term of an algorithm).
The first subsection will focus on the fundamentals we need in this thesis. The second subsection will review some more developed concepts.

2.1 Fundamentals

As already stated before, in this subsection we will review some of the fundamentals of BST, which we need for the more developed concepts in the subsection "Necessary extensions". Compared to [4] or [6] some ideas will be simplified or only special cases are explained.

We will begin with the primitives of BST. These are the expressions which can't be defined in terms of the other ones and which we use for their definition:

1. The collection of natural numbers $\mathbb{N}$, together with its equality $=_{\mathbb{N}}$, its operations and order.

2. The equality by definition ":=".

3. The equivalence by definition ":$\Leftrightarrow$".

4. Pairing. For two terms a, b we have the ordered pair (a, b). We can refer to the entries with the also primitive projections $\mathrm{pr}_1(a, b) := a$ and $\mathrm{pr}_2(a, b) := b$.

5. A *totality* is either the set $\mathbb{N}$ or the totality X is defined by a *membership condition* $\mathcal{M}_X$ i.e. $x \in X :\Leftrightarrow \mathcal{M}_X(x)$. $\mathcal{M}_X(x)$ is called the *membership formula* for X. One can say that a membership condition describes what has to be done in order to *construct* an element of the totality. It is not necessary that the membership condition is decidable for any object.

6. The notion of a *finite routine* or an *algorithm* is left undefined in [4]. In [2] p. 154 an informal explanation of an operation is given, which fits our notion of a finite routine or algorithm:

   > "The notion of an operation (rule, algorithm, algorithmic process, finite routine, mechanical operation) is fundamental. We understand that an operation is a <u>finite</u>, concrete object which can be given by a <u>list of instructions</u>. These instructions must be explicit in nature, and must call only for the performance of "mechanical" computations – no creative activity (such as solving

problems) must be required, and no element of chance (tossing a die) or of free choice must be involved. ... but these are partial operations, i.e. they may at some arguments be undefined." (underlining differs from [2])

The logical framework of BST is first-order intuitionistic logic with equality. The other fundamentals can be defined in terms of the primitives.

Definition 1 (Collections)**.** Let X, Y be totalities with membership formulas $\mathcal{M}_X, \mathcal{M}_Y$. We say X and Y are *definitionally equal* $X := Y$ if $[\mathcal{M}_X(x) :\Leftrightarrow \mathcal{M}_Y(x)]$.

X is *inhabited* if there is a x such that $\mathcal{M}_X(x)$ holds. If we defined an equality on X, we have:

$$X \text{ is inhabited} \Leftrightarrow \exists_{x \in X}(\ x =_X x\ )$$

We may define an *equality* on X, "$=_X$". In order to do so, we define an *equality formula* $\mathtt{Eq}_X(x, y)$ which satisfies the properties of a equivalence relation. We define $x =_X y :\Leftrightarrow \mathtt{Eq}_X(x, y)$. If the related totality is clear from the context, we also write $x = y$. We identify the totality with equality with the pair $(X, =_X)$. Most of the time we just write X and think about the related equality $=_X$ as implicitly given.

The totality of sets $\mathbb{V}_0$ with its equality $=_{\mathbb{V}_0}$, defined as followed

$$\begin{aligned} X \in \mathbb{V}_0 &:\Leftrightarrow X \text{ is a set} \\ X =_{\mathbb{V}_0} Y &:\Leftrightarrow \exists_{f \in \mathbb{F}(X,Y)} \exists_{g \in \mathbb{F}(Y,X)}(\ (f, g) : X =_{\mathbb{V}_0} Y\ ) \end{aligned} \tag{3.1}$$

where

$$(f, g) : X =_{\mathbb{V}_0} Y :\Leftrightarrow\ g \circ f = id_X\ \ \&\ \ f \circ g = id_Y$$

is a *class*. All other totalities with equality, which membership condition includes quantification over $\mathbb{V}_0$, are classes as well. We omit the simple proof for the equality $=_{\mathbb{V}_0}$ satisfying the properties of an equivalence relation.

A *preset* is a totality, for which we know what it means to prove its membership formula. In particular for a formula that only uses bounded quantification (quantification over sets) the BHK-interpretation expresses what we have to do. A *set* X is either $\mathbb{N}$ or it is a preset together with an equality defined on it. This definition of a set is not a formal definition, but a recommendation. Precisely, a set is an element of $\mathbb{V}_0$; our "definition" tells us what we should define to be in the totality $\mathbb{V}_0$.

For two sets X, Y, we define their *product* $X \times Y$ by

$$\begin{aligned} z \in X \times Y &:\Leftrightarrow \exists_{x \in X} \exists_{y \in Y}(\ z := (x, y)\ ) \\ z =_{X \times Y} z' &:\Leftrightarrow \mathrm{pr}_1(z) =_X \mathrm{pr}_1(z')\ \ \&\ \ \mathrm{pr}_2(z) =_Y \mathrm{pr}_2(z'). \end{aligned}$$

$X \times Y$ is a set, as the equality obviously satisfies the conditions of an equivalence relation and the construction of an element reduces to the construction of an element of X and an element of Y.

Remark. In the definition of the preset, it is not necessary that the membership formula is decidable. For example lets define the following formulas on the natural numbers:

$$G(n) :\Leftrightarrow 2n \text{ is the sum of two primes}$$
$$P(n) :\Leftrightarrow \forall_{m>n}(\ G(m)\ )$$

As we don't know if the Goldbach conjecture is provable, we don't know if $P(n)$ is decidable for any $n \in \mathbb{N}^+$. Anyway we know what we have to do to prove $P(n)$ for a concrete $n \in \mathbb{N}^+$. Thus the totality X, with

$$n \in X :\Leftrightarrow n \in \mathbb{N}^+ \quad \& \quad P(n)$$

is a preset.
Also whenever the membership formula of a totality includes quantification over $\mathbb{V}_0$, we don't know how to interpret this quantification and thus don't know how to prove it. Hence a class can't be a set.
The totality $\mathbb{V}_0$ is defined in a open ended way. This means that whenever we define a totality X, which reflects the aforementioned intuition of being a set, we add to the definition of $\mathbb{V}_0$ that X is an element of $\mathbb{V}_0$.

Definition 2 (Non dependent assignment routines)**.** Let X, Y be totalities.
A *non dependent assignment routine* $\alpha : X \rightsquigarrow Y$ from X to Y is a a finite routine, that assigns to each element $x \in X$ an element $y \in Y$. We write $\alpha(x) := y$ and write *assignment routine*, instead of non dependent assignment routine unless we want to highlight the non dependency .
Now assume X, Y being sets. Then an assignment routine $f : X \rightsquigarrow Y$ is called an *operation*. An operation $f : X \rightsquigarrow Y$ is called a *function* if it respects equality, i.e.

$$\forall_{x,x' \in X}(\ x =_x x' \Rightarrow f(x) =_Y f(x')\ )$$

holds. We write $f : X \to Y$.
A function $f : X \to Y$ is an *embedding* from X to Y, if $x =_X x'$, whenever $f(x) =_Y f(x')$. We will write $f : X \hookrightarrow Y$.

Let X, Y, Z be totalities.
Two assignment routines $f : X \rightsquigarrow Y$ and $g : X \rightsquigarrow Y$ are *definitionally equal*, we write $f := g$, if

$$\forall_{x \in X}(\ f(x) := g(x)\ )$$

For two assignment routines $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$, we define the *composition* assignment routine $g \circ f$ by

$$g \circ f : X \rightsquigarrow Z, \qquad\qquad g \circ f(x) := g(f(x))$$

for every $x \in X$.

For any set X we define the identity function id_X

$$id_X : X \to X, \qquad\qquad id_X(x) := x$$

for every $x \in X$.

Definition 3 (The set of functions)**.** The totality of all functions from the set X to the set Y, is denoted by $\mathbb{F}(X,Y)$ and equipped with the point-wise equality.

$$\begin{aligned} f \in \mathbb{F}(X,Y) &:\Leftrightarrow \exists_{g:X\to Y}(\ f := g\ ) \\ f =_{\mathbb{F}(X,Y)} g &:\Leftrightarrow \forall_{x\in X}(\ f(x) =_Y g(x)\ ) \end{aligned}$$

Obviously $=_{\mathbb{F}(X,Y)}$ satisfies the properties of a equivalence relation. Therefore this totality is a set, because $\mathcal{M}_{\mathbb{F}(X,Y)}$ expresses: to prove $\mathcal{M}_{\mathbb{F}(X,Y)}(f)$ one has to define an appropriate function and prove its definitional equality to f. So its clear what we have to do in order to prove the membership formula.
Since every function $f : X \to Y$ we define is definitionally equal to itself, $f \in \mathbb{F}(X,Y)$

Definition 4. Let I be a set and $\mu_0 : I \rightsquigarrow \mathbb{V}_0$ a non dependent assignment routine. A *dependent assignment routine* over μ_0 is a finite routine μ_1, which assigns to each $i \in I$ an element $\mu_1(i)$ in $\mu_0(i)$. We write

$$\mu_1 : \bigcurlywedge_{i\in I} \mu_0(i)$$

Their totality is denoted by $\mathbb{A}(I,\mu_0)$. It is equipped with the point-wise equality

$$\mu_1 =_{\mathbb{A}} \nu_1 :\Leftrightarrow \forall_{i\in I}(\ \mu_1(i) =_{\mu_0(i)} \nu_1(i)\ )$$

and is a set with an argument similar to the one that the totality of all functions is a set.

Definition 5. A *subset* of a set X is a pair (A, i_A), where A is a set and $i_A : A \hookrightarrow X$. Usually we write A instead of (A, i_A) and implicitly think of i_A as given.
The totality with equality of all subsets of X, is denoted by $\mathcal{P}(X)$. It is defined by

$$\begin{aligned} (A, i_A) \in \mathcal{P}(X) &:\Leftrightarrow A \text{ is a set } \ \&\ \ i_A : A \hookrightarrow X \\ (A, i_A) =_{\mathcal{P}(X)} (B, i_B) &:\Leftrightarrow \exists_{f:A\to B}\exists_{g:B\to A}(\ i_A \circ g =_{\mathbb{F}(B,X)} i_B\ \ \&\ \ i_B \circ f =_{\mathbb{F}(A,X)} i_A\ ) \end{aligned}$$

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[dr, hook, "i_A"'] & & B \arrow[ll, "g"] \arrow[dl, hook', "i_B"] \\ & X & \end{tikzcd}$$

Since the membership condition of $\mathcal{P}(X)$ contains quantification over $\mathbb{V}_0$, $\mathcal{P}(X)$ is a class.

Definition 6. A formula $P(x)$ on a set X is called an *extensional property*, if it satisfies

$$\forall_{x,y\in X}(\ [x =_X y\ \ \&\ \ P(x)] \Rightarrow P(y)\ ).$$

The *extensional subset of X generated by P*, X_P, denotes the totality with equality that is defined by

$$x \in X_P :\Leftrightarrow x \in X \;\; \& \;\; P(x)$$
$$x =_{X_P} x' :\Leftrightarrow x =_X x'$$

Clearly $X_P := \{x \in X \mid P(x)\}$ is a set, since X is a set.
The notion subset is justified, because the pair (X_P, i_{X_P}), with $i_{X_P} : X_P \hookrightarrow X$; $i_{X_P}(x) := x$, is in $\mathcal{P}(X)$.
Important to mention, is that with this definition the property $P(x)$ on X is equivalent to the membership to the set X_P.
As an example lets consider the set X and the property $P(x, x') :\Leftrightarrow x =_X x'$. $P(x)$ then is an extensional property, since equality is transitive, and generates the subset $D(X)$ of $X \times X$

$$D(X) := \{(x, x') \in X \times X \mid x =_X x'\}$$

which we call the *diagonal of X*. It inherits the equality from $X \times X$.

Note that it is also possible to define a new equality on X_P. E.g. in section 5 the set of real numbers will be defined as a subset with a different equality than the superset. If we do so, X_P still is a set and for the same reason, as before, a subset of X.

2.2 Necessary extensions

This subsection deals with the idea of set indexed families and important concepts that depend on them.
We will use these new concepts to interpret the logical connectives in section 3.

2.2.1 Set-indexed families of sets

Definition 7. Let I be a set and $D(I)$ as before. Then an I-$family$, or a $family\ of\ sets$ indexed by I, is a pair $\Lambda := (\lambda_0, \lambda_1)$.

$$\lambda_0 : I \rightsquigarrow \mathbb{V}_0 \tag{7.1}$$
$$\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}(\lambda_0(i), \lambda_0(j)), \qquad \lambda_1(i,j) := \lambda_{ij} \text{ for } (i,j) \in D(I) \tag{7.2}$$

such that the following properties hold:

D7.1 For every $i \in I$, we have $\lambda_{ii} := id_{\lambda_0(i)}$

D7.2 For $i, j, k \in I$ such that $i =_I j$ and $j =_I k$, the following diagram commutes

$$\begin{tikzcd} \lambda_0(i) \arrow[rr, "\lambda_{ij}"] \arrow[dr, "\lambda_{ik}"'] & & \lambda_0(j) \arrow[dl, "\lambda_{jk}"] \\ & \lambda_0(k) & \end{tikzcd}$$

For $(i,j) \in D(I)$ the function λ_{ij} is called the *transport map* from $\lambda_0(i)$ to $\lambda_0(j)$ and the dependant assignment routine λ_1 the *modulus of function-likeness* of λ_0. The last term comes from the following property:

$$(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$$

Proof. By definition we have that $id_{\lambda_0(i)} := \lambda_{ii} := \lambda_{ji} \circ \lambda_{ij}$ and
$id_{\lambda_0(j)} := \lambda_{jj} := \lambda_{ij} \circ \lambda_{ji}$

Example 1. Let I, J be sets and $\lambda := (\lambda_0, \lambda_1)$ be an I-family. We define $\mu := (\mu_0, \mu_1)$ by:

$$\mu_0 : I \times J \to \mathbb{V}_0, \qquad \mu_0(i,j) := \lambda_0(i)$$

for every $(i,j) \in I \times J$, and

$$\begin{aligned}
&\mu_1 : \bigcurlywedge_{((i,j),(i',j')) \in D(I \times J)} \mathbb{F}(\mu_0(i,j), \mu_0(i',j')),\\
&\mu_{(i,j)(i',j')} := \mu_1((i,j),(i',j')) : \lambda_0(i) \to \lambda_0(i'),\\
&\mu_{(i,j)(i',j')}(x) := \lambda_{ii'}(x)
\end{aligned}$$

Then $\mu := (\mu_0, \mu_1)$ is an $I \times J$-family, as we have:

D7.1 for every $(i,j) \in I \times J$, we have $\mu_{(i,j)(i,j)} := \lambda_{ii} := id_{\lambda_0(i)} := id_{\mu_0(i,j)}$

D7.2 for $(i,j), (i',j'), (i''), j'') \in I \times J$ such that $(i,j) =_{I \times J} (i',j')$ and $(i',j') =_{I \times J} (i'',j'')$, we have $i =_I i' =_I i''$. Hence:

$$\mu_{(i,j)(i'',j'')} := \lambda_{ii''} := \lambda_{i'i'' \circ \lambda_{ii'}} := \mu_{(i',j')(i''j'')} \circ \mu_{(i,j)(i',j')}$$

We call μ the *canonical extension* of the I-family λ onto $I \times J$. We will write $\lambda(i,j)$ instead of $\mu(i,j)$ for any $(i,j) \in I \times J$ and get

$$\forall_{(i,j) \in I \times J}(\ \lambda_0(i,j) := \lambda_0(i)\ )$$

2.2.2 The exterior union of a family of sets

We will use the exterior union of set indexed families to interpret the existential quantifier. So we use the the notion of a Σ-set for it, referring to the type theoretic Σ-type.

Definition 8. If I is a set and $\Lambda := (\lambda_0, \lambda_1)$ an I-family of sets, then the totality $\Sigma_{i \in I} \lambda_0(i)$ is called the *exterior* or *disjoint union* of Λ or the Σ-*set* of Λ. It is defined by the following membership condition and equality

$$\begin{aligned}
w \in \Sigma_{i \in I} \lambda_0(i) &:\Leftrightarrow \exists_{i \in I} \exists_{x \in \lambda_0(i)}(\ w := (i,x)\ )\\
(i,x) =_{\Sigma_{i \in I} \lambda_0(i)} (j,x') &:\Leftrightarrow i =_I j \ \ \&\ \ \lambda_{ij}(x) =_{\lambda_0(j)} x'
\end{aligned} \tag{8.1}$$

It is natural to consider the totality $\Sigma_{i \in I} \lambda_0(i)$ to be a set.

We proof that the equality $=_{\Sigma_{i\in I}\lambda_0(i)}$ satisfies the properties of an equivalence relation.

Proof. Let $(i,x),(j,y),(k,z)\in\Sigma_{i\in I}\lambda_0(i)$. Then since $i =_I i$ and $\lambda_{ii}(x) := id_{\lambda_0(i)}(x) := x$, we have

$$(i,x) =_{\Sigma_{i\in I}\lambda_0(i)} (i,x).$$

Next we assume $(i,x) =_{\Sigma_{i\in I}\lambda_0(i)} (j,y)$. So we have $i =_I j$ and therefore $j =_I i$, because "$=_I$" is an equivalence relation. Additionally by the definition of the transport maps:

$$x := id_{\lambda_0(i)}(x) := \lambda_{ii}(x) := \lambda_{ji}(\lambda_{ij}(x)) \overset{(i,x)=(j,y)}{=_{\lambda_0(i)}} \lambda_{ji}(x')$$

holds. Hence we have $(j,y) =_{\Sigma_{i\in I}\lambda_0(i)} (i,x)$.
Finally, if we have $(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y)$ and $(j,y) =_{\sum_{i\in I}\lambda_0(i)} (k,z)$ by definition and transitivity of "$=_I$", we get $i =_I k$. The validity of

$$\lambda_{ik}(x) := \lambda_{jk}(\lambda_{ij}(x)) =_{\lambda_0(k)} \lambda_{jk}(y) =_{\lambda_0(k)} z$$

follows from the definition of equality 8.1 and D7.2. Thus we have $(i,x) =_{\Sigma_{i\in I}\lambda_0(i)} (k,z)$. Hence the equality satisfies the properties of an equivalence relation. □

Example 2. Let X, Y be sets. $\mathbf{2} := \{0,1\}$ clearly is a set. The coproduct $X+Y$ can be defined as a Σ-set.
First we define the $\mathbf{2}$-family of X and Y $\Lambda(X,Y) := (\lambda_0^{X,Y}, \lambda_1^{X,Y})$ by:

$$\lambda_0^{X,Y} : 0,1 \rightsquigarrow \mathbb{V}_0, \qquad \lambda_0^{X,Y}(0) := X,\ \ \lambda_0^{X,Y}(1) := Y$$

and

$$\lambda_1^{X,Y} : \bigwedge_{(i,i')\in D(\{0,1\})} \lambda_0^{X,Y}$$
$$\lambda_1^{X,Y}(0,0) := id_X,\ \ \lambda_1^{X,Y}(1,1) := id_Y$$

Now we can define the coproduct:

$$X+Y := \sum_{i\in\mathbf{2}} \lambda_0^{X,Y}(i)$$

This set has the required property, we expect from the coproduct

$$\forall_{x\in X}\forall_{y\in Y}(\ (0,x),(1,y) \in \sum_{i\in\mathbf{2}} \lambda_0^{X,Y}(i)\ )$$

First and second projection of the Σ-set of Λ, $\mathrm{pr}_1(\Lambda)$ and $\mathrm{pr}_2(\Lambda)$, are important to formalize choosing an object of the pair. Therefore it is interesting to make their definitions precise. The first one is implemented as supposed

$$\mathrm{pr}_1(\Lambda) : \sum_{i\in\mathbf{2}} \lambda_0^{X,Y}(i) \to \mathbf{2},$$
$$\mathrm{pr}_1(\Lambda)(i,z) := i$$

for every $(i,z) \in \sum_{i\in\mathbf{2}} \lambda_0^{X,Y}(i)$. The second one is a dependent assignment routine

$$\mathrm{pr}_2(\Lambda) : \bigcurlywedge_{(i,z)\in\sum_{i\in\mathbf{2}} \lambda_0^{X,Y}(i)} \lambda_0(i),$$

$$\mathrm{pr}_2(\Lambda)(i,z) := \mathrm{pr}_2(i,z) := z$$

for every $(i,z) \in \sum_{i\in\mathbf{2}} \lambda_0^{X,Y}(i)$.

2.2.3 Dependent functions over a family of sets

There are several reasons to invent dependent functions. One is to generalise the cartesian product, this is shown in detail in [4] and the special case of the product of two sets is shown in the example below.
The main application of this concept in this thesis is the use in the interpretation of the universal quantifier.

As functions are a special kind of non dependent assignment routines, dependent functions are a special kind of dependent assignment routines.

Definition 9. Let I be a set, $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets. The membership condition of and the equality on the totality $\Pi_{i\in I}\lambda_0(i)$, called the set of *dependent functions* over Λ or the Π-*set* of Λ, are

$$\phi \in \prod_{i\in I} \lambda_0(i) :\Leftrightarrow \phi \in \mathbb{A}(I,\lambda_0) \ \ \& \ \ \forall_{(i,j)\in D(I)}(\ \phi(j) := \phi_j =_{\lambda_0(j)} \lambda_{ij}(\phi_i)\ ) \tag{9.1}$$

$$\phi =_{\Pi_{i\in I}\lambda_0(i)} \varphi :\Leftrightarrow \forall_{i\in I}(\ \phi_i =_{\lambda_0(i)} \varphi_j\ ) \tag{9.2}$$

It is natural to consider the totality $\prod_{i\in I} \lambda_0(i)$ to be a set

The equivalence $=_{\Pi_{i\in I}\lambda_0(i)}$ satisfies the properties of an equivalence relation, since the set of dependent functions over a family of sets is an extensional subset of $\mathbb{A}(I,\lambda_0)$.

Example 3. Like mentioned above the Π-set can be seen as a generalisation of the cartesian product. Therefore we will proof the equality to the cartesian product of two sets

$$\prod_{i\in\mathbf{2}} \lambda_0(i)^{X,Y} =_{\mathbb{V}_0} X \times Y$$

where X, Y are sets and $\Lambda(X,Y) := (\lambda_0^{X,Y}, \lambda_1^{X,Y})$ as in example 2. To proof the statement, we have to show that 3.1 holds. Let $X, Y,\ \Lambda(X,Y)$ be as given. We define

$$f : \prod_{i\in\mathbf{2}} \lambda_0(i)^{X,Y} \to X \times Y \qquad\qquad f(\phi) := (\phi_0, \phi_1)$$

$$g : X \times Y \to \prod_{i\in\mathbf{2}} \lambda_0(i)^{X,Y} \qquad\qquad g(x,y) := \varphi_{xy}$$

where

$$\varphi_{xy} : \bigcurlywedge_{i \in \mathbf{2}} \lambda_0(i),$$

$$\varphi_{xy}(0) := x \in \lambda_0^{X,Y}(0) \qquad\qquad \varphi_{xy}(1) := y \in \lambda_0^{X,Y}(1)$$

for every $x \in X$ and $y \in Y$. So f, g are well defined and as can easily be seen they respect equality. In particular they are functions. Hence

$$f \circ g(x, y) := f(g(x, y)) := f(\varphi_{xy}) := (\varphi_{xy}(0), \varphi_{xy}(1)) := (x, y)$$

$$g \circ f(\phi) := g(f(\phi)) := g(\phi_0, \phi_1) := \varphi_{\phi_0\phi_1} \overset{(*)}{:=} \phi$$

$$(*) \quad \varphi_{\phi_0\phi_1}(0) := \phi_0 := \phi(0) \quad \& \quad \varphi_{\phi_0\phi_1}(1) := \phi_1 := \phi(1)$$

for $(x, y) \in X \times Y$, $\phi \in \Pi_{i\in\mathbf{2}}\lambda_0^{X,Y}(i)$.
So $f \circ g = id_{X\times Y}$ and $g \circ f = id_{\Pi_{i\in\mathbf{2}}\lambda_0^{X,Y}(i)}$ and therefore $X \times Y =_{\mathbb{V}_0} \Pi_{i\in\mathbf{2}}\lambda_0^{X,Y}(i)$.

Example 4. The second projection of a Σ-set is even a dependent function.
Let the definitions and premises as in example 2. We define the $\sum_{i\in\mathbf{2}}\lambda_0(i)$-family $\sum^{\Lambda} := (\sigma_0^{\Lambda}, \sigma_1^{\Lambda})$ by

$$\sigma_0^{\Lambda} : \sum_{i\in\mathbf{2}} \lambda_0(i) \rightsquigarrow \mathbb{V}_0,$$
$$\sigma_0^{\Lambda}(i, z) := \lambda_0(i),$$
$$\sigma_1^{\Lambda}((i, z), (i', z')) := \lambda_{ii'}.$$

σ_0^{Λ} is well defined, what follows from the definition of Λ, and σ_1^{Λ} satisfies D7.1 & D7.2:

$$\sigma^{\Lambda}_{(i,z)(i,z)} := \lambda_{ii} := id_{\lambda_0(i)} := id_{\sigma_0^{\Lambda}(i,z)},$$
$$\sigma^{\Lambda}_{(i,z)(k,z'')} := \lambda_{ik} := \lambda_{mk} \circ \lambda_{im} := \sigma^{\Lambda}_{(m,z')(k,z'')} \circ \sigma^{\Lambda}_{(i,z)(m,z')}$$

for $(i, z), (m, z'), (k, z'') \in \sum_{i\in\mathbf{2}}\lambda_0(i)$, such that $(i, z) =_{\sum_{i\in\mathbf{2}}\lambda_0(i)} (m, z')$ and $(m, z') =_{\sum_{i\in\mathbf{2}}\lambda_0(i)} (k, z'')$.
Moreover, if $(i, z) =_{\sum_{i\in\mathbf{2}}\lambda_0(i)} (i', z')$, then

$$\mathrm{pr}_2(\Lambda)(i, z) := z \overset{\text{Def 9.2}}{=}_{\lambda_0(i)} \lambda_{i'i}(z') := \lambda_{i'i}(\mathrm{pr}_2(\Lambda)(i', z'))$$

Therefore the second projection $\mathrm{pr}_2(\Lambda)$ is a dependent function over $\sum^{\Lambda}$.

2.2.4 Basic families of sets

In the following we will discuss some general schemes to generate new set indexed families of sets.
We start with the extension of some basic concepts, that we already defined for sets, on I-families. Namely functions between I-families, the construction of the disjoint union of two I-families and the construction of the product of two I-families. All of these concepts are "point-wise generalisations" of the basic ideas.

Proposition 10. Let I be a set, $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1)$ be I-families of sets.

i) The pair $\mathbb{F}(\Lambda, M) := (\mathbb{F}(\lambda_0, \mu_0), \mathbb{F}(\lambda_1, \mu_1))$ is an I-familiy of sets, we define for every $i \in I$

$$\mathbb{F}(\lambda_0, \mu_0) : I \rightsquigarrow \mathbb{V}_0, \qquad \mathbb{F}(\lambda_0, \mu_0)(i) := \mathbb{F}(\lambda_0(i), \mu_0(i))$$

and

$$\begin{aligned} \mathbb{F}(\lambda_1, \mu_1) : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}(\ \mathbb{F}(\lambda_0, \mu_0)(i), \mathbb{F}(\lambda_0, \mu_0)(j)\ ), \\ \mathbb{F}(\lambda_1, \mu_1)_{ij} := \mathbb{F}(\lambda_1, \mu_1)(i,j) : \mathbb{F}(\lambda_0(i), \mu_0(i)) \to \mathbb{F}(\lambda_0(j), \mu_0(j)), \\ \mathbb{F}(\lambda_1, \mu_1)_{ij}(f) := \mu_{ij} \circ f \circ \lambda_{ji}, \end{aligned}$$

for every $f \in \mathbb{F}(\lambda_0(i), \mu_0(i))$.

ii) The pair $\Lambda + M := (\lambda_0 + \mu_0, \lambda_1 + \mu_1)$ is an I-familiy of sets, we define for every $i \in I$

$$\lambda_0 + \mu_0 : I \rightsquigarrow \mathbb{V}_0, \qquad (\lambda_0 + \mu_0)(i) := \lambda_0(i) + \mu_0(i)$$

and

$$\begin{aligned} \lambda_1 + \mu_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}(\ \lambda_0(i) + \mu_0(i), \lambda_0(j) + \mu_0(j)\ ), \\ (\lambda_1 + \mu_1)_{ij} := (\lambda_1 + \mu_1)(i,j) : \lambda_0(i) + \mu_0(i) \to \lambda_0(j) + \mu_0(j), \\ (\lambda_1 + \mu_1)_{ij}(0, x) := (0, \lambda_{ij}(x)), \\ (\lambda_1 + \mu_1)_{ij}(1, y) := (1, \lambda_{ij}(y)), \end{aligned}$$

for $x \in \lambda_0(i)$ and $y \in \mu_0(i)$.

iii) The pair $\Lambda \times M := (\lambda_0 \times \mu_0, \lambda_1 \times \mu_1)$ is an I-familiy of sets, we define for every $i \in I$

$$\lambda_0 \times \mu_0 : I \rightsquigarrow \mathbb{V}_0, \qquad (\lambda_0 \times \mu_0)(i) := \lambda_0(i) \times \mu_0(i)$$

and

$$\begin{aligned} \lambda_1 \times \mu_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}(\ \lambda_0(i) \times \mu_0(i), \lambda_0(j) \times \mu_0(j)\ ), \\ (\lambda_1 \times \mu_1)_{ij} := (\lambda_1 \times \mu_1)(i,j) : \lambda_0(i) \times \mu_0(i) \to \lambda_0(j) \times \mu_0(j), \\ (\lambda_1 \times \mu_1)_{ij}(x, y) := (\lambda_{ij}(x), \mu_{ij}(y)), \end{aligned}$$

for $x \in \lambda_0(i)$ and $y \in \mu_0(i)$.

Proof. First we notice that I is a set by premise and all of the pairs are of the correct form. An assignment routine $I \rightsquigarrow \mathbb{V}_0$ and a corresponding dependent assignment routine. So to proof the proposition we have to show for every pair that it fulfils the two conditions D7.1 and D7.2.

i) D7.1: Let $i \in I, f \in \mathbb{F}(\lambda_0(i), \mu_0(i))$ be arbitrary. Then we have:

$$\forall_{x \in \lambda_0(i)}(\ \mathbb{F}(\lambda_1, \mu_1)_{ii}(f)(x) := \mu_{ii} \circ f \circ \lambda_{ii}(x) := id_{\mu_0(i)} \circ f \circ id_{\lambda_0(i)}(x) := f(x) \ )$$

and thus $\mathbb{F}(\lambda_1, \mu_1)_{ii}(f) := f$. Since $f \in \mathbb{F}(\lambda_0(i), \mu_0(i))$ was chosen freely, $\mathbb{F}(\lambda_1, \mu_1)_{ii} := id_{\mathbb{F}(\lambda_0(i), \mu_0(i))}$ is constructively valid.
D7.2: For $i, j, k \in I$ such that $i =_I j$ and $j =_I k$, we have for all $f \in \mathbb{F}(\lambda_0(i), \mu(i))$:

$$\begin{aligned}\forall_{x \in \lambda_0(k)}(\ \mathbb{F}(\lambda_1,\mu_1)_{jk} \circ \mathbb{F}(\lambda_1, \mu_1)_{ij}(f)(x) & \\ &:= \mathbb{F}(\lambda_1, \mu_1)_{jk}(\mu_{ij} \circ f \circ \lambda_{ji})(x) \\ &:= \mu_{jk} \circ \mu_{ij} \circ f \circ \lambda_{ji} \circ \lambda_{kj}(x) \\ &:= \mu_{ik} \circ f \circ \lambda_{ki}(x) \\ &:= \mathbb{F}(\lambda_1, \mu_1)_{ik}(f)(x) \ )\end{aligned}$$

and thus $\mathbb{F}(\lambda_1, \mu_1)_{jk} \circ \mathbb{F}(\lambda_1, \mu_1)_{ij}(f) := \mathbb{F}(\lambda_1, \mu_1)_{ik}(f)$. Since $f \in \mathbb{F}(\lambda_0(i), \mu_0(i))$ was chosen freely, $\mathbb{F}(\lambda_1, \mu_1)_{jk} \circ \mathbb{F}(\lambda_1, \mu_1)_{ij} := \mathbb{F}(\lambda_1, \mu_1)_{ik}$ is constructively valid.
Therefore the pair $\mathbb{F}(\Lambda, M) := (\mathbb{F}(\lambda_0, \mu_0), \mathbb{F}(\lambda_1, \mu_1))$ is an I-family.

ii) D7.1: Let $i \in I$ be arbitrary. Then we have:

$$\begin{aligned}\forall_{x \in \lambda_0(i)}(\ (\lambda_1{+}\mu_1)_{ii}(0, x) & \\ &:= (0, \lambda_{ii}(x)) := (0, id_{\lambda_0(i)}(x)) := (0, x) \\ &:= id_{\lambda_0(i)+\mu_0(i)}(0, x) \ ) \\ \forall_{y \in \mu_0(i)}(\ (\lambda_1{+}\mu_1)_{ii}(1, y) & \\ &:= (1, \mu_{ii}(y)) := (1, id_{\mu_0(i)}(y)) := (1, y) \\ &:= id_{\lambda_0(i)+\mu_0(i)}(1, y) \ )\end{aligned}$$

and thus $(\lambda_1 + \mu_1)_{ii} := id_{\lambda_0(i)+\mu_0(i)}$.
D7.2: For $i, j, k \in I$ such that $i =_I j$ and $j =_I k$, we have:

$$\begin{aligned}\forall_{x \in \lambda_0(i)}(\ (\lambda_1{+}\mu_1)_{jk} \circ (\lambda_1 + \mu_1)_{ij}(0, x) & \\ &:= (\lambda_1 + \mu_1)_{jk}((0, \lambda_{ij}(x)) := (0, \lambda_{jk} \circ \lambda_{ij}(x)) \\ &:= (0, \lambda_{ik}(x)) := (\lambda_1 + \mu_1)_{ik}(0, x) \ ) \\ \forall_{y \in \mu_0(i)}(\ (\lambda_1{+}\mu_1)_{jk} \circ (\lambda_1 + \mu_1)_{ij}(1, y) & \\ &:= (\lambda_1 + \mu_1)_{jk}((1, \mu_{ij}(y)) := (1, \mu_{jk} \circ \mu_{ij}(y)) \\ &:= (1, \mu_{ik}(y)) := (\lambda_1 + \mu_1)_{ik}(1, y) \ )\end{aligned}$$

and thus $(\lambda_1 + \mu_1)_{jk} \circ (\lambda_1 + \mu_1)_{ij} := (\lambda_1 + \mu_1)_{ik}$ is constructively valid.
Therefore the pair $\Lambda + M := (\lambda_0 + \mu_0, \lambda_1 + \mu_1)$ is an I-family.

iii) D7.1: Let $i \in I$ be arbitrary. Then we have:

$$\begin{aligned}\forall_{x\in\lambda_0(i),y\in\mu_0(i)}(\; (\lambda_1{\times}\mu_1)_{ii}(x,y) &\\ &:= (\lambda_{ii}(x),\mu_{ii}(y)) := (id_{\lambda_0(i)}(x), id_{\mu_0(i)}(y)) \\ &:= (x,y) := id_{\lambda_0(i)\times\mu_0(i)}(x,y) \;)\end{aligned}$$

and thus $(\lambda_1 \times \mu_1)_{ii} := id_{\lambda_0(i)\times\mu_0(i)}$.
D7.2: For $i, j, k \in I$ such that $i =_I j$ and $j =_I k$, we have:

$$\begin{aligned}\forall_{x\in\lambda_0(i),y\in\mu_0(i)}(\; (\lambda_1{\times}\mu_1)_{jk} \circ (\lambda_1 \times \mu_1)_{ij}(x,y) &\\ &:= (\lambda_1 \times \mu_1)_{jk}(\lambda_{ij}(x), \mu_{ij}(y)) \\ &:= (\lambda_{jk} \circ \lambda_{ij}(x), \mu_{jk} \circ \mu_{ij}(y)) \\ &:= (\lambda_{ik}(x), \mu_{ik}(y)) := (\lambda_1 \times \mu_1)_{ik}(x,y) \;)\end{aligned}$$

and thus $(\lambda_1 \times \mu_1)_{jk} \circ (\lambda_1 \times \mu_1)_{ij} := (\lambda_1 \times \mu_1)_{ik}$ is constructively valid.
Therefore the pair $\Lambda \times M := (\lambda_0 \times \mu_0, \lambda_1 \times \mu_1)$ is an I-family.

□

So far we discussed families that are indexed by any regular set. As stated in definition 1 for two (index) sets their product is a set. Consequently all the above assertions apply to the product, too. It is also possible to construct new families that are uniquely defined for such special index sets, though.

Proposition 11. Let I, J be sets and $R := (\rho_0, \rho_1)$ be an $(I \times J)$-family of sets, where:

$$\rho_0 : I \times J \rightsquigarrow \mathbb{V}_0, \qquad \rho_1 : \bigcurlywedge_{((i,j),(i',j'))\in D(I\times J)} \mathbb{F}(\rho_0(i,j), \rho_0(i',j')),$$
$$\rho_1\left((i,j),(i',j')\right) := \rho_{(i,j)(i',j')}$$

i) For a fixed $j \in J$, we can construct the "restriction" of R to $I \times \{j\}$.
This new family $\Lambda^j := (\lambda_0^j, \lambda_1^j)$ is an I-family. We define, for every $i \in I$:

$$\lambda_0^j : I \rightsquigarrow \mathbb{V}_0, \qquad \lambda_0^j(i) := \rho_0(i,j)$$

and

$$\lambda_1^j : \bigcurlywedge_{(i,i'\in D(I)} \mathbb{F}\left(\rho_0(i,j), \rho_0(i',j)\right),$$
$$\lambda_1^j(i,i') := \lambda_{ii'}^j : \rho_0(i,j) \to \rho_0(i',j), \qquad \lambda_{ii'}^j := \rho_{(i,j)(i',j)}.$$

ii) For a fixed $i \in I$, we can construct the "restriction" of R to $\{i\} \times J$.
This new family $M^i := (\mu_0^i, \mu_1^i)$ is an J-family. We define, for every $j \in J$:

$$\mu_0^i : J \rightsquigarrow \mathbb{V}_0, \qquad \mu_0^i(j) := \rho_0(i,j)$$

and

$$\mu_1^i : \bigcurlywedge_{(j,j' \in D(J)} \mathbb{F}\left(\rho_0(i,j), \rho_0(i,j')\right),$$

$$\mu_1^i(j,j') := \mu_{jj'}^i : \rho_0(i,j) \to \rho_0(i,j'), \qquad \mu_{jj'}^i := \rho_{(i,j)(i,j')}.$$

Proof. We will proof case i). The case ii) is done similarly.
To proof that the pair $\lambda^j : (\lambda_0^j, \lambda_1^j)$ is an I-set for every $j \in J$, we have to show that λ_0^j and λ_1^j are well-defined and the properties D7.1 & D7.2 hold. All of these statements follow immediately from the definition of R.
Let $j \in J$ be arbitrary, but fiexd.
λ_0^j *is well-defined*: By definition of R, $\rho_0(i,j)$ is a set, for every $i \in I$.
Because $\rho_0(i,j) := \lambda_0(i)$, for every $i \in I$, we immediately get that λ_0^j is well-defined.
λ_1^j *is well-defined*: Let $i, i' \in I$. We have to show that $\lambda_{ii'}^j$ is a function. Clearly this is valid, since $\lambda_{ii'}^j := \rho_{(i,j)(i',j)}$, where the later one is a function by definition of R. Therefore λ_1^j is well-defined.
D7.1:For $i \in I$, we have:

$$\lambda_{ii'}^j := \rho_{(i,j)(i,j)} := id_{\rho_0(i,j)}$$

D7.2: For $i, k, m \in I$ such that $i = k$ and $k = m$ we have:

$$\lambda_{km}^j \circ \lambda_{ik}^j := \rho_{(k,j)(m,j)} \circ \rho_{(i,j)(k,j)} := \rho_{(i,j)(m,j)} := \lambda_{im}^j.$$

Therefore the pair $\lambda^j : (\lambda_0^j, \lambda_1^j)$ is an I-set for every $j \in J$. □

Proposition 12. Let I, J be sets and $R := (\rho_0, \rho_1)$ be an $(I \times J)$-family of sets, where:

$$\rho_0 : I \times J \rightsquigarrow \mathbb{V}_0, \qquad \rho_1 : \bigcurlywedge_{((i,j),(i',j')) \in D(I \times J)} \mathbb{F}(\rho_0(i,j), \rho_0(i',j')),$$

$$\rho_1\left((i,j),(i',j')\right) := \rho_{(i,j)(i',j')}$$

i) The pair $\sum^1 R := (\Sigma^1 \rho_0, \Sigma^1 \rho_1)$ is an I-family of sets. We define, for $i \in I$:

$$\sum^1 \rho_0 : I \rightsquigarrow \mathbb{V}_0, \qquad \left(\sum^1 \rho_0\right)(i) := \sum_{j \in J} \rho_0(i,j)$$

and

$$\sum^1 \rho_1 : \bigcurlywedge_{(i,i') \in D(I)} \mathbb{F}\left(\sum_{j \in J} \rho_0(i,j), \sum_{j \in J} \rho_0(i',j)\right),$$

$$\left(\sum^1 \rho_1\right)_{ii'} := \left(\sum^1 \rho_1\right)(i,i') : \sum_{j \in J} \rho_0(i,j) \to \sum_{j \in J} \rho_0(i',j),$$

$$\left(\sum^1 \rho_1\right)_{ii'} (j,x) := (j, \rho_{(i,j)(i',j)}(x))$$

for $j \in J$ and $x \in \rho_0(i,j)$.

ii) The pair $\sum^2 R := (\Sigma^2\rho_0, \Sigma^2\rho_1)$ is a J-family of sets. We define, for $j \in J$:

$$\sum^2 \rho_0 : J \rightsquigarrow \mathbb{V}_0, \qquad \left(\sum^2 \rho_0\right)(j) := \sum_{i\in I} \rho_0(i,j)$$

and

$$\sum^2 \rho_1 : \bigcurlywedge_{(j,j')\in D(J)} \mathbb{F}\left(\sum_{i\in I}\rho_0(i,j), \sum_{i\in I}\rho_0(i,j')\right),$$

$$\left(\sum^2 \rho_1\right)_{jj'} := \left(\sum^2 \rho_1\right)(j,j') : \sum_{i\in I}\rho_0(i,j) \to \sum_{I\in I}\rho_0(i,j'),$$

$$\left(\sum^2 \rho_1\right)_{jj'}(i,x) := (i, \rho_{(i,j)(i,j')}(x))$$

for $i \in I$ and $x \in \rho_0(i,j)$.

iii) The pair $\prod^1 R := (\prod^1 \rho_0, \prod^1 \rho_1)$, is an I-family of sets. We define, for $i \in I$:

$$\prod^1 \rho_0 : I \rightsquigarrow \mathbb{V}_0, \qquad \left(\prod^1 \rho_0\right)(i) := \prod_{j\in J} \rho_0(i,j)$$

and

$$\prod^1 \rho_1 : \bigcurlywedge_{(i,i')\in D(I)} \mathbb{F}\left(\prod_{j\in J}\rho_0(i,j), \prod_{j\in J}\rho_0(i',j)\right),$$

$$\left(\prod^1 \rho_1\right)_{ii'} := \left(\prod^1 \rho_1\right)(i,i') : \prod_{j\in J}\rho_0(i,j) \to \prod_{j\in J}\rho_0(i',j),$$

$$\left[\left(\prod^1 \rho_1\right)_{ii'}(\phi)\right]_j := \rho_{(i,j)(i',j)}(\phi_j),$$

for $j \in J$ and $\phi \in \prod_{j\in J}\rho_0(i,j)$.

iv) The pair $\prod^2 R := (\prod^2 \rho_0, \prod^2 \rho_1)$, is a J-family of sets. We define, for $j \in J$:

$$\prod^2 \rho_0 : J \rightsquigarrow \mathbb{V}_0, \qquad \left(\prod^2 \rho_0\right)(j) := \prod_{i\in I} \rho_0(i,j)$$

and

$$\prod^2 \rho_1 : \bigcurlywedge_{(j,j')\in D(J)} \mathbb{F}\left(\prod_{i\in I}\rho_0(i,j), \prod_{i\in I}\rho_0(i,j')\right),$$

$$\left(\prod^2 \rho_1\right)_{jj'} := \left(\prod^2 \rho_1\right)(j,j') : \prod_{i\in I}\rho_0(i,j) \to \prod_{i\in I}\rho_0(i,j'),$$

$$\left[\left(\prod^2 \rho_1\right)_{jj'}(\phi)\right]_i := \rho_{(i,j)(i,j')}(\phi_i),$$

for $i \in I$ and $\phi \in \prod_{i\in I}\rho_0(i,j)$.

Proof. As in the proof for the previous proposition we have to show for all the constructed pairs, that both assignment routines are well-defined and that the properties D7.1 & D7.2 in the definition of a set indexed family are satisfied.

i) $\underline{\left(\Sigma^1\rho_0\right)\ \textit{is well-defined}}$: In the previous proposition we constructed the J-family $M^i := (\mu_0^i, \mu_1^i)$ for a fixed $i \in I$. Therefore $\left(\Sigma^1\rho_0\right)(i) := \Sigma_{j\in J}\mu_0^i(j)$, for every $i \in I$, is a set and $\left(\Sigma^1\rho_0\right)$ is well-defined.
$\underline{\left(\Sigma^1\rho_1\right)\ \textit{is well-defined}}$: Let $(i,i') \in D(I)$ be arbitrary. We have to show that $\left(\sum^1 \rho_1\right)_{ii'}$ is a function.
First we show that the assignment routine is well-defined. For every $(j,x) \in \sum_{j\in J}\rho_0(i,j)$:

$$\left(\sum^1 \rho_1\right)_{ii'}(j,x) := (j, \rho_{(i,j)(i',j)}(x)) \in \sum_{j\in J}\rho_0(i',j)$$

and so the assignment routine is well-defined.
Now we show that the assignment routine is a function. Let $(j,x),(j',x') \in \sum_{j\in J}\rho_0(i,j)$ be such that $(j,x) =_{\Sigma_{j\in J}\rho_0(i,j)} (j',x')$. Thus by definition we have

$$j =_J j' \ \ \& \ \ \rho_{(i,j)(i,j')}(x) =_{\rho_0(i,j')} x$$

and, since $i =_I i'$ and $\rho_{(i,j')(i',j')}$ is a function

$$\rho_{(i,j)(i',j')}(x) := \rho_{(i,j')(i',j')} \circ \rho_{(i,j)(i,j')}(x) =_{\rho_0(i',j')} \rho_{(i,j')(i',j')}(x')$$

and therefore

$$\rho_{(i',j)(i',j')}(\rho_{(i,j)(i',j)}(x)) := \rho_{(i,j)(i',j')}(x) =_{\rho_0(i',j')} \rho_{(i,j')(i',j')}(x'). \qquad (*)$$

With that in mind, it is easy to show that $\left(\Sigma^1\rho_1\right)_{ii'}$ is a function:

$$\begin{aligned}\left(\sum^1\rho_1\right)_{ii'}(j,x) &:= (j,\rho_{(i,j)(i',j)}(x))\\ &\overset{(*)}{=}_{\sum_{i\in J}\rho_0(i',j)} (j',\rho_{(i,j')(i',j')}(x'))\\ &:= \left(\sum^1\rho_1\right)_{ii'}(j',x')\end{aligned}$$

where $(i,i'),(j,x),(j',x')$ are characterized as above.
Hence $\left(\Sigma^1\rho_1\right)_{ii'}$ is a function.
<u>*D7.1*</u>: Let $i\in I$ be arbitrary, likewise $(j,x)\in\Sigma_{j\in J}\rho_0(i,j)$.

$$\begin{aligned}\left(\sum^1\rho_1\right)_{ii}(j,x) &:= (j,\rho_{(i,j)(i,j)}(x)) := (j,id_{\rho_0(i,j)}(x))\\ &:= (j,x)\\ &:= id_{\left(\sum_{j\in J}\rho_0(i,j)\right)}(j,x) := id_{\left(\sum^1\rho_0\right)(i)}(j,x)\end{aligned}$$

Because i and (j,x) were chosen freely,

$$\left(\sum^1\rho_1\right)_{ii} := id_{\left(\sum^1\rho_0\right)(i)}$$

is constructively valid.
<u>*D7.2*</u>: Let $i,k,m\in I$ such that $i=_I k\ \ \&\ \ k=_I m$. Then:

$$\begin{aligned}\left(\sum^1\rho_1\right)_{km}\circ\left(\sum^1\rho_1\right)_{ik}(j,x) &:= \left(\sum^1\rho_1\right)_{km}(j,\rho_{(i,j)(k,j)}(x))\\ &:= (j,\rho_{(k,j)(m,j)}(\rho_{(i,j)(k,j)}(x)))\\ &:= (j,\rho_{(i,j)(m,j)}(x)) := \left(\sum^1\rho_1\right)_{im}(j,x)\end{aligned}$$

for every $(j,x)\in\left(\sum^1\rho_0\right)(i)$ and therefore

$$\left(\sum^1\rho_1\right)_{km}\circ\left(\sum^1\rho_1\right)_{ik} := \left(\sum^1\rho_1\right)_{im}$$

So the pair $\sum^1 R := (\Sigma^1\rho_0,\Sigma^1\rho_1)$ is an I-family of sets.

ii) The proof is similar to the one of the case i).

iii) $\underline{\left(\prod^1 \rho_0\right)}$ *is well-defined*: In the previous lemma we constructed the J-family $M^i := (\mu_0^i, \mu_1^i)$ for a fixed $i \in I$. Therefore $\left(\prod^1 \rho_0\right)(i) := \prod_{j\in J} \mu_0^i(j)$, for every $i \in I$, is a set and $\left(\prod^1 \rho_0\right)$ is well-defined.
$\underline{\left(\prod^1 \rho_1\right)}$ *is well-defined*: Let $(i, i') \in D(I)$ be arbitrary. We have to show that $\left(\prod^1 \rho_1\right)_{ii'}$ is a function. First we show that $\left(\prod^1 \rho_1\right)_{ii'}$ is a well-defined assignment routine. Therefor let $\phi \in \prod_{j\in J} \rho_0(i, j)$ and $j, j' \in D(J)$, then:

$$\rho_{(i,j)(i,j')}(\phi_j) = \phi_{j'}$$

and thus, since $\rho_{(i,j')(i',j)}$ is a function:

$$\rho_{(i,j')(i',j)} \circ \rho_{(i,j)(i,j')}(\phi_j) = \rho_{(i,j')(i',j)}(\phi_{j'})$$

For $\phi \in \prod_{j\in J} \rho_0(i, j)$ and $j \in J$, we have

$$\begin{aligned}\rho_{(i',j)(i',j')}\left(\left[\left(\prod^1 \rho_1\right)_{ii'}(\phi)\right]_j\right) &:= \rho_{(i',j)(i',j')}(\rho_{(i,j)(i',j)}(\phi_j)) \\ &:= \rho_{(i',j)(i',j')}(\rho_{(i,j')(i',j)} \circ \rho_{(i,j)(i,j')}(\phi_j)) \\ &= \rho_{(i',j)(i',j')}(\rho_{(i,j')(i',j)}(\phi_{j'})) \\ &:= \rho_{(i,j')(i',j')}(\phi_{j'}) \\ &:= \left[\left(\prod^1 \rho_1\right)_{ii'}(\phi)\right]_{j'}\end{aligned}$$

So $\left(\prod^1 \rho_0\right)_{ii'}(\phi)$ is a dependent function and $\left(\prod^1 \rho_0\right)_{ii'}$ is a well-defined assignment routine.
It is also a function as for $\phi, \varphi \in \prod_{j\in J} \rho_0(i, j)$, such that $\phi = \varphi$, holds:

$$\phi_j = \varphi_j$$

for every $j \in J$, and

$$\begin{aligned}\left[\left(\prod^1 \rho_1\right)_{ii'}(\phi)\right]_j &:= \rho_{(i,j)(i',j)}(\phi_j) \\ &= \rho_{(i,j)(i',j)}(\varphi_j) := \left[\left(\prod^1 \rho_1\right)_{ii'}(\varphi)\right]_j\end{aligned}$$

and therefore

$$\left(\prod^1 \rho_1\right)_{ii'}(\phi) = \left(\prod^1 \rho_1\right)_{ii'}(\varphi).$$

Hence $\left(\prod^1 \rho_1\right)$ is well-defined.
<u>*D7.1*</u>: Let $i \in I$ be arbitrary, likewise $j \in J$ and $\phi \in \prod_{j\in J} \rho_0(i,j)$. Then

$$\begin{aligned}\left[\left(\prod^1 \rho_1\right)_{ii}(\phi)\right]_j &:= \rho_{(i,j)(i,j)}(\phi_j) \\ &:= id_{\rho_0(i,j)}(\phi_j) \\ &:= \phi_j := \left[id_{\prod_{j\in J}\rho_0(i,j)}(\phi)\right]_j\end{aligned}$$

Because i, j and ϕ were chosen freely,

$$\left(\prod^1 \rho_1\right)_{ii} := id_{\prod_{j\in J}\rho_0(i,j)}$$

is constructively valid.
<u>*D7.2*</u>: Let $i, k, m \in I$ such that $i =_I k \;\;\&\;\; k =_I m$. Then:

$$\begin{aligned}\left[\left(\prod^1 \rho_1\right)_{km} \circ \left(\prod^1 \rho_1\right)_{ik}(\phi)\right]_j &:= \rho_{(k,j)(m,j)} \circ \rho_{(i,j)(k,j)}(\phi_j) \\ &:= \rho_{(i,j)(m,j)}(\phi_j) := \left[\left(\prod^1 \rho_1\right)_{im}(\phi)\right]_j\end{aligned}$$

for every $j \in J$ and therefore

$$\left(\prod^1 \rho_1\right)_{km} \circ \left(\prod^1 \rho_1\right)_{ik} := \left(\prod^1 \rho_1\right)_{im}$$

So the pair $\prod^1 R := (\prod^1 \rho_0, \prod^1 \rho_1)$, is an I-family of sets.

iv) The proof is similar to the one of case iii)

$\square$

Definition 13. Let X, Y be sets. We define:

$$\sum_{x\in X} Y := \sum_{x\in X} \lambda_0(x)$$

for every $x \in X$, where

$$\lambda_0 : X \to \mathbb{V}_0, \qquad\qquad \lambda_0(x) := Y, \tag{13.1}$$

and

$$\begin{aligned}
&\lambda_1 : \bigcurlywedge_{(x,x')\in D(X)} \mathbb{F}(\lambda_0(x), \lambda_0(x')), \\
&\lambda_{xx'} := \lambda_1(x,x') : Y \to Y, \\
&\lambda_{xx'}(y) := y
\end{aligned} \tag{13.2}$$

Thus $\lambda := (\lambda_0, \lambda_1)$ clearly is a X-family and $\sum_{x\in X} Y$ is a well defined set.
Now let X, Y be sets and $\lambda := (\lambda_0, \lambda_1)$ be a Y-family. Then the canonical extension of λ onto $Y \times X$ is a $Y \times X$-family and

$$\sum_{x\in X} \lambda_0(y) := \left(\sum^{1} \lambda_0\right)(y) \tag{13.3}$$

is well defined. $\sum_{x\in X} \lambda_0(y)$ is a Y-family by proposition 12.
For the sets $Y_1, ..., Y_n$, the $Y_1 \times ... \times Y_n$-family λ and $i \in \{1, ..., n\}$ by definition of the exterior union and proposition 12 we have that

$$\sum_{y_i\in Y_i} \lambda_0(y_1, ...y_n) \tag{13.4}$$

is well-defined and either a $Y_1 \times ... \times Y_{i-1} \times Y_{i+1} \times ... \times Y_n$-family or (in the case $n = 1$) a set.

Similarly we can define for dependent functions:
The set

$$\prod_{x\in X} Y := \prod_{x\in X} \lambda_0(y)$$

where λ is like in (13.1) and (13.2).
The Y-family

$$\prod_{x\in X} \lambda_0(y) := \left(\prod^{1} \lambda_0\right)(y)$$

where λ is like in (13.3).
And we get the set, $Y_1 \times ... \times Y_{i-1} \times Y_{i+1} \times ... \times Y_n$-family respectively,

$$\prod_{y_i\in Y_i} \lambda_0(y_1, ..., y_n)$$

by the definition 9 of dependent functions, proposition 12 respectively, where λ is like in (13.4).

Let $k, l, n, m \in \mathbb{N}^+$, $\Lambda^i \in \{\sum, \prod\}$ for $1 \leq i \leq k$, $Y_1, \dots, Y_n, X_1, \dots, X_m$ be sets and $Z_1, \dots, Z_l$ be sets such that

$$\begin{aligned} &\forall_{i \in \{1,\dots,l\}} \exists_{j \in \{1,\dots,\mathtt{max}\{n,m\}\}} (\ Z_i := Y_j \vee Z_i := X_j\ ) \\ \& \quad &\forall_{1 \leq j \leq n} \exists_{1 \leq i \leq l} (\ Y_j := Z_i\ ) \end{aligned}$$

Then we will write

$$\Lambda_{z_{i_1} \in Z_{i_1}} \dots \Lambda_{z_{i_k} \in Z_{i_k}} \lambda_0(z_{\pi(1)}, \dots, z_{\pi(n)})$$

instead of

$$\Lambda_{z_{i_1} \in Z_{i_1}} \left(\dots \left(\Lambda_{z_{i_k} \in Z_{i_k}} \lambda_0(z_{\pi(1)}, \dots, z_{\pi(n)}) \right) \dots \right)$$

where $i_1, \dots, i_k \in \{1, \dots, l\}$ and $\pi : \{1, \dots, n\} \to \mathbb{N}^+$, $\pi(q) := i$ s.t. $Z_i := Y_q$ for every $q \in \{1, \dots, n\}$. With the definitions we made this is well defined.
So if we combine Π- and Σ-operators this combination is well defined, as long as the underlying object is a set or an arbitrary set-indexed family.

3 The BHK-interpretation of BISH within BST

Now we have all the notions to formulate the BHK-interpretation of BISH within BST, as done in [6]. Important to mention is the "set-dependence" of this formulation i.e. the totality of proofs for any given formula ϕ we consider has to be a set. In particular it has to be clear what a proof for the given formula is. In return any totality of proofs we obtain by the logical connectives is a set as well (definition 14). In the case, that a formula $\phi(x)$ depends on a given variable x of a set X, the totality of proofs of $\phi(x)$ for any given $x \in X$ has to be a set. Therefore we also consider set-indexed families of proof sets (definition 15).
As an example of application and a suggestion how to define a proof set for "atomic" formulas, in section 4 and 5 we present parts of a proof-relevant approach to Bishop Constructive Analysis (BCA).

Definition 14 (set dependent BHK-interpretation - Part I)**.** Let ϕ, φ be formulas in BISH such that the corresponding totalities of all proofs $\mathtt{Prf}(\phi)$ and $\mathtt{Prf}(\varphi)$ are sets. We define:

$$\mathtt{Prf}(\phi \wedge \varphi) := \mathtt{Prf}(\phi) \times \mathtt{Prf}(\varphi),$$
$$\mathtt{Prf}(\phi \vee \varphi) := \mathtt{Prf}(\phi) + \mathtt{Prf}(\varphi),$$
$$\mathtt{Prf}(\phi \Rightarrow \varphi) := \mathbb{F}(\mathtt{Prf}(\phi), \mathtt{Prf}(\varphi))$$

Given the X-family $\mathtt{Prf}^{\phi} := (\mathtt{Prf}_0^{\phi}, \mathtt{Prf}_1^{\phi})$ of a formula $\phi(x)$ on a set X, we can define the proof set of a quantification of $\phi(x)$ over the set X.
Therefore let X be a set, $\phi(x)$ be a formula on X and the family $\mathtt{Prf}^{\phi} := (\mathtt{Prf}_0^{\phi}, \mathtt{Prf}_1^{\phi})$ be given, i.e.

$$\mathtt{Prf}_0^{\phi} : X \rightsquigarrow \mathbb{V}_0, \qquad \mathtt{Prf}_0^{\phi}(x) := \mathtt{Prf}(\phi(x))$$

for every $x \in X$, and

$$\mathtt{Prf}_1^{\phi} : \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}(\ \mathtt{Prf}_0^{\phi}(x), \mathtt{Prf}_0^{\phi}(x')\ ),$$
$$\mathtt{Prf}_{xx'}^{\phi} := \mathtt{Prf}_1^{\phi}(x, x') : \mathtt{Prf}(\phi(x)) \to \mathtt{Prf}(\phi(x'))$$

for $(x, x') \in D(X)$ satisfies the properties D7.1 & D7.2.
We define:

$$\mathtt{Prf}\ (\forall_{x \in X} \phi(x)) := \prod_{x \in X} \mathtt{Prf}_0^{\phi}(x) := \prod_{x \in X} \mathtt{Prf}(\phi(x)),$$
$$\mathtt{Prf}\ (\exists_{x \in X} \phi(x)) := \sum_{x \in X} \mathtt{Prf}_0^{\phi}(x) := \sum_{x \in X} \mathtt{Prf}(\phi(x))$$

Note that the definition of $\exists$ is a generalisation of $\vee$, while the definition of $\forall$ is a generalisation of $\wedge$.

Remark (Treatment of negation)**.** In [6] Petrakis mentions that the treatment of a negated formula $\neg Q$ as the implication $Q \Rightarrow \perp$ is problematic in BST. That is because it has to be $\mathtt{Prf}(\perp) := \emptyset$, if we accept the BHK-interpretation as presented in the introduction. The use of the empty set in BISH is problematic though, thus the status of $\mathbb{F}(\mathtt{Prf}(Q), \emptyset)$ is problematic, too. Anyway Bishop managed to avoid most of negation, so we will avoid its treatment in this thesis.

The constructions below generalise the definitions we made in definition 14 on proof families indexed by the set which the formula is defined on. They make use of the families we defined in section 2.2.4.

Definition 15 (set dependent BHK-interpretation - Part II)**.** Let X, Y be sets, $\phi(x), \varphi(x)$ be formulas on X and the X-families $\mathtt{Prf}^{\phi} := (\mathtt{Prf}_0^{\phi}, \mathtt{Prf}_1^{\phi})$, $\mathtt{Prf}^{\varphi} := (\mathtt{Prf}_0^{\varphi}, \mathtt{Prf}_1^{\varphi})$ be given. To the formulas

$$(\phi \wedge \varphi)(x) :\Leftrightarrow \phi(x) \wedge \varphi(x),$$
$$(\phi \vee \varphi)(x) :\Leftrightarrow \phi(x) \vee \varphi(x),$$
$$(\phi \Rightarrow \varphi)(x) :\Leftrightarrow \phi(x) \Rightarrow \varphi(x)$$

we associate the following X-families:

$$\mathtt{Prf}^{\phi\wedge\varphi} := \mathtt{Prf}^{\phi} \times \mathtt{Prf}^{\varphi},$$
$$\mathtt{Prf}^{\phi\vee\varphi} := \mathtt{Prf}^{\phi} + \mathtt{Prf}^{\varphi},$$
$$\mathtt{Prf}^{\phi\Rightarrow\varphi} := \mathbb{F}(\mathtt{Prf}^{\phi}, \mathtt{Prf}^{\varphi})$$

Let the $X \times Y$-family $\mathtt{Prf}^{\theta}$ of a formula $\theta(x, y)$ on $X \times Y$ be given, i.e.

$$\mathtt{Prf}_0^{\theta} : X \times Y \rightsquigarrow \mathbb{V}_0, \qquad \mathtt{Prf}_0^{\theta}(x, y) := \mathtt{Prf}(\theta(x, y))$$

for every $(x, y) \in X \times Y$, and

$$\mathtt{Prf}_1^{\theta} : \bigcurlywedge_{((x,y),(x',y')) \in D(X\times Y)} \mathbb{F}(\mathtt{Prf}_0^{\theta}(x, y), \mathtt{Prf}_0^{\theta}(x', y'),$$
$$\mathtt{Prf}_{(x,y)(x',y')}^{\theta} := \mathtt{Prf}_1^{\theta}((x, y), (x', y')) : \mathtt{Prf}(\theta(x, y)) \to \mathtt{Prf}(\theta(x', y'))$$

for $((x, y), (x', y')) \in D(X)$ satisfies the properties D7.1 & D7.2.
We define for $x \in X$

$$(\forall_y \theta)(x) :\Leftrightarrow \forall_{y\in Y} \theta(x, y),$$
$$(\exists_y \theta)(x) :\Leftrightarrow \exists_{y\in Y} \theta(x, y)$$

and associate the following X-families in a canonical way to these formulas

$$\mathtt{Prf}^{\forall_y \theta} := \prod^{1} \mathtt{Prf}^{\theta},$$
$$\mathtt{Prf}^{\exists_y \theta} := \sum^{1} \mathtt{Prf}^{\theta}.$$

For the sake of clarity we will write down the last two families more detailed. We have:

$$\mathtt{Prf}^{\forall_y \theta} := \left(\prod^1 \mathtt{Prf}_0^\theta, \prod^1 \mathtt{Prf}_1^\theta \right),$$

$$\left(\prod^1 \mathtt{Prf}_0^\theta \right)(x) := \prod_{y \in Y} \mathtt{Prf}_0^\theta(x, y) := \prod_{y \in Y} \mathtt{Prf}(\theta(x, y)),$$

$$\left(\prod^1 \mathtt{Prf}_1^\theta \right)_{xx'} : \prod_{y \in Y} \mathtt{Prf}(\theta(x, y)) \to \prod_{y \in Y} \mathtt{Prf}(\theta(x', y))$$

and

$$\mathtt{Prf}^{\exists_y \theta} := \left(\sum^1 \mathtt{Prf}_0^\theta, \sum^1 \mathtt{Prf}_1^\theta \right),$$

$$\left(\sum^1 \mathtt{Prf}_0^\theta \right)(x) := \sum_{y \in Y} \mathtt{Prf}_0^\theta(x, y) := \sum_{y \in Y} \mathtt{Prf}(\theta(x, y)),$$

$$\left(\sum^1 \mathtt{Prf}_1^\theta \right)_{xx'} : \sum_{y \in Y} \mathtt{Prf}(\theta(x, y)) \to \sum_{y \in Y} \mathtt{Prf}(\theta(x', y))$$

for $x \in X$ and $(x, x') \in D(X)$.

Remark. For any formula ϕ we call and element $p \in \mathtt{Prf}(\phi)$ of its proof set a *witness* of ϕ. We write $p : \phi$.

This definitions make precise that in BST whenever we use the existential quantifier in a property $P(x)$ on a set X, we demand a witness that this existence holds. This means we have to construct an appropriate object. Therefore we can avoid the use of the Axiom of Choice in our proofs, as we already defined how to choose such an element in the construction of the proof element of a statement.
For example the following statement:
Let $(x_n)_n \in \mathbb{F}(\mathbb{N}^+, \mathbb{R})$. Then:

(x_n) is convergent if and only if it is a Cauchy sequence

expressed as a formula we have:

$$\text{Conv}(x_n) \Leftrightarrow \text{Cauchy}(x_n)$$

The set dependent BHK-interpretation tells us how a proof element $p := (p_1, p_2)$ of this formula looks like. Specifically:

$$p \in \mathbb{F}(\mathtt{Prf}(\text{Conv}(x_n)), \mathtt{Prf}(\text{Cauchy}(x_n))) \times \mathbb{F}(\mathtt{Prf}(\text{Cauchy}(x_n)), \mathtt{Prf}(\text{Conv}(x_n)))$$

where p_1 and p_2 are assigned in the canonical way.
Next we notice that

$$\text{Conv}(x_n) :\Leftrightarrow \exists_{x\in\mathbb{R}}(\ x_n \xrightarrow{n} x\ )$$

and thus by the BHK-interpretation, proofing this statement means to construct an element of

$$\texttt{Prf}(\text{Conv}(x_n)) := \sum_{x\in\mathbb{R}} \texttt{Prf}(x_n \xrightarrow{n} x).$$

All in all, to construct the element p_2 we have to define a specific function that assigns to every member of $\texttt{Prf}(\text{Cauchy}(x_n))$ a specific element of $\texttt{Prf}(\text{Conv}(x_n))$ $:= \sum_{x\in\mathbb{R}} \texttt{Prf}(x_n \xrightarrow{n} x)$. So to construct p_2 we have to give a construction of such an element $x \in \mathbb{R}$, for every element of $\texttt{Prf}(\text{Cauchy}(x_n))$. Therefore we don't need the axiom of choice in this case, because we specify the choice function we use.
As Petrakis mentioned in [5]: "The use of some choice is an indication of missing witnessing-data".

4 A proof-relevant approach: general schemes

In the definition of our BHK-interpretation, we have to define the proof sets of our underlying formulas (atomic formulas). In this section we define such proof sets, for some special kind of formulas. In order to assign the set of all proofs to every $x \in X$, for a given set X, we use a X-family.
Additionally we point out some simplifications for the construction of members of proof sets. The main part is carried out in [6] and [5].

Proposition 16 (General Scheme 1 (GS1))**.** Let X, Y be sets and $Q(x,p)$ be an extensional property on $X \times Y$, i.e. for $x, x' \in X$ and $p, p' \in Y$

$$[x =_X x' \ \ \& \ \ p =_Y p' \ \ \& \ \ Q(x,p)] \Rightarrow Q(x',p').$$

In addition let the property $P(x)$ defined by

$$P(x) :\Leftrightarrow \exists_{p \in Y}(Q(x,p)).$$

Then $P(x)$ is extensional and we can define the X-family

$$\texttt{PrfMemb}^P := (\texttt{PrfMemb}_0^P, \texttt{PrfMemb}_1^P)$$

by

$$\texttt{PrfMemb}_0^P : X \rightsquigarrow \mathbb{V}_0 \qquad\qquad \texttt{PrfMemb}_0^P(x) := \{p \in Y | Q(x,p)\}$$

for every $x \in X$, and for $p \in \texttt{PrfMemb}_0^P(x)$ and $(x,x') \in D(X)$

$$\begin{aligned}
&\texttt{PrfMemb}_1^P : \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}(\ \texttt{PrfMemb}_0^P(x), \texttt{PrfMemb}_0^P(x')\ ) \\
&\texttt{PrfMemb}_{xx'}^P := \texttt{PrfMemb}_1^P(x,x') : \texttt{PrfMemb}_0^P(x) \to \texttt{PrfMemb}_0^P(x'), \\
&\texttt{PrfMemb}_{xx'}^P(p) := p.
\end{aligned}$$

For every $x \in X$, $\texttt{PrfMemb}_0^P(x)$ is by definition the extensional subset of Y generated by $Q(x,.)$. An element p of $\texttt{PrfMemb}_0^P(x)$ is called a *witness* or *modulus* for $P(x)$ and we will write $p : x \in X_P :\Leftrightarrow Q(x,p)$; $p : x \in X_P$ can be read as: "p witnessing that $x \in X_P$".

Proof. First we proof that $P(x)$ is an extensional property. If $x =_X x'$ & $p =_Y p'$ such that $Q(x,p)$ holds, we get $Q(x',p')$ by the extensionality of Q and hence $P(x')$.
Next we proof that $\texttt{PrfMemb}^P := (\texttt{PrfMemb}_0^P, \texttt{PrfMemb}_1^P)$ is a X-family. To do so, we show that $\texttt{PrfMemb}_0^P$ and $\texttt{PrfMemb}_1^P$ are well-defined and the properties D7.1 and D7.2 are satisfied. For every $x \in X$, $\texttt{PrfMemb}_0^P(x)$ is an extensional subset of Y. So $\texttt{PrfMemb}_0^P$ is well-defined.
If $(x,x') \in D(X)$ and $p \in \texttt{PrfMemb}_0^P(x)$, then by the extensionality of Q we get $Q(x',p)$ and thus $Q(x', \texttt{PrfMemb}_{xx'}^P(p))$. So $p := \texttt{PrfMemb}_{xx'}^P(p) \in \texttt{PrfMemb}_0^P(x')$ and $\texttt{PrfMemb}_{xx'}^P$

is well-defined. $\texttt{PrfMemb}^P_{xx'}$ is also a function, since for every $p' \in \texttt{PrfMemb}^P_0(x)$ such that $p =_Y p'$, clearly

$$\texttt{PrfMemb}^P_{xx'}(p) := p =_Y p' := \texttt{PrfMemb}^P_{xx'}(p')$$

holds. As $(x, x') \in D(X)$ was chosen freely, $\texttt{PrfMemb}^P_1$ is well-defined.
The properties D7.1 and D7.2 remain to show.
<u>D7.1 & D7.2</u>: For $(x, x'), (x', x'') \in D(X)$ we have:

$$\texttt{PrfMemb}^P_{xx}(p) := p := id_{\texttt{PrfMemb}^P_0(x)}(p)$$
$$\texttt{PrfMemb}^P_{x'x''} \circ \texttt{PrfMemb}^P_{xx'}(p) := p := \texttt{PrfMemb}^P_{xx''}(p)$$

for all $p \in \texttt{PrfMemb}^P_0(x)$. Hence the properties are satisfied. □

Example 5. Let X, Y be sets and $R(x_1, x_2)$ a relation on X such that

$$R(x_1, x_2) :\Leftrightarrow \exists_{p \in Y}(\ Q(x_1, x_2, p)\ )$$

where $Q(x_1, x_2, p)$ is an extensional property on $X \times X \times Y$.
Then $R(x_1, x_2)$ is extensional and GS1 defines the $X \times X$-family

$$\texttt{PrfMemb}^R := (\texttt{PrfMemb}^R_0, \texttt{PrfMemb}^R_1).$$

There is a trivial generalisation, which we will use later in the examples. For simplification we present it in the following.

Proposition 17 (General Scheme 2 (GS2))**.** Let X, Y, Z be sets and $Q(x, p, z)$ be an extensional property on $X \times Y \times Z$, i.e. for $x, x' \in X$, $p, p' \in Y$ and $z, z' \in Z$

$$[x =_X x' \ \ \&\ \ p =_Y p' \ \ \&\ \ z =_Z z' \ \ \&\ \ Q(x, p, z)] \Rightarrow Q(x', p', z').$$

In addition let the property $P(x)$ defined by

$$P(x) :\Leftrightarrow \exists_{p \in Y} \exists_{z \in Z}(Q(x, p, z)).$$

Then $P(x)$ is extensional and we can define the X-family

$$\texttt{PrfMemb}^P := (\texttt{PrfMemb}^P_0, \texttt{PrfMemb}^P_1)$$

by

$$\texttt{PrfMemb}^P_0 : X \rightsquigarrow \mathbb{V}_0 \qquad \texttt{PrfMemb}^P_0(x) := \{(p, z) \in Y \times Z | Q(x, p, z)\}$$

for every $x \in X$, and for $(p, z) \in \texttt{PrfMemb}^P_0(x)$ and $(x, x') \in D(X)$

$$\texttt{PrfMemb}^P_1 : \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}(\ \texttt{PrfMemb}^P_0(x), \texttt{PrfMemb}^P_0(x')\ )$$
$$\texttt{PrfMemb}^P_{xx'} := \texttt{PrfMemb}^P_1(x, x') : \texttt{PrfMemb}^P_0(x) \to \texttt{PrfMemb}^P_0(x'),$$
$$\texttt{PrfMemb}^P_{xx'}(p, z) := (p, z).$$

For every $x \in X$, $\texttt{PrfMemb}^P_0(x)$ is by definition the extensional subset of $Y \times Z$ generated by $Q(x, ., .)$. An element (p, z) of $\texttt{PrfMemb}^P_0(x)$ is called a *witness* or *modulus* for $P(x)$ and we will write $(p, z) : x \in X_P :\Leftrightarrow Q(x, p, z)$; $(p, z) : x \in X_P$ can be read as: "(p, z) witnessing that $x \in X_P$".

Proof. Since GS2 is the special case of GS1, where we substitute the set Y in GS1 by the set $Y \times Z$, the validity of GS2 is immediate. $\square$

Remark. Of course this scheme can be generalised on a property $P(x)$ on X of the form

$$P(x) :\Leftrightarrow \exists_{p_1 \in X_1} \dots \exists_{p_n \in X_n} (\ Q(x, p_1, ..., p_n)\ )$$

for an extensional property $Q(x, p_1, ..., p_n)$ on $X \times X_1 \times \cdots \times X_n$.

Definition 18. So we can define proof sets for properties given by an existential formula independently from the exact BHK-interpretation we defined in the previous section. We will use this idea, to define the proof sets of atomic formulas i.e.

$$\begin{aligned} \mathtt{Prf}(x \in X_P) &:= \mathtt{PrfMemb}_0^P(x), \\ \mathtt{Prf}(R(x_1, x_2) &:= \mathtt{PrfMemb}_0^R(x_1, x_2) \end{aligned}$$

where P is a property on a set X given by an existential formula like in proposition 16, and R is a relation on a set X given by an existential formula like in example 5.
Therefore the fundamental relation of our framework is not membership, but *membership-with-evidence*.

The following proposition points out how to define an assignment routine from one (extensional sub-)set to an other extensional subset. This may be very obvious, but it underlines the idea behind GS1 and GS2.

Proposition 19 (General Scheme 3 (GS3))**.** Let X, X', Y, Y' be sets and the properties $P(x), P'(x')$, defined by

$$\begin{aligned} P(x) &:\Leftrightarrow \exists_{p \in Y} (\ Q(x, p)\ ) \\ P'(x') &:\Leftrightarrow \exists_{p' \in Y'} (\ Q'(x', p')\ ) \end{aligned}$$

where $Q(x, p)$ is extensional property on $X \times Y$ and $Q'(x', p')$ is extensional property on $X' \times Y'$.
To define an assignment routine $f_{PP'} : X_P \rightsquigarrow X'_{P'}$ it is sufficient to define:

1. an assignment routine
$$f : X_P \rightsquigarrow X'$$

2. a dependent assignment routine
$$\Phi_f : \bigcurlywedge_{x \in X} \bigcurlywedge_{p \in \mathtt{PrfMemb}_0^P(x)} \mathtt{PrfMemb}_0^{P'}(f(x))$$

Then $f_{PP'} : X_P \rightsquigarrow X'_{P'}$, $f_{PP'}(x) := f(x) \in X'_{P'}$ is well-defined. If f is a function, $f_{PP'}$ is a function as well.

Note that for $P(x) :\Leftrightarrow \exists_{p\in\mathbb{N}}(p =_{\mathbb{N}} 1)$ we have $X_P := X$. This holds because

$$\mathcal{M}_{X_P}(x) :\Leftrightarrow \mathcal{M}_X(x) \;\;\&\;\; P(x) \overset{\text{P(x) is tautology}}{:\Leftrightarrow} \mathcal{M}_X(x)$$

Then in 2. the d.a.r. reduces to $\Phi_f : \curlywedge_{x\in X} \mathtt{PrfMemb}_0^{P'}(f(x))$, since $\mathtt{PrfMemb}_0^{P}(x)$ doesn't depend on x. In this case $f_{PP'} : X \rightsquigarrow X_{P'}$
Of course every assignment routine (function) on X, can be easily restricted to an assignment routine (function) on $X_{P'}$. Therefore in 1. it is also possible to define $f : X \rightsquigarrow X'$. With the implicitly given embedding $i_{X_P} : X_P \hookrightarrow X$, we get $f \circ i_{X_P} : X_P \rightsquigarrow X', f \circ i_{X_P}(x) := f(x)$, what satisfies our needs.

Proof. Let $x \in X_P$ be arbitrary and $p \in \mathtt{PrfMemb}_0^{P}(x)$ its witness. Since f is a an assignment routine, $f(x) \in X'$ and $\Phi_f(x)(p) \in \mathtt{PrfMemb}_0^{P'}(f(x))$ is a witness for $P'(f(x))$. Consequently $f(x) \in X_{P'}$ and $f_{PP'}$ is well-defined.
If f is a function $f_{PP'}$ clearly is a function. □

Corollary. Let X, Y be sets and $P(x)$ defined as in GS1, then:

$$\omega \in \prod_{x\in X} \mathtt{PrfMemb}_0^{P}(x) :\Leftrightarrow \begin{cases} \omega : \curlywedge_{x\in X} \mathtt{PrfMemb}_0^{P}(x) \\ \& \;\; \forall_{(x,x')\in D(X)}(\; \omega_x =_{\mathtt{PrfMemb}_0^{P}(x)} \omega_{x'} \;) \end{cases}$$

Proof. Let X, Y be sets and the property $P(x)$, defined by

$$P(x) :\Leftrightarrow \exists_{p\in Y}(\; Q(x,p) \;)$$

where $Q(x,p)$ is extensional property on $X \times Y$.
Then:

$$\begin{aligned} \omega \in \prod_{x\in X} \mathtt{PrfMemb}_0^{P}(x) :\Leftrightarrow & \begin{cases} \omega \in \mathbb{A}(X, \mathtt{PrfMemb}_0^{P}) \\ \& \;\; \forall_{(x,x'\in D(X)}(\; \omega_x =_{\mathtt{PrfMemb}_0^{P}(x)} \mathtt{PrfMemb}_{x'x}^{P}(\omega_{x'}) \;) \end{cases} \\ :\Leftrightarrow & \begin{cases} \omega : \curlywedge_{x\in X} \mathtt{PrfMemb}_0^{P}(x) \\ \& \;\; \forall_{(x,x'\in D(X)}(\; \omega_x =_{\mathtt{PrfMemb}_0^{P}(x)} \omega_{x'} \;) \end{cases} \end{aligned}$$

□

Proposition 20 (General Scheme 4 (GS4))**.** Let X, X', Y, Y' be sets and the properties $P(x), P'(x')$, defined by

$$P(x) :\Leftrightarrow \exists_{p\in Y}(\; Q(x,p) \;)$$
$$P'(x') :\Leftrightarrow \exists_{p'\in Y'}(\; Q'(x',p') \;)$$

where $Q(x,p)$ is extensional property on $X \times Y$ and $Q'(x',p')$ is extensional property on $X' \times Y'$.
If

$$\omega : \curlywedge_{x\in X} \mathbb{F}(\mathtt{PrfMemb}_0^{P}(x), \mathtt{PrfMemb}_0^{P'}(x)),$$
$$\omega(x)(p) := T(p)$$

for every $x \in X$ and every $p \in \texttt{PrfMemb}_0^P(x)$ and $T \in \mathbb{F}(Y, Y')$ is well-defined, then ω is in the Π-set of $\texttt{PrfMemb}^{P \Rightarrow P'} := (\texttt{PrfMemb}_0^{P \Rightarrow P'}, \texttt{PrfMemb}_1^{P \Rightarrow P'}) := \mathbb{F}(\texttt{PrfMemb}^P, \texttt{PrfMemb}^{P'})$.

Proof. By proposition 10, we have

$$\texttt{PrfMemb}_0^{P \Rightarrow P'} : X \rightsquigarrow \mathbb{V}_0, \qquad \texttt{PrfMemb}_0^{P \Rightarrow P'}(x) := \mathbb{F}(\texttt{PrfMemb}_0^P(x), \texttt{PrfMemb}_0^{P'}(x))$$

and

$$\begin{aligned}
&\texttt{PrfMemb}_1^{P \Rightarrow P'} : \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}(\texttt{PrfMemb}_0^{P \Rightarrow P'}(x), \texttt{PrfMemb}_0^{P \Rightarrow P'}(x')), \\
&\texttt{PrfMemb}_{xx'}^{P \Rightarrow P'} := \texttt{PrfMemb}_1^{P \Rightarrow P'}(x, x') : \texttt{PrfMemb}_0^{P \Rightarrow P'}(x) \to \texttt{PrfMemb}_0^{P \Rightarrow P'}(x'), \\
&\texttt{PrfMemb}_{xx'}^{P \Rightarrow P'}(p) := \texttt{PrfMemb}_{xx'}^{P'} \circ p \circ \texttt{PrfMemb}_{x'x}^{P} \overset{(*)}{:=} p
\end{aligned}$$

To show (*), let $(x, x') \in D(X)$, $p \in \texttt{PrfMemb}_0^{P \Rightarrow P'}(x)$. Then we have:

$$\begin{aligned}
\forall_{z \in \texttt{PrfMemb}_0^P(x')} \big(\, \texttt{PrfMemb}_{xx'}^{P \Rightarrow P'}(p)(z) &:= \texttt{PrfMemb}_{xx'}^{P'} \circ p \circ \texttt{PrfMemb}_{x'x}^{P}(z) \\
&:= \texttt{PrfMemb}_{xx'}^{P'} \circ p(\texttt{PrfMemb}_{x'x}^{P}(z)) \\
&:= \texttt{PrfMemb}_{xx'}^{P'}(p(z)) := p(z) \, \big)
\end{aligned}$$

and (*) holds.
Next we will prove that <u>ω is in the Π-set of $\texttt{PrfMemb}^{P \Rightarrow P'}$</u>.
By assumption ω is already a d.a.r.. Therefore, we only have to show:

$$\forall_{(x,x') \in D(X)} \big(\, \omega_x = \texttt{PrfMemb}_{x'x}^{P \Rightarrow P'}(\omega_{x'}) := \omega_{x'} \, \big) \tag{**}$$

Thus for every $(x, x') \in D(X)$ we have to prove the equality of the two functions ω_x and $\omega_{x'}$.
Let $(x, x') \in D(X)$ and $p \in \texttt{PrfMemb}_0^P(x)$ be arbitrary. We have:

$$\omega_x(p) := T(p) := \omega_{x'}(p)$$

Since p was freely chosen, $\omega_x := \omega_{x'}$. Hence (**) is satisfied and

$$\omega \in \prod_{x \in X} \mathbb{F}(\texttt{PrfMemb}_0^P(x), \texttt{PrfMemb}_0^{P'}(x))$$

. $\square$

5 A proof-relevant approach: examples

In this chapter we give some examples for the general schemes and also try to provide an idea how our proof-relevant approach to BCA looks like. The numbers at definitions and propositions refer to the ones in BB85. Remarks and notations are additional and manly for simplicity. We use paragraphs of the form **P**... ■ in definitions, for simple proofs. Being familiar with BB85 is only needed if one wants to see how the presentation of proofs differs from Bishop's. Otherwise this chapter is for illustration only.

Definition 2.1. (The real numbers) A *real number* is a sequence of rationals $x := (x_n)_{n\in\mathbb{N}^+}$ which is associated to a function $p \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$ such that p witnessing x to be in $\mathbb{R}$. So the membership condition for $\mathbb{R}$ is given by:

$$x \in \mathbb{R} :\Leftrightarrow x \in \mathbb{F}(\mathbb{N}^+, \mathbb{Q}) \;\&\; \exists_{p\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}(p : x \in \mathbb{R})$$
$$p : x \in \mathbb{R} :\Leftrightarrow \forall_{k\in\mathbb{N}^+}\forall_{m,n\geq p(k)}(|x_m - x_n| \leq \frac{1}{k}) \tag{2.1.1}$$

P $p : x \in \mathbb{R}$ is extensional: Let $x, x' \in \mathbb{F}(\mathbb{N}^+, \mathbb{Q})$ and $p, p' \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$ such that $p : x \in \mathbb{R} \;\&\; p = p' \;\&\; x = x'$. Because the equality on the set of functions from one set to another is pointwise, we have:

$$\forall_{k\in\mathbb{N}^+}\forall_{m,n\geq p(k)=_{\mathbb{N}^+}p'(k)}(\frac{1}{k} \geq |x_m - x_n| =_{\mathbb{Q}} |x'_m - x'_n|$$

Thus $p' : x' \in \mathbb{R}$. ■

We define $X := \mathbb{F}(\mathbb{N}^+, \mathbb{Q}), Y := \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$ and $Q(x, p) :\Leftrightarrow (p : x \in \mathbb{R})$, then from GS1 we get the corresponding $\mathbb{F}(\mathbb{N}^+, \mathbb{Q})$-family of proofs:

$$\texttt{PrfMemb}^{\mathbb{R}} := (\texttt{PrfMemb}^{\mathbb{R}}_0, \texttt{PrfMemb}^{\mathbb{R}}_1),$$
$$\texttt{PrfMemb}^{\mathbb{R}}_0(x) := \{p \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \mid p : x \in \mathbb{R}\}$$

for every $x \in \mathbb{F}(\mathbb{N}^+, \mathbb{Q})$.
Note that we use the equality of the set $\mathbb{F}(\mathbb{N}^+, \mathbb{Q})$.
If we construct $x \in \mathbb{R}$ we have to define an appropriate $p \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$. Therefore every constructed $x \in \mathbb{R}$ can be seen as a pair (x, p) s.t. $p : x \in \mathbb{R}$. We will refer to this p for any given $x \in \mathbb{R}$ by p_x.

Additionally we define a new equality on $\mathbb{R}$. Otherwise $\mathbb{R}$ would inherit the equality of its superset $\mathbb{F}(\mathbb{N}^+, \mathbb{Q})$, which doesn't satisfy our needs. For two reals $x, y \in \mathbb{R}$ the equality is defined by:

$$x =_{\mathbb{R}} y :\Leftrightarrow \exists_{\omega\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}(\omega : x =_{\mathbb{R}} y)$$
$$\omega : x =_{\mathbb{R}} y :\Leftrightarrow \omega \geq id_{\mathbb{N}^+} \;\&\; \forall_{j\in\mathbb{N}^+}\forall_{n\geq\omega(j)}(|x_n - y_n| \leq \frac{1}{j})$$

By the same argument as above, $\omega : x =_{\mathbb{R}} y$ is an extensional property and therefore we get the $\mathbb{R} \times \mathbb{R}$-family (here $\mathbb{R}$ refers to $(\mathbb{R}, =_{\mathbb{F}(\mathbb{N}^+,\mathbb{Q})})$):

$$\begin{aligned}&\texttt{PrfMemb}^{=_{\mathbb{R}}} := (\texttt{PrfMemb}_0^{=_{\mathbb{R}}}, \texttt{PrfMemb}_1^{=_{\mathbb{R}}}),\\ &\texttt{PrfMemb}_0^{=_{\mathbb{R}}}(x,y) := \texttt{Prf}(x =_{\mathbb{R}} y) := \texttt{PrfEq}^{\mathbb{R}}(x,y) := \{\omega \in \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+) \mid \omega : x =_{\mathbb{R}} y\}\end{aligned}$$

for every $(x,y) \in \mathbb{R} \times \mathbb{R}$.

Proposition 2.2. The equality of real numbers is an equivalence relation
Proof. For the reflexivity $id : \mathbb{N}^+ \to \mathbb{N}^+, id(k) := k$ satisfies our needs, for symmetry of $x =_{\mathbb{R}} y$ the assigned witness is clearly also the witness of $y =_{\mathbb{R}} x$. To prove the transitivity, let $x, y, z \in \mathbb{R}$ such that $x =_{\mathbb{R}} y$ and $y =_{\mathbb{R}} z$ and ω_{xy}, ω_{yz} in $\texttt{PrfEq}^{\mathbb{R}}(x,y)$ and $\texttt{PrfEq}^{\mathbb{R}}(y,z)$ respectively. With $\omega_{xz} : \mathbb{N}^+ \to \mathbb{N}^+,\ \omega_{xz}(n) := \texttt{max}\{\omega_{xy}(2n), \omega_{yz}(2n)\}$ we have

$$\forall_{j\in\mathbb{N}^+}\forall_{n\geq\omega_{xz}(j)}(|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n| \leq \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j})$$

and obviously $\omega_{xz} \geq id_{\mathbb{N}^+}$. Therefore $x =_{\mathbb{R}} z$ and our new equality is transitive. Thus the equality is an equivalence relation.

Remark. From now on we denote by $\mathbb{R}$ the set $(\mathbb{R}, =_{\mathbb{R}})$ and by $\mathbb{R}_{\mathbb{F}}$ the set $(\mathbb{R}, =_{\mathbb{F}(\mathbb{N}^+,\mathbb{Q})})$.

Definition 2.4. Let $x := (x_n)_n, y := (y_n)_n \in \mathbb{R}$ and $\alpha \in \mathbb{Q}$. We define:
(i) $x + y := (x_n + y_n)_{n\in\mathbb{N}^+}$
(ii) $\alpha^* := (\alpha, \alpha, \alpha, ...)_{n\in\mathbb{N}^+}$
(iii) $-x := (-x_n)_{n\in\mathbb{N}^+}$
(iv) $\texttt{max}\{x, y\} := (\texttt{max}\{x_n, y_n\})_{n\in\mathbb{N}^+}$
(v) $|x| := \texttt{max}\{-x, x\}$

Proposition 2.5. The sequences we defined in Definition 2.4 are real numbers.
Proof. To proof the proposed, we have to find a witnessing function for each sequence. Let $\chi, \varphi \in \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$ s.t. $\chi : x \in \mathbb{R}$ and $\varphi : y \in \mathbb{R}$
(i). We define $\omega : \mathbb{N}^+ \to \mathbb{N}^+, \omega(k) := \texttt{max}\{\chi(2k), \varphi(2k)\}$. Then $\omega : x + y \in \mathbb{R}$, since we have:

$$\begin{aligned}\forall_{k\in\mathbb{N}^+}\forall_{m,n\geq\omega(k)}(\ |(x+y)_m - (x+y)_n| &= |x_m + y_m - x_n - y_n|\\ &\leq |x_m - x_n| + |y_m - y_n| \leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}\ )\end{aligned}$$

(ii). We define $\omega : \mathbb{N}^+ \to \mathbb{N}^+, \omega(k) := k$, which trivially satisfies (2.1.1).
(iii). We define $\omega : \mathbb{N}^+ \to \mathbb{N}^+, \omega(k) := \chi(k)$. Thus we have

$$\forall_{k\in\mathbb{N}^+}\forall_{n,m\geq\omega(k)}(|(-x)_n - (-x)_m| = |x_m - x_n| \overset{\omega(k)=\chi(k)}{\underset{\&\ \chi:x\in\mathbb{R}}{\leq}} \frac{1}{k})$$

and therefore $\omega : -x \in \mathbb{R}$
(iv). We define $\omega : \mathbb{N}^+ \to \mathbb{N}^+, \omega(k) := \mathtt{max}\{\chi(2k), \varphi(2k)\}$. Then $\omega : \mathtt{max}\{x, y\} \in \mathbb{R}$, since we have:

$$\forall_{k\in\mathbb{N}^+}\forall_{m,n\geq\omega(k)}(\ |\mathtt{max}\{x,y\}_n - \mathtt{max}\{x,y\}_m| \\ \leq |x_n - x_m| + |y_n - y_m| \leq \frac{1}{2k} + \frac{1}{2k} \leq \frac{1}{k}\ )$$

(v). Follows immediately from (iii) and (iv).

The defined assignment routines in (i)-(v) all respect equality, we omit the simple constructions of the witnessing functions to the reader, and therefore are functions by GS3.

Notation. For a totality X we write X^2 for $X \times X$. For $z \in \mathbb{R}^2_{\mathbb{F}}$ we write:

$$z_1 := \mathtt{pr}_1(z)\ \ \&\ \ z_2 := \mathtt{pr}_2(z)$$

Analogue for $\mathbb{R}$.

Definition 2.10. Let $x \in \mathbb{R}$. We define:

$$x \in \mathbb{R}^{\geq} :\Leftrightarrow \exists_{\omega\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}(\ \omega : x \in \mathbb{R}^{\geq}\ ),$$
$$\omega : x \in \mathbb{R}^{\geq} :\Leftrightarrow \forall_{k\in\mathbb{N}^+}\forall_{m\geq\omega(k)}(\ x_{p_x(m)} \geq -\frac{1}{k}\ ) \\ \&\quad \omega \geq id_{\mathbb{N}^+}$$

P $\omega : x \in \mathbb{R}^{\geq}$ is extensional: Let $(x, y) \in D(\mathbb{R})$ and $\omega_{xy} : x =_{\mathbb{R}} y$. Further let $(\omega, \omega') \in D(\mathbb{F}(\mathbb{N}^+, \mathbb{N}^+))$. We define:

$$\theta : \mathbb{N}^+ \to \mathbb{N}^+ \qquad\qquad \theta(t) := \mathtt{max}\{p_x(\omega(t)), p_y(t), id_{\mathbb{N}^+}(t), \omega_{xy}(t)\}$$

for every $t \in \mathbb{N}^+$. If $\omega : x \in \mathbb{R}^{\geq}$ we have, for any $t, k \in \mathbb{N}^+$ and $m \in \mathbb{N}^+$, s.t. $m \geq \omega(k)$

$$\begin{aligned} y_{p_y(m)} &\geq x_{\theta(t)} - |y_{p_y(m)} - x_{\theta(t)}| \\ &\geq x_{\theta(t)} - (|y_{p_y(m)} - y_{\theta(t)}| + |y_{\theta(t)} - x_{\theta(t)}|) \\ &\geq x_{\theta(t)} - |y_{p_y(m)} - y_{\theta(t)}| - |y_{\theta(t)} - x_{\theta(t)}| \\ &\geq -\frac{1}{t} - \frac{1}{\mathtt{min}\{m,t\}} - \frac{1}{t} \end{aligned}$$

Since $t \in \mathbb{N}^+$ is arbitrary, we have

$$\begin{aligned} y_{p_y(m)} &\geq -\frac{1}{m} \\ &\geq -\frac{1}{\omega(k)} \\ &\geq -\frac{1}{k} \end{aligned}$$

Hence $\omega' : y \in \mathbb{R}^{\geq}$, since the equality on $\mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$ is pointwise. ■

For $x, y \in \mathbb{R}$, we define:

$$x \leq_{\mathbb{R}} y :\Leftrightarrow y - x \in \mathbb{R}^{\geq}$$

Let $X := \mathbb{R}^2, Y := \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$ and $Q((x, y), \omega) :\Leftrightarrow (\omega : x \leq_{\mathbb{R}} y)$. GS1 defines the $\mathbb{R}^2$-family

$$\begin{aligned}&\texttt{PrfMemb}^{\leq_{\mathbb{R}}} := (\texttt{PrfMemb}_0^{\leq_{\mathbb{R}}}, \texttt{PrfMemb}_1^{\leq_{\mathbb{R}}}),\\ &\texttt{PrfMemb}_0^{\leq_{\mathbb{R}}}(x, y) := \{\omega \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \mid \omega : x \leq_{\mathbb{R}} y\}\end{aligned}$$

Definition 3.1. (Convergent sequences) Let $x_0 \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}^+} \in \mathbb{F}(\mathbb{N}^+, \mathbb{R})$. We define:

$$\mathrm{Conv}_{\mathrm{x}_0}((x_n)_{n \in \mathbb{N}^+}) :\Leftrightarrow \exists_{C \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)}(C : x_n \xrightarrow{n} x_0)$$
$$C : x_n \xrightarrow{n} x_0 :\Leftrightarrow \forall_{k \in \mathbb{N}^+} \forall_{m, n \geq C(k)}(|x_n - x_0| \leq \frac{1}{k})$$

P $C : x_n \xrightarrow{n} x_0$ is extensional: Let $((x_n), (y_n)) \in D(\mathbb{F}(\mathbb{N}^+, \mathbb{R}))$ and $(C, C') \in D(F(\mathbb{N}^+, \mathbb{N}^+))$. If $C : x_n \xrightarrow{n} x_0$, we have:

$$\forall_{k \in \mathbb{N}^+} \forall_{m, n \geq C(k) =_{\mathbb{N}^+} C'(k)}(|y_n - x_0| =_{\mathbb{R}} |x_n - x_0| \leq \frac{1}{k})$$

Thus we have: $C' : y_n \xrightarrow{n} x_0$. ■

Let $X := \mathbb{F}(\mathbb{N}^+, \mathbb{R}), Y := \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$ and $Q_{x_0}((x_n), C) :\Leftrightarrow C : x_n \xrightarrow{n} x_0$. By GS1 we get the $\mathbb{F}(\mathbb{N}^+, \mathbb{R})$-family

$$\begin{aligned}&\texttt{PrfMemb}^{\mathrm{Conv}_{x_0}} := (\texttt{PrfMemb}_0^{\mathrm{Conv}_{x_0}}, \texttt{PrfMemb}_1^{\mathrm{Conv}_{x_0}}),\\ &\texttt{PrfMemb}_0^{\mathrm{Conv}_{x_0}}((x_n)) := \{C \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \mid C : x_n \xrightarrow{n} x_0\}.\end{aligned}$$

Further we define for a given sequence $(x_n)_n \in \mathbb{F}(\mathbb{N}^+, \mathbb{R})$:

$$\mathrm{Conv}((x_n)_n) :\Leftrightarrow \exists_{x \in \mathbb{R}} \exists_{p_x \in \texttt{PrfMemb}_0^{\mathbb{R}}(x)}(\ \mathrm{Conv}_x((x_n)_n)\ )$$

Consequently the associated proof set is given by the formal BHK-interpretation:

$$\texttt{Prf}(\mathrm{Conv}(x_n)) := \sum_{x \in \mathbb{R}} \sum_{p_x \in \texttt{PrfMemb}_0^{\mathbb{R}}(x)} \texttt{PrfMemb}_0^{\mathrm{Conv}_x}(x_n)$$

From now on we write $\leq$ for $\leq_{\mathbb{R}}$. Definition 2.4(ii) gives rise to an embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$ and therefore every element of $\mathbb{Q}$ can be identified with an element in $\mathbb{R}$.

Definition 3.2. (Cauchy sequences) Let $(x_n)_{n \in \mathbb{N}^+} \in \mathbb{F}(\mathbb{N}^+, \mathbb{R})$. We define:

$$\mathrm{Cauchy}((x_n)_{n \in \mathbb{N}^+}) :\Leftrightarrow \exists_{C \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)}(C : \mathrm{Cauchy}((x_n)_{n \in \mathbb{N}^+}))$$
$$C : \mathrm{Cauchy}((x_n)_{n \in \mathbb{N}^+}) :\Leftrightarrow \forall_{k \in \mathbb{N}^+} \forall_{m, n \geq C(k)}(|x_n - x_m| \leq \frac{1}{k})$$

P $C : \text{Cauchy}((x_n)_{n\in\mathbb{N}^+})$ is extensional: Let $((x_n),(y_n)) \in D(\mathbb{F}(\mathbb{N}^+,\mathbb{R}))$ and $(C,C') \in D(F(\mathbb{N}^+,\mathbb{N}^+))$. If $C : \text{Cauchy}((x_n)_{n\in\mathbb{N}^+})$, we have:

$$\forall_{k\in\mathbb{N}^+}\forall_{m,n\geq C(k)=_{\mathbb{N}^+}C'(k)}(|y_n - y_m| =_{\mathbb{R}} |x_n - x_m| \leq \frac{1}{k})$$

Thus we have: $C' : \text{Cauchy}((y_n)_{n\in\mathbb{N}^+})$. ■

Let $X := \mathbb{F}(\mathbb{N}^+,\mathbb{R}), Y := \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)$ and $Q((x_n),C) :\Leftrightarrow C : \text{Cauchy}((x_n)_{n\in\mathbb{N}^+})$. By GS1 we get the $\mathbb{F}(\mathbb{N}^+,\mathbb{R})$-family

$$\begin{aligned}&\texttt{PrfMemb}^{\text{Cauchy}} := (\texttt{PrfMemb}_0^{\text{Cauchy}}, \texttt{PrfMemb}_1^{\text{Cauchy}}),\\ &\texttt{PrfMemb}_0^{\text{Cauchy}}((x_n)) := \{C \in \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+) \mid C : \text{Cauchy}((x_n)_{n\in\mathbb{N}^+})\}.\end{aligned}$$

Theorem 3.3. A sequence $(x_n)_n \in \mathbb{F}(\mathbb{N}^+,\mathbb{R})$ converges if and only if it is a Cauchy sequence.
Proof. Let $(x_n)_n \in \mathbb{F}(\mathbb{N}^+,\mathbb{R})$. For every $n \in \mathbb{N}^+$, $p_n : x_n \in \mathbb{R}$ are the associated witnesses to the x_n. To proof the statement we have to construct an element

$$\zeta_1 \in \mathbb{F}(\texttt{Prf}(\text{Conv}(x_n)), \texttt{PrfMemb}_0^{\text{Cauchy}}(x_n)) \qquad \text{(part I)}$$

and an element

$$\zeta_2 \in \mathbb{F}(\texttt{PrfMemb}_0^{\text{Cauchy}}(x_n), \texttt{Prf}(\text{Conv}(x_n))) \qquad \text{(part II)}$$

<u>Part I</u>: We define the function

$$\begin{aligned}&\zeta_1 : \texttt{Prf}(\text{Conv}(x_n)) \to \texttt{PrfMemb}_0^{\text{Cauchy}}(x_n),\\ &\zeta_1(x,p_x,C) : \mathbb{N}^+ \to \mathbb{N}^+,\\ &\zeta_1(x,p_x,C)(k) := p_x(2k)\end{aligned}$$

for every $k \in \mathbb{N}^+$ and $(x,p_x,C) \in \sum_{x\in\mathbb{R}}\sum_{p_x\in\texttt{PrfMemb}_0^{\mathbb{R}}(x)} \texttt{PrfMemb}_0^{\text{Conv}_x}(x_n)$. Since p_x is a function, $\zeta_1(x,p_x,C)$ is a function, too. ζ_1 is well defined, because we have:

$$\begin{aligned}\forall_{k\in\mathbb{N}^+}\forall_{m,n\geq\zeta_1(x,p_x,C)(k)=_{\mathbb{N}^+}p_x(2k)}(\ &|x_n - x_m|\\ &= |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x|\\ &\leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}\ )\end{aligned}$$

and hence $\zeta_1(x,p_x,C) \in \texttt{PrfMemb}_0^{\text{Cauchy}}(x_n)$. Therefore ζ_1 is a function, because for every

$$\big((x,p_x,C),(y,p_y,C')\big) \in D\left(\sum_{x\in\mathbb{R}}\sum_{p_x\in\texttt{PrfMemb}_0^{\mathbb{R}}(x)} \texttt{PrfMemb}_0^{\text{Conv}_x}(x_n)\right)$$

we have

$$\forall_{k\in\mathbb{N}^+}(\ \zeta_1(x,p_x,C)(k) := p_x(2k) = p_y(2k) := \zeta_1(y,p_y,C')(k)\ )$$

Thus $\zeta_1(x,p_x,C) =_{\texttt{PrfMemb}_0^{\text{Cauchy}}(x_n)} \zeta_1(y,p_y,C')$.
Part II: We define the function

$$\zeta_2 : \texttt{PrfMemb}_0^{\text{Cauchy}}(x_n) \to \texttt{Prf}(\text{Conv}(x_n)),$$
$$\zeta_2(q) := (y, id, C)$$

where $y \in \mathbb{R}$, $id \in \texttt{PrfMemb}_0^{\mathbb{R}}(x)$ and $C \in \texttt{PrfMemb}_0^{\text{Conv}_x}(x_n)$ are defined as follows. Let $q \in \texttt{PrfMemb}_0^{\text{Cauchy}}(x_n)$, we define

$$\mu : \mathbb{N}^+ \to \mathbb{N}^+, \qquad \mu(k) := q(3k)$$

and

$$y : \mathbb{N}^+ \to \mathbb{Q}, \qquad y_k := \underbrace{[\underbrace{x_{\mu(k)}}_{\in\mathbb{R}}]_{p_{\mu(k)}(3k)}}_{p_{\mu(k)}(3k)\text{-th rational approximation of } x_{\mu(k)}}$$

Then with

$$id : \mathbb{N}^+ \to \mathbb{N}^+, \qquad id(k) := k$$

,we have

$$\begin{aligned}
\forall_{k\in\mathbb{N}^+}\forall_{m,n\geq id(k):=k}(\ |y_m - y_n| &= |(x_{\mu(m)})_{p_{\mu(m)}(3m)} - (x_{\mu(n)})_{p_{\mu(n)}(3n)}| \\
&= |(x_{\mu(m)})_{p_{\mu(m)}(3m)} - (x_{\mu(n)})_{p_{\mu(n)}(3n)} \\
&\quad + x_{\mu(m)} - x_{\mu(m)} - x_{\mu(n)} + x_{\mu(n)}| \\
&\leq |(x_{\mu(m)})_{p_{\mu(m)}(3m)} - x_{\mu(m)}| + |x_{\mu(n)} \\
&\quad - (x_{\mu(n)})_{p_{\mu(n)}(3n)}| + |x_{\mu(m)} - x_{\mu(n)}| \\
&\leq \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = \frac{1}{k}\ )
\end{aligned}$$

and therefore $id : y \in \mathbb{R}$.
Now we will construct the modulus of convergence C. We begin by defining

$$C : \mathbb{N}^+ \to \mathbb{N}^+, \qquad C(k) := \texttt{max}\{q(3k), 3k\}$$

Thus we get

$$\begin{aligned}
\forall_{k\in\mathbb{N}^+}\forall_{n\geq C(k)}(\ |x_n - y| &= |x_n - y_n + y_n - y| \\
&\leq |x_n - x_{\mu(n)} + x_{\mu(n)} - (x_{\mu(n)})_{p_{\mu(n)}(3n)}| + |y_n - y| \\
&\leq |x_n - x_{\mu(n)}| + |x_{\mu(n)} - (x_{\mu(n)})_{p_{\mu(n)}(3n)}| + |y_n - y| \\
&\leq \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = \frac{1}{k}\ )
\end{aligned}$$

and $C : x_n \xrightarrow{n} y$.
So the a.r. ζ_2 is well defined. Clearly it also is a function, since the equality on $\texttt{PrfMemb}_0^{\text{Cauchy}}(x_n)$ is pointwise.
All in all ζ_1 and ζ_2 fulfil our needs and (ζ_1, ζ_2) is our witness of the statement. .

Definition 4.1. Let $a, b \in \mathbb{R}$. We define the the *finite closed interval* $[a, b]$ as an extensional subset of $\mathbb{R}$. Therefore we define the property $P(x)$ on $\mathbb{R}$.

$$P(x) :\Leftrightarrow \; a \leq x \;\&\; x \leq b$$

This property is extensional, because the property " $\leq$ " is extensional on $\mathbb{R}$. Now we can define:

$$[a, b] := \mathbb{R}_P$$
$$x \in [a, b] :\Leftrightarrow \; \exists_{\alpha,\beta \in \mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}(\alpha : a \leq x \;\;\&\;\; \beta : x \leq b)$$

GS2 implies the corresponding $\mathbb{R}$-family of witnesses:

$$\texttt{PrfMemb}^{[a,b]} := (\texttt{PrfMemb}_0^{[a,b]}, \texttt{PrfMemb}_1^{[a,b]}),$$
$$\texttt{PrfMemb}_0^{[a,b]}(x) := \{(\alpha, \beta) \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \times \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \mid \alpha : a \leq x \;\;\&\;\; \beta : x \leq b\}.$$

We say the interval $[a, b]$ is *nonvoid*, if we can construct a real number $x \in [a, b]$. The finite closed interval $[a, b]$ is *compact* if it is nonvoid.

Definition 4.5. (Continuous functions) Let $a, b \in \mathbb{R}$ and $f \in \mathbb{F}([a, b], \mathbb{R})$. We define:

$$\text{Cont}(f) :\Leftrightarrow \exists_{\delta \in \mathbb{F}(\mathbb{N}^+,\mathbb{R}^+)}(\delta : \text{Cont}(f))$$
$$\delta : \text{Cont}(f) :\Leftrightarrow \forall_{k \in \mathbb{N}^+} \forall_{x,y \in [a,b]}(\; |x - y| \leq \delta(k) \Rightarrow |f(x) - f(y)| \leq \frac{1}{k} \;)$$

P $\delta : \text{Cont}(f)$ is extensional: Let $(f, f') \in D(\mathbb{F}([a, b], \mathbb{R}))$ and $(\delta, \delta') \in D(\mathbb{F}(\mathbb{N}^+, \mathbb{R}^+))$ such that $\delta : \text{Cont}(f)$. Then:

$$\forall_{k \in \mathbb{N}^+} \forall_{x,y \in [a,b]}(\; |x - y| \leq \delta(k) =_{\mathbb{N}^+} \delta'(k) \Rightarrow |f'(x) - f'(y)| =_{\mathbb{R}} |f(x) - f(y)| \leq \frac{1}{k} \;)$$
$$\forall_{k \in \mathbb{N}^+} \forall_{x,y \in [a,b]}(\; |x - y| \leq \delta'(k) =_{\mathbb{N}^+} \delta(k) \Rightarrow |f'(x) - f'(y)| =_{\mathbb{R}} |f(x) - f(y)| \leq \frac{1}{k} \;)$$

holds. Thus $\delta' : \text{Cont}(f')$. ■

Let $a, b \in \mathbb{R}$, $X := \mathbb{F}([a, b], \mathbb{R}), Y := \mathbb{F}(\mathbb{N}^+, \mathbb{R}^+)$ and $Q(f, \delta) := \delta : \text{Cont}(f)$. Then GS1 defines the $\mathbb{F}([a, b], \mathbb{R})$-family:

$$\texttt{PrfMemb}^{\text{Cont}} := (\texttt{PrfMemb}_0^{\text{Cont}}, \texttt{PrfMemb}_1^{\text{Cont}}),$$
$$\texttt{PrfMemb}_0^{\text{Cont}}(f) := \{\delta \in \mathbb{F}(\mathbb{N}^+, \mathbb{R}^+) \mid \delta : \text{Cont}(f)\}.$$

We denote by $C([a,b],\mathbb{R})$ the totality with equality of continuous functions on $[a,b]$. From above it immediately follows that this totality is an extensional subset of $\mathbb{F}([a,b,],\mathbb{R})$.

Definition 6.1. For the formulation of the integral below, we need some abbreviations.
Let $n,m \in \mathbb{N}$ and $a = a_1 \leq a_2 \leq ... \leq a_n = b \in \mathbb{R}$, then the sequence $P := (a_1, ..., a_n)$ is called a *partition* of $[a,b]$.
A partition $Q := (b_1, ..., b_m)$ of $[a,b]$ is a *refinement* of P, if the following holds:

$$\forall_{i\in\{1,\ldots,n\}}\exists_{j\in\{1,\ldots,n\}}(\ b_j = a_i\ )$$

In addition to the above variables, let f be a continuous function on $[a,b]$. We define

$$S(f,P) := \sum_{i=1}^{n-1} f(a_i)(a_{i+1} - a_i),$$

and

$$S(f,a,b,n) := \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n})$$

Thus if we define the partition $P := (a, a + 1\frac{b-a}{n}, ..., a + n\frac{b-a}{n})$, we have

$$S(f,a,b,n) := S(f,P).$$

Theorem 6.3. (The integral) Let $a,b \in \mathbb{R}$ such that the interval $[a,b]$ is compact. Then

$$\forall_{f\in \mathrm{C}([a,b],\mathbb{R})}\forall_{\delta_f\in \mathtt{PrfMemb}_0^{\mathrm{Cont}}(f)}\exists_{s\in\mathbb{R}}(\ \mathrm{Conv}_s((S(f,a,b,n))_{n\in\mathbb{N}^+})\ ) \tag{6.1}$$

holds.
We call this limit of $(S(f,a,b,n))_n$ the *integral* of f from a to b and write

$$\int_a^b f(x)dx.$$

Consequently we define the *integral* on a given compact interval $[a,b]$ as the dependent function:

$$\int_a^b : \prod_{f\in C([a,b],\mathbb{R})}\ \prod_{\delta_f\in\mathtt{PrfMemb}_0^{\mathrm{Cont}}(f)}\ \sum_{s\in\mathbb{R}} \mathtt{PrfMemb}_0^{\mathrm{Conv}_s}(S(f,a,b,n))$$

Proof. The witness of our statement (6.1) is an element of

$$\prod_{f\in C([a,b],\mathbb{R})}\ \prod_{\delta_f\in\mathtt{PrfMemb}_0^{\mathrm{Cont}}(f)}\ \sum_{s\in\mathbb{R}}\ \sum_{p\in\mathtt{PrfMemb}_0^{\mathbb{R}}(s)} \mathtt{PrfMemb}_0^{\mathrm{Conv}_s}((S(f,a,b,n))_n) \tag{6.2}$$

Theorem 3.3.part II gives us an element ζ_2 of

$$\mathbb{F}\left(\mathtt{PrfMemb}_0^{\mathrm{Cauchy}}((S(f,a,b,n))_n), \sum_{s\in\mathbb{R}} \sum_{p\in\mathtt{PrfMemb}_0^{\mathbb{R}}(s)} \mathtt{PrfMemb}_0^{\mathrm{Conv}_s}((S(f,a,b,n))_n\right).$$

So it suffices to construct an element σ of

$$\prod_{f\in C([a,b],\mathbb{R})} \prod_{\delta_f\in\mathtt{PrfMemb}_0^{\mathrm{Cont}}(f)} \mathtt{PrfMemb}_0^{\mathrm{Cauchy}}((S(f,a,b,n))_n)$$

because then the dependent function $\zeta_2\circ\sigma$ is an element of (6.2).
We define

$$\sigma : \prod_{f\in C([a,b],\mathbb{R})} \prod_{\delta_f\in\mathtt{PrfMemb}_0^{\mathrm{Cont}}(f)} \mathtt{PrfMemb}_0^{\mathrm{Cauchy}}((S(f,a,b,n))_n),$$

$$\sigma(f,\delta_f) : \mathbb{N}^+ \to \mathbb{N}^+,$$

$$\sigma_{f,\delta_f}(k) := \sigma(f,\delta_f)(k) := \lceil \frac{b-a}{\delta_f(k\cdot 2\lceil b-a\rceil)} \rceil$$

for every $k\in\mathbb{N}^+$, where $\lceil . \rceil$ denotes the ceiling function.
Next we prove that σ is well defined. Let $a,b\in\mathbb{R}$ and $[a,b]$ a compact interval. Additionally assume f to be a continuous function on $[a,b]$ and $\delta_f \in \mathtt{PrfMemb}_0^{\mathrm{Cont}}(f)$ an arbitrary modulus of continuity of f.
Let $t,m,n\in\mathbb{N}^+$ such that $\frac{b-a}{m}, \frac{b-a}{n} \le \delta_f(t)$. We define the partitions:

$$C := (c_0,...,c_n) \qquad \forall_{i\in\{0,...,n\}}(\ c_i := a + i\frac{b-a}{n}\ )$$

$$D := (d_0,...,d_m) \qquad \forall_{i\in\{0,...,m\}}(\ d_i := a + i\frac{b-a}{m}\ )$$

and the common refinement of C and D, obtained by ordering the numbers $c_0,...,c_n,d_0,...,d_m$ in an increasing sequence.

$$E := (e_0,...,e_k) \qquad \forall_{i\in\{0,...,k-1\}}(\ e_i \le e_{i+1}\ )$$

Since we can order the sequences $(\frac{i}{n})_{i=1}^n$ and $(\frac{i}{m})_{i=1}^m$ on $\mathbb{Q}$ in an increasing sequence, this is also possible for the sequences C and D.
We write $\sum_i^C$ to denote the summation over all indices $j\in\{0,...,k\}$ such that $c_i \le e_j < c_{i+1}$.
Then

$$\begin{aligned}|S(f,C)-S(f,E)| &= \left|\sum_{i=0}^{n-1} f(c_i)(c_{i+1}-c_i) - \sum_{j=0}^{k-1} f(e_j)(e_{j+1}-e_j)\right| \\ &= \left|\sum_{i=0}^{n-1}\sum_i^C (f(c_i)-f(e_j))(e_{j+1}-e_j)\right| \\ &\overset{c_i-e_j\le\delta(t)}{\le} \left|\sum_{i=0}^{n-1}\sum_i^C t^{-1}\,(e_{j+1}-e_j)\right| = t^{-1}\,(b-a)\end{aligned}$$

Analogue we have

$$|S(f,D) - S(f,E)| \leq t^{-1}\,(b-a)$$

thus by triangle inequality

$$|S(f,C) - S(f,D)| \leq |S(f,C) - S(f,E)| + |S(f,D) - S(f,E)| \leq 2(b-a)\,t^{-1}$$

Hence if we choose $t := k \cdot 2\lceil b-a \rceil$, we get

$$|S(f,a,b,n) - S(f,a,b,m)| = |S(f,C) - S(f,D)| \leq 2(b-a)\,t^{-1} \leq \frac{1}{k} \tag{6.3}$$

In particular we have:

$$\begin{aligned}\forall_{k\in\mathbb{N}^+}\forall_{m\geq\sigma_{f,\delta_f}(k)}:\ \frac{1}{m} &\leq \big(\sigma_{f,\delta_f}(k)\big)^{-1} \leq \frac{\delta_f(k\cdot 2\lceil b-a\rceil)}{b-a}\\ &\Rightarrow \frac{b-a}{m} \leq \delta_f(k\cdot 2\lceil b-a\rceil)\end{aligned}$$

Therefore with (6.3) we get:

$$\sigma_{f,\delta_f} : \mathrm{Cauchy}((S(f,a,b,n))_n)$$

Also $\sigma(f)$ is a dependent function because we have:

$$\begin{aligned}\sigma_{f,\delta_f}(k) &:= \lceil \frac{b-a}{\delta_f(k\cdot 2\lceil b-a\rceil)} \rceil\\ &= \lceil \frac{b-a}{\delta_{f'}(k\cdot 2\lceil b-a\rceil)} \rceil := \sigma_{f,\delta'_f}\end{aligned}$$

for every $k \in \mathbb{N}^+$, $f \in C([a,b],\mathbb{R})$ and $(\delta_f, \delta'_f) \in D(\texttt{PrfMemb}_0^{\mathrm{Cont}}(f))$.
Finally we conclude that σ is a dependent function, since for $(f,f') \in D(C([a,b],\mathbb{R}))$

$$\begin{aligned}&\forall_{\delta_f\in\texttt{PrfMemb}_0^{\mathrm{Cont}}(f)}(\ \delta_f := \texttt{PrfMemb}_{ff'}^{\mathrm{Cont}}(\delta_f) \in \texttt{PrfMemb}_0^{\mathrm{Cont}}(f')\ )\\ &\forall_{\delta_{f'}\in\texttt{PrfMemb}_0^{\mathrm{Cont}}(f')}(\ \delta_{f'} := \texttt{PrfMemb}_{f'f}^{\mathrm{Cont}}(\delta_{f'}) \in \texttt{PrfMemb}_0^{\mathrm{Cont}}(f)\ )\end{aligned}$$

and

$$\begin{aligned}&\forall_{\delta_f\in\texttt{PrfMemb}_0^{\mathrm{Cont}}(f)}(\ \sigma(f)(\delta_f) := \sigma_{f,\delta_f} \underset{\text{depends on }\delta_f}{\overset{\sigma_{f,\delta_f}\text{only}}{=}} \sigma_{f',\delta_f} := \sigma(f')(\delta_f)\ )\\ &\forall_{\delta_{f'}\in\texttt{PrfMemb}_0^{\mathrm{Cont}}(f')}(\ \sigma(f)(\delta_{f'}) := \sigma_{f,\delta_{f'}} \underset{\text{depends on }\delta_f}{\overset{\sigma_{f,\delta_f}\text{only}}{=}} \sigma_{f',\delta_{f'}} := \sigma(f')(\delta_{f'})\ )\end{aligned}$$

holds. So $\sigma(f) = \sigma(f')$ and σ is a dependent function.

6 Conclusion

The exact BHK-interpretation we presented in this thesis, reveals the algorithmic structure of several constructive proofs. Namely the proof of all statements, which fundamental property can be expressed as an extensional existential formula. This holds for fundamental properties like continuity of a function, that are witnessed by a modulus of some sort. But it is also possible to give proof-relevant definitions for several concepts of BISH, as we did for the set of reals $\mathbb{R}$ or the relation $\leq_{\mathbb{R}}$. Since every extensional property P on a set X is equivalent to the membership to the set X_P, our proof-relevant approach extends the exact BHK-interpretation. Thus it provides an interpretation of prime formulas that express membership with evidence. Moreover, through the "introduction" of proof-relevance we can avoid the use of choice in the corresponding proofs. So all in all this more precise treatment of BCM, helps us to better understand how proofs of subformulas of a given formula ϕ build the set of proofs, thus a witness of ϕ. We note that the relevant information of a proof of a formula, the witness, is a precisely described mathematical object. Hence the exact BHK-interpretation helps us to better understand the computational content of proofs in CM and therefore provides a better understanding of what constructive mathematics is.
However further research about the treatment of negation has to be done.

The semi formal framework (BST) we used for our interpretation also closes ranks between BISH, MLTT and HoTT. It enables the translation of several concepts and results from MLTT and HoTT into BISH, as carried out in [6].

References

[1] P. Aczel, M. Rathjen: *Constructive Set Theory,* book draft, 2010.

[2] M. J. Beeson: Formalizing constructive mathematics: why and how, in [7], 1981, 146-190.

[3] E. Bishop and D. S. Bridges: *Constructive Analysis, Grundlehren der math. Wissenschaften 279,* Springer-Verlag, Heidelberg-Berlin-New York, 1985.

[4] Iosif Petrakis: Dependent sums and dependent products in Bischop's set theory, in P. Dybjer et. al. (Eds) TYPES 2018, LIPIcs, Vol. 130, Article No. 3, 2019 DOI 10.4230/LIPIcs.TYPES.2018.3

[5] Iosif Petrakis: On the BHK-interpretation of BISH within BISH, Arbeitstagung Bern, München und Verona, LMU 12.12.2019

[6] I. Petrakis: *Families of Sets in Bishop Set Theory,* Habilitation Thesis, LMU 2020.

[7] F. Richman: *Constructive Mathematics,* LNM 873, Springer-Verlag, 1981.

[8] H. Schwichtenberg: *Constructive analysis with witnesses,* Lecture notes, LMU 2019.

[9] The Univalent Foundations Programm: *Homotopy Type Theory: Univalent Foundations of Mathematics,* Institute for Advanced Study, Princeton 2013.