Bachelor Thesis

The Yoneda lemma

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Abstract

This thesis discusses the Yoneda lemma, which is considered one of the central theorems in category theory. The first chapter of this thesis introduces the topic by raising the question how can the set of natural transformations between a Hom-functor and another set-valued functor be described, which the Yoneda lemma gives the answer to. The second chapter deals with basic concepts of category theory and some important examples, which will be needed later for the formulation and the proof of the lemma, as well as for some of its applications. In Chapter 3 the Yoneda lemma is formulated and proven. Chapter 4 is devoted to some applications of the Yoneda lemma. The final chapter refers to the question of the introduction and summarizes the impact of the Yoneda lemma.

1 Introduction

This thesis deals with the Yoneda lemma and some of its applications. The Yoneda lemma goes back to a conversation between the American mathematician Saunders Mac Lane (1909 - 2005) and the Japanese mathematician Nobuo Yoneda (1930-1996). Regarding the fact that nowadays it is considered as one of the most important results of category theory the circumstances surrounding the emergence of the Yoneda lemma are all the more astonishing. In the years 1954/1955 Yoneda went on a one-year research trip to France, where he met Mac Lane, who at that time was collecting information for a book on category theory. Category theory was a new theory in that time, as it was first formulated by Mac Lane and Eilenberg in 1945. The myth states that Mac Lane and Yoneda met in a café at the Gare du Nord in Paris, where Mac Lane interviewed Yoneda, continuing their conversation even on Yoneda's train until he left. It is remarkable that such an important result emerged in such a short meeting and also that Yoneda was only about 24 years old at that time. The content of this meeting was then named by Mac Lane as the "Yoneda lemma". [Kinoshita, 1997]

The Yoneda lemma provides an answer to the question: how can the set $\mathrm{Hom}(y_a, F)$ of natural transformations between a Hom-functor y_a and another set-valued functor F, be described?

The aim of this work is to present the Yoneda lemma and to clarify its use for different mathematical problems. This thesis consists of three main parts. Chapter 2 introduces the essential concepts of category theory, that are based on the book *Category Theory* by Steve Awodey [Awodey, 2010, chapter 1-3, 5-7] and the lecture notes by Iosif Petrakis [Petrakis, 2021, chapter 1]. The approach to the Yoneda lemma in chapter 3 and 4 with the given definitions, theorems, proofs and examples is mainly based on the book *Category Theory* by Steve Awodey, [Awodey, 2010, chapter 8]. Chapter 3 introduces the Yoneda lemma and provides a proof. Chapter 4 deals with some consequences and applications of the Yoneda lemma, whereas in comparison to Awodey, the proof in 4.9 is given more explicit. Where indicated, amendments from additional literature are included. The text is adapted and explained in such a way that this should serve as a self-contained approach to the topic for readers on the bachelor level.

2 Background in category theory

In this section, basic terminology from category theory will be defined and explained. This chapter serves as a general introduction and summary of those definitions necessary for conducting the proof of the Yoneda Lemma.

2.1 Definition of a category

Definition 2.1. A *category* $\mathbf{C}$ is a structure $(\mathrm{C}_0, \mathrm{C}_1, \mathrm{dom}, \mathrm{cod}, \circ, 1)$, where

(i) C_0 is the collection of the objects of $\mathbf{C}$, denoted by A, B, C,...

(ii) C_1 is the collection of the arrows of $\mathbf{C}$,denoted by f,g,h,...

(iii) For every f in C_1, there are given two objects

$$\mathrm{dom}(f),\ \mathrm{cod}(f)$$

called the *domain* and *codomain* of f. We write

$$f : A \to B,$$

where $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.

(iv) If $f : a \to b,\ g : b \to c$ arrows in $\mathbf{C}$, that is, with

$$\mathrm{cod}(f) = \mathrm{dom}(g),$$

there is an arrow

$$g \circ f : A \to C$$

called the *composite* of f and g.

(v) For every a $\in \mathrm{C}_0$, there is given an arrow

$$1_A : A \to A$$

called the *identity arrow* of A.

These data need to satisfy the following conditions:

(a) Unit:

$$f \circ 1_A = f = 1_B \circ f$$

for all $f : A \to B$.

(b) Associativity:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for all $f : A \to B,\ g : B \to C,\ h : C \to D.$

Given any objects A, B in $\mathbf{C}$, we write $\mathrm{Hom}_{\mathbf{C}}(A, B)$, or sometimes just $\mathrm{Hom}(A, B)$, if $\mathbf{C}$ is clear, to denote the collection of arrows f in C_1 with $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$.

Example 2.2 (The category of sets)**.** We write **Set** to denote the category which has sets as objects and functions as arrows between objects.

To prove that **Set** is indeed a category, we need to show that **Set** satisfies the following data and properties:

(i) *Objects*: sets $A, B, C, \ldots$ in Set_0.

(ii) *arrows*: functions $f, g, h, \ldots$ in Set_1.

(iii) Every arrow $f : A \to B$ is an function from a set

$$A = \mathrm{dom}(f)$$

to a set

$$B = \mathrm{cod}(f).$$

(iv) If $f : A \to B, g : B \to C$, there is a composite function

$$g \circ f : A \to C,$$

given by

$$(g \circ f)(a) = g(f(a)) \text{ for any } a \in A. \tag{1}$$

(v) For every $A \in \mathrm{Set}_0$, there is an identity function

$$1_A : A \to A$$

given by

$$1_A(a) = a. \tag{2}$$

We need to show that the following conditions are satisfied:

(a) Unit: If $f : A \to B,$ then

$$f \circ 1_A = f = 1_B \circ f$$

since for any $a \in A$, we have that

$$(f \circ 1_A)(a) = f(1_A(a)) = f(a),$$
$$(1_B \circ f)(a) = 1_B(f(a)) = f(a)$$

using equations (1) and (2).

(b) Associativity: If $f : A \to B,\ g : B \to C,\ h : C \to D$, then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

always holds, since for any $a \in A$, we have

$$((h \circ g) \circ f)(a) = h(g(f(a))) = (h \circ (g \circ f))(a)$$

using equation (1).

Some other important examples of categories that will be discussed in the course of this thesis include the *functor category*, which is discussed in chapter 2.3.2, the *category of cones*, which is defined in chapter 2.8 and the *category of elements*, defined in theorem 4.9.

Definition 2.3. A category $\mathbf{C}$ is called *small*, if the the collections C_0 and C_1 are both sets. If one of them is a proper class i.e., a class that is not a set, then $\mathbf{C}$ is called *large*. A category $\mathbf{C}$ is said to be *locally small* if for every $A, B \in \mathrm{C}_0$ the collection $\mathrm{Hom}(A, B)$ form a set.

Example 2.4. A lots of interesting categories have a proper class of objects, for example the category of sets, which we will discuss next. The category of sets is locally small, since the collection of all functions from a set A to a set B do indeed form a set.

Remark 2.5. All sets are themselves classes, but it is possible for a class not to form a set. This is made clear by the so called Russell's paradox. One might think that every collection of things forms a set. The Russell's paradox shows that such naive construction leads to a contradiction. Let V denote the class of all sets x. Suppose V is a set, then by the axiom schema of separation $R = \{x \in V : x \notin x\}$ were set. But then $R \in R \Leftrightarrow R \notin R$, which is not possible. So, the assumption that V is a set must be wrong.

2.2 The opposite (or "dual") category

In this section, we consider a new category that can be constructed from existing ones.

Definition 2.6. The *opposite* (or *"dual"*) category $\mathbf{C}^{op}$ denotes the category obtained by taking the category $\mathbf{C}$ and reversing the direction of all arrows. That means, that $\mathbf{C}^{op}$ has the same collection of objects as $\mathbf{C}$ has, but the collection of arrows of $\mathbf{C}^{op}$ is the collection of $f^{op} : d \to c$ such that $f : c \to d$ is an arrow of $\mathbf{C}$. Here f^{op} is the reversal of the arrow f in $\mathbf{C}$.
We can define units and composition in $\mathbf{C}^{op}$ in terms of the corresponding operations in $\mathbf{C}$ in following way, namely, for any c in C_0

$$(1_c)^{op} = 1_c$$

and if $f : a \to b, g : b \to c$ in C_1, we have

$$(g \circ f)^{op} = f^{op} \circ g^{op}.$$

Thus, a diagram in **C**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

looks like this is $\mathbf{C}^{op}$

$$\begin{tikzcd} A \arrow[r, "f^{op}"] \arrow[dr, "g^{op} \circ f^{op}"'] & B \arrow[d, "g^{op}"] \\ & C, \end{tikzcd}$$

where we have left out the identity arrows in both drawings.

2.3 Functors

Just like functions allow one to compare and relate sets, we have the concept of functors to relate categories.

Definition 2.7. [Petrakis, 2021, chapter 1.14]

Let **C** and **D** be categories. A *covariant* functor from **C** to **D** is a pair

$$F = (F_0, F_1),$$

such that

(i) $F_0 : \mathrm{C}_0 \to \mathrm{D}_0$ is a function that maps an object A in **C** to an object $F_0(A)$ of **D**,

(ii) $F_1 : \mathrm{C}_1 \to \mathrm{D}_1$ is a function that maps an arrow $f : A \to B$ of **C** to an arrow

$$F_1(f) : F_0(A) \to F_0(B)$$

of **D**, which satisfy the following properties:

(a) For every A in C_0 we have that

$$F_1(1_A) = 1_{F_0(A)} \tag{3}$$

(b) If $f : A \to B$ and $g : B \to C$, then

$$F_1(g \circ f) = F_1(g) \circ F_1(f) \tag{4}$$

i.e., the diagram above marked by # commutes,

$$
\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\
\mathbf{C} \arrow[dd, "F"'] & & C \\
 & & F(B) \arrow[d, "F(g)"] \\
\mathbf{D} & F(A) \arrow[ur, "F(f)"] \arrow[r, "F(g \circ f)"'] & F(C). \arrow[ul, phantom, "\#"]
\end{tikzcd}
$$

In this case we write $F : \mathbf{C} \to \mathbf{D}$.

A *contravariant* functor from $\mathbf{C}$ to $\mathbf{D}$ is a pair $F = (F_0, F_1)$, such that

(i) $F_0 : \mathrm{C}_0 \to \mathrm{D}_0$ is a function that maps an object A in $\mathbf{C}$ to an object $F_0(A)$ of $\mathbf{D}$,

(ii') $F_1 : \mathrm{C}_1 \to \mathrm{D}_1$ is a function that maps an arrow $f : A \to B$ of $\mathbf{C}$ to an arrow

$$F_1(f) : F_0(B) \to F_0(A)$$

of $\mathbf{D}$, which satisfy the following properties:

(a) For every A in $\mathbf{C}_0$ we have that

$$F_1(1_A) = 1_{F_0(A)} \tag{5}$$

(b') If $f : A \to B$ and $g : B \to C$, then

$$F_1(g \circ f) = F_1(f) \circ F_1(g) \tag{6}$$

i.e., the following diagram marked by # commutes,

$$
\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\
\mathbf{C} \arrow[ddd, "F"'] & & C \\
 & & F(B) \arrow[dl, "F(f)"'] \\
\mathbf{D} & F(A) & F(C). \arrow[l, "F(g \circ f)"] \arrow[u, "F(g)"'] \arrow[ul, phantom, "\divideontimes"]
\end{tikzcd}
$$

In this case we write $F : \mathbf{C}^{op} \to \mathbf{D}$.

Remark 2.8. For more clarity, we omit the identity arrows in drawing categories.

A covariant functor is $F : \mathbf{C}^{op} \to D$ is exactly a contravariant functor from $\mathbf{C}$ to $\mathbf{D}$.

Definition 2.9. Let $\mathbf{C}$ be a locally small category. A functor $F : \mathbf{C} \to \mathbf{D}$ is called

- *injective (res. surjective) on objects* if the object part $F_0 : \mathrm{C}_0 \to \mathrm{D}_0$ is injective (res. surjective).
- *injective (res. surjective) on arrows* if the arrow part $F_1 : \mathrm{C}_1 \to \mathrm{D}_1$ is injective (res. surjective).
- *faithful* if for all a, b in C_0 the map

  $$F_{ab} : \mathrm{Hom}_{\mathbf{C}}(a, b) \to \mathrm{Hom}_{\mathbf{D}}(F(a), F(b))$$

  defined by $f \mapsto F(f)$ is injective i.e. a faithful functor is only injective with respect to pairs of arrows that had the same domain and codomain to begin with. That is, if $f, g : a \to b$ and F is a faithful functor,

  $$F(f) = F(g) \Rightarrow f = g.$$
- *full* if F_{ab} is always surjective for every a,b.

2.3.1 Natural transformations

We talked about functors as mappings between categories, now a *natural transformation* is a "map" between two functors from a category $\mathbf{C}$ to a category $\mathbf{D}$.

Definition 2.10. Let **C**, **D** be categories. Suppose $F = (F_0, F_1)$ and $G = (G_0, G_1)$ are functors from **C** to **D**. A *natural transformation* from F to G is a family of arrows in **D** of the form

$$\tau_c : F_0(c) \to G_0(c),$$

such that for every c in C_0, and every $f : c \to c'$ in C_1 the diagram below marked with ※ commutes,

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=L}] & \mathbf{D} \arrow[Rightarrow, from=U, to=L, "\tau"] \end{tikzcd}
\qquad
\begin{tikzcd} F_0(c) \arrow[r, "F_1(f)"] \arrow[d, "\tau_c"'] \arrow[dr, phantom, "※"] & F_0(c') \arrow[d, "\tau_{c'}"] \\ G_0(c) \arrow[r, "G_1(f)"'] & G_0(c') \end{tikzcd}$$

i.e.

$$\tau_{c'} \circ F_1(f) = G_1(f) \circ \tau_c. \tag{7}$$

We denote a *natural transformation* τ from F to G by $\tau : F \Rightarrow G$.

2.3.2 The functor category

Definition 2.11. For categories **C**, **D** we define the functor category $Fun(\mathbf{C},\ \mathbf{D})$[1] to be the category where the objects are the functors from **C** to **D** and the arrows are natural transformation between them i.e., if $F, G : \mathbf{C} \to \mathbf{D}$, an arrow from F to G is a natural transformation from F to G. The identity arrow

$$1_F : F \Rightarrow F$$

is the family of arrows

$$(1_F)_c : F_0(c) \to F_0(c),$$

where

$$(1_F)_c = 1_{F_0(c)},$$

and the following diagram trivially commutes in **D**

$$\begin{tikzcd} F_0(c) \arrow[r, "F_1(f)"] \arrow[d, "1_{F_0(c)}"'] & F_0(c') \arrow[d, "1_{F_0(c')}"] \\ F_0(c) \arrow[r, "F_1(f)"'] & F_0(c'). \end{tikzcd}$$

If $F, G, H : \mathbf{C} \to \mathbf{D},\ \tau : F \Rightarrow G$ and $\sigma : G \Rightarrow H,$ the composite arrow $\sigma \circ \tau$ is

[1] In literature, the functor category $Fun(\mathbf{C}, \mathbf{D})$ is often denoted by $\mathbf{D}^{\mathbf{C}}$.

defined by

$$(\sigma \circ \tau)_c = \sigma_c \circ \tau_c : F_0(c) \to H_0(c), \tag{8}$$

for every c in C_0, and , if $f : \mathbf{C} \to \mathbf{C'}$ in C_1, the following outer diagram commutes

$$\begin{tikzcd}
F_0(c) \arrow[r, "F_1(f)"] \arrow[d, "\tau_c"'] \arrow[dd, bend right, "(\sigma \circ \tau)_c"'] & F_0(c') \arrow[d, "\tau_{c'}"] \arrow[dd, bend left, "(\sigma \circ \tau)_{c'}"] \\
G_0(c) \arrow[r, "G_1(f)"'] \arrow[d, "\sigma_c"'] & G_0(c') \arrow[d, "\sigma_{c'}"] \\
H_0(c) \arrow[r, "H_1(f)"'] & H_0(c'),
\end{tikzcd}$$

since

$$\begin{aligned}
(\sigma \circ \tau)_{c'} \circ F_1(f) &= (\sigma_{c'} \circ \tau_{c'}) \circ F_1(f) \\
&= \sigma_{c'} \circ (\tau_{c'}) \circ F_1(f)) \\
&= \sigma_{c'} \circ (G_1(f) \circ \tau_c) \qquad (9) \\
&= (\sigma_{c'} \circ G_1(f)) \circ \tau_c) \\
&= (H_1(f) \circ \sigma_c) \circ \tau_c \qquad (10) \\
&= H_1(f) \circ (\sigma_c \circ \tau_c) \\
&= H_1(f) \circ (\sigma \circ \tau)_c.
\end{aligned}$$

In equation (9) we used that τ is a natural transformation form F to G and in equation (10) we used that σ is a natural transformation form G to H.

Definition 2.12. Let $F, G : \mathbf{C} \to \mathbf{D}$ functors. A *natural isomorphism* is a natural transformation

$$\vartheta : F \Rightarrow G$$

which is an isomorphism in the functor category Fun($\mathbf{C}$, $\mathbf{D}$).

Lemma 2.13. A natural transformation $\vartheta : F \Rightarrow G$ is a natural isomorphism iff each component $\vartheta_c : F(c) \to G(c)$ is an isomorphism.

Proof. Suppose $\vartheta : F \Rightarrow G$ is a natural isomorphism. Then by definition there exist an inverse $\vartheta^{-1} : G \Rightarrow F$ such that

$$\vartheta^{-1} \circ \vartheta = 1_F \text{ and } \vartheta \circ \vartheta^{-1} = 1_G.$$

Therefore,

$$[\vartheta^{-1} \circ \vartheta]_c = [1_F]_c \text{ and } [\vartheta \circ \vartheta^{-1}]_c = [1_G]_c.$$

Because of (8), this is equivalent to

$$\vartheta_c^{-1} \circ \vartheta_c = 1_{F(c)} \text{ and } \vartheta_c \circ \vartheta_c^{-1} = 1_{G(c)}.$$

So, the components of ϑ are isomorphisms. If conversely each component $\vartheta_c : F(c) \to G(c)$ is an isomorphism for each $c \in C_0$. That means, there exist an inverse $(\vartheta_c)^{-1} : G(c) \to F(c)$ such that

$$(\vartheta_c)^{-1} \circ \vartheta_c = 1_{F(c)} \quad \text{and} \quad \vartheta_c \circ (\vartheta_c)^{-1} = 1_{G(c)}.$$

We define

$$(\vartheta^{-1})_c := (\vartheta_c)^{-1}.$$

Knowing that ϑ is an natural transformation, for any $f : c \to c'$ we have

$$\begin{aligned} & G(f) \circ \vartheta_c = \vartheta_{c'} \circ F(f) \\ \Leftrightarrow \quad & (\vartheta^{-1})_{c'} \circ G(f) \circ \vartheta_c \circ (\vartheta^{-1})_c = (\vartheta^{-1})_{c'} \vartheta_b \circ F(f) \circ (\vartheta^{-1})_c \\ \Leftrightarrow \quad & (\vartheta^{-1})_{c'} \circ G(f) = F(f) \circ (\vartheta^{-1})_c \end{aligned}$$

$\Rightarrow$ ϑ^{-1} is a natural transformation, which is the inverse of ϑ. $\square$

From now on, we will denote the objects with lowercase letters instead of uppercase letters.

2.4 The covariant representable functors

Example 2.14. We assume that the category $\mathbf{C}$ is locally small. For each object a of $\mathbf{C}$,

$$\mathrm{Hom}(a, -) : \mathbf{C} \to \mathbf{Set}$$

is a covariant functor, called *representable functor* of a.

To show that this indeed determines a functor, we need to show that

(i) For any object b in $\mathbf{C}$,

$$[\mathrm{Hom}(a, -)]_0 (b) = \mathrm{Hom}(a, b).$$

Since $\mathbf{C}$ is locally small, $\mathrm{Hom}(a,b)$ is a set and therefore the function

$$[\mathrm{Hom}(a,-)]_0 : b \mapsto \mathrm{Hom}(a,b)$$

is well defined.

(ii) Any arrow $f : b \to b'$ in $\mathbf{C}$ induces a function

$$[\mathrm{Hom}(a,-)]_1(f) = [\mathrm{Hom}(a,f)]_1 : \overbrace{[\mathrm{Hom}(a,-)]_0(b)}^{\mathrm{Hom}(a,b)} \to \overbrace{[\mathrm{Hom}(a,-)]_0(b')}^{\mathrm{Hom}(a,b')}$$
$$(g : a \to b) \mapsto (f \circ g : a \to b')$$

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & b \arrow[r, "f"] & b' \end{tikzcd}$$

Thus,

$$[\mathrm{Hom}(a,f)]_1(g) = f \circ g. \tag{11}$$

In order to be a functor we require $\mathrm{Hom}(a,-)$ to satisfy the following conditions:

(a) For any $c \in C_0$,

$$[\mathrm{Hom}(a,1_c)]_1 = 1_{\underbrace{[\mathrm{Hom}(a,-)]_0(c)}_{\mathrm{Hom}(a,c)}} \tag{12}$$

and that

(b) if $f : c \to c', g : c' \to c''$, then

$$[\mathrm{Hom}(a,g \circ f)]_1 = [\mathrm{Hom}(a,g)]_1 \circ [\mathrm{Hom}(a,f)]_1 \tag{13}$$

We show the equations (12) and (13), by taking an argument $x : a \to c$, then for any $c \in C_0$

$$\begin{aligned} [\mathrm{Hom}(a,1_c)]_1(x) &= 1_c \circ x \\ &= x \\ &= 1_{\mathrm{Hom}(a,c)}(x) \end{aligned}$$

and for $f : c \to c',\ g : c' \to c'',\ g \circ f : c \to c''$,

$$\begin{aligned} [\mathrm{Hom}(a,g \circ f)](x) &= (g \circ f) \circ x \\ &= g \circ (f \circ x) \\ &= [\mathrm{Hom}(a,g)]_1(f \circ x) \\ &= [\mathrm{Hom}(a,g)]_1([\mathrm{Hom}(a,f)]_1(x)) \\ &= [\mathrm{Hom}(a,g)]_1 \circ [\mathrm{Hom}(a,f)]_1(x). \end{aligned}$$

2.5 The contravariant representable functors

Example 2.15. We assume that the category $\mathbf{C}$ is locally small. For each object a of $\mathbf{C}$,

$$\mathrm{Hom}(-, a) : \mathbf{C}^{op} \to \mathbf{Set}$$

is a contravariant functor, called *representable functor* of a.

To show that this indeed determines a functor, we need to show that

(i) For any object b in $\mathbf{C}$,

$$[\mathrm{Hom}(-, a)]_0\,(b) = \mathrm{Hom}(b, a)$$

note that $b \mapsto \mathrm{Hom}(b, a)$ is well defined, since $\mathbf{C}$ is locally small.

(ii') Any arrow $f : b \to b'$ in $\mathbf{C}$ induces a function

$$\begin{aligned}[\mathrm{Hom}(-, a)]_1\,(f) = [\mathrm{Hom}(f, a)]_1 : \mathrm{Hom}(b', a) &\to \mathrm{Hom}(b, a)\\ (g : b' \to a) &\mapsto (g \circ f : b \to a)\end{aligned}$$

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & b' \arrow[r, "g"] & a \end{tikzcd}$$

Thus,

$$[\mathrm{Hom}(f, a)]_1\,(g) = g \circ f. \tag{14}$$

In order to be a functor we require $\mathrm{Hom}(-, a)$ to satisfy the following conditions:

(a) For any $c \in C_0$,

$$[\mathrm{Hom}(1_c, a)]_1 = 1_{\mathrm{Hom}(c,a)} \tag{15}$$

and that

(b') if $f : c \to c', g : c' \to c''$, then

$$[\mathrm{Hom}(g \circ f, a)]_1 = [\mathrm{Hom}(f, a)]_1 \circ [\mathrm{Hom}(g, a)]_1 \tag{16}$$

We show equation (15), by taking an argument $x : c \to a$, then for any $c \in C_0$

$$\begin{aligned}[\mathrm{Hom}(1_c, a)]_1\,(x) &= x \circ 1_c\\ &= x\\ &= 1_{\mathrm{Hom}(c,a)}(x)\\ &= 1_{[\mathrm{Hom}(-,a)]_0(c)}(x).\end{aligned}$$

We show equation (16), by taking an argument $x' : c \to c''$, then for $f : c \to c'$, $g : c' \to c''$, $g \circ f : c \to c''$,

$$\begin{aligned}
[\mathrm{Hom}(g \circ f, a)]_1 (x') &= x' \circ (g \circ f) \\
&= (x' \circ g) \circ f \\
&= [\mathrm{Hom}(f.a)]_1 (x' \circ g) \\
&= [\mathrm{Hom}(f, a)]_1 ([\mathrm{Hom}(g, a)]_1 (x')) \\
&= [\mathrm{Hom}(f, a)]_1 \circ [\mathrm{Hom}(g, a)]_1 (x').
\end{aligned}$$

2.6 Some basic notions in categories

2.6.1 Isomorphism

Definition 2.16. In any category $\mathbf{C}$, an arrow $f : a \to b$ in $\mathbf{C}$ is called an *isomorphism*, or an *iso*, if there exists an arrow $g : b \to a$ in $\mathbf{C}$ such that

$$g \circ f = 1_a \text{ and } f \circ g = 1_b.$$

In this case we say that a is *isomorphic* to b, and write $a \cong b$.

For example an isomorphism of sets is a bijective function. Isomorphic objects in **Set** correspond to sets with the same number of elements.

2.6.2 Initial and terminal objects

Definition 2.17. In any category $\mathbf{C}$, an object
0 is **initial** if for any object c there is a unique morphism

$$0 \to c,$$

1 is **terminal** if for any object c there is a unique morphism

$$c \to 1.$$

Note that there is a kind of "duality" in these definitions. Precisely, a terminal object in $\mathbf{C}$ is exactly an initial object in $\mathbf{C}^{op}$.

Proposition 2.18. Initial and terminal objects are unique up to isomorphism.

Proof. In fact, if c and c' are both initial (terminal) in the same category, then there is a unique isomorphism $c \to c'$. Indeed, suppose that 0 and 0' are both initial objects in some category $\mathbf{C}$; the following diagram makes it clear that 0 and 0' are uniquely isomorphic,

$$
\begin{tikzcd}
0 \arrow[r, "u"] \arrow[dr, "1_0"'] & 0' \arrow[d, "v"] \arrow[dr, "1_{0'}"] & \\
& 0 \arrow[r, "u"'] & 0'
\end{tikzcd}
$$

since $v \circ u = 1_c$ and $u \circ v = 1_{c'}$, and u, v are unique. For terminal objects, apply the foregoing to $\mathbf{C}^{op}$. □

Example 2.19. In the category **Set**, the empty set $\varnothing$ is an initial object because, for every set c, there always exists an unique function, namely the empty function from the empty set $\varnothing$ to c. The graph of an empty function is the empty set itself. If $\{x\}$ is a singleton set, then, for every set c, there is an unique function from c to $\{x\}$, namely the function that maps every element of c to the unique element of $\{x\}$. Therefore, $\{x\}$ is a terminal object in **Set**.

2.6.3 Products

Definition 2.20. Let $\mathbf{C}$ be a category and a, b objects of $\mathbf{C}$. A *(binary) product*[2] *of a and b* consists of an object

$$a \times b$$

of $\mathbf{C}$ together with two arrows

$$pr_a : a \times b \to a \text{ and } pr_b : a \times b \to b,$$

such that the universal property of product is satisfied i.e., if c is an object in $\mathbf{C}$ and $f_a : c \to a$ and $f_b : c \to b$, there is a unique arrow

$$h = \langle f_a, f_b \rangle : c \to a \times b,$$

such that the following inner diagrams commute i.e.,

$$pr_a \circ h = f_a \text{ and } pr_b \circ h = f_b.$$

$$
\begin{tikzcd}
& c \arrow[dl, "f_a"'] \arrow[d, dashed, "h"'] \arrow[dr, "f_b"] & \\
a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b
\end{tikzcd}
$$

A category $\mathbf{C}$ has products, if for every objects a, b of $\mathbf{C}$, there is a product $a \times b$ in $\mathbf{C}$ (for simplicity we avoid to mention the corresponding projection arrows).

In any category with binary products, we can show, that

$$a \times (b \times c) \cong (a \times b) \times c.$$

[2] products are unique up to isomorphism, so one may speak of *the* product.

Proof. Let $p := a \times (b \times c)$ be the iterated product with the maps $p_1 : p \to a$, $p_2 : p \to (b \times c) \to b$ and $p_3 : p \to (b \times c) \to c$,. Define $q := (a \times b) \times c$ with the maps $q_1 : q \to a \times b \to a$, $q_2 : q \to b \times c \to b$ and $q_3 : q \to c$. By the universal property of product we get a unique map $p_1 \times p_2 : p \to a \times b$. Applying the universal property of product again, we get a unique map $f := (p_1 \times p_2) \times p_3 : p \to q$ with $q_i \circ f = p_i$ for $i = 1, 2, 3$. With a similar argument we get $g := p_1 \times (p_2 \times p_3) : q \to p$. By composing, we get $g \circ f : p \to p$ which respects the p_i. By the universal property of product, such a map is unique, but the identity is another such map. Thus they must be the same, therefore $g \circ f = 1_p$. Similarly we get $f \circ g = 1_q$, so f and g are inverse and $p \cong q$. $\square$

Definition 2.21. If a category **C** has products, $f : a \to b$ and $f' : a' \to b'$ are in C_1, then

$$f \times f' = \langle f \circ pr_a, f' \circ pr_{a'} \rangle : a \times a' \to b \times b$$

$$\begin{tikzcd}
 & & a \times a' \arrow[dl, "pr_a"'] \arrow[dr, "pr_{a'}"] \arrow[dd, dashed, "f \times f'"] & & \\
 & a \arrow[dl, "f"'] & & a' \arrow[dr, "f'"] & \\
b & & b \times b' \arrow[ll, "pr_b"] \arrow[rr, "pr_{b'}"'] & & b'
\end{tikzcd}$$

2.6.4 Equalizers

Definition 2.22. In any category **C**, given parallel arrows

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b
\end{tikzcd}$$

an *equalizer* of f and g consists of an object eq and an arrow $e : eq \to a$, universal such that

$$f \circ e = g \circ e.$$

That is, given any $z : c \to a$ with $f \circ z = g \circ z$, there is a unique $u : c \to eq$ with $e \circ u = z$, as in the diagram below:

$$\begin{tikzcd}
eq \arrow[r, "e"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
c \arrow[u, dashed, "u"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

2.7 Exponentials

Definition 2.23. If b, c are objects of a category $\mathbf{C}$ with products, an *exponential of b and c* is an object

$$c^b$$

in $\mathbf{C}$ together with an arrow

$$\mathrm{eval}_{b,c} : c^b \times b \to c,$$

such that for any object d in $\mathbf{C}$ and every arrow

$$f : d \times b \to c$$

there is a unique arrow

$$\hat{f} : d \to c^b \tag{17}$$

such that

$$\mathrm{eval}_{b,c} \circ (\hat{f} \times 1_b) = f,$$

all as in the diagram above

$$\begin{tikzcd}
c^b & c^b \times b \arrow[r, "\mathrm{eval}_{b,c}"] & c \\
d \arrow[u, "\hat{f}"] & d \times b \arrow[u, "\hat{f}\times 1_b"] \arrow[ur, "f"'] &
\end{tikzcd}$$

where the arrow $\hat{f} \times 1_b$ is determined in definition 2.21

$$\begin{tikzcd}
 & & d \times b \arrow[dl, "pr_d"'] \arrow[dr, "pr_b"] \arrow[dd, dashed, "\hat{f}\times 1_b"] & & \\
 & d \arrow[dl, "\hat{f}"'] & & b \arrow[dr, "1_b"] & \\
c^b & & c^b \times b \arrow[ll, "pr_{c^b}"] \arrow[rr, "pr_b"'] & & b.
\end{tikzcd}$$

The arrow $\hat{f}$ is called the (exponential) *transpose* of f. A category has exponentials, if for every b, c in $\mathbf{C}$ there is an exponential c^b in $\mathbf{C}$.

Given any arrow

$$g : d \to c^b$$

we write

$$\bar{g} := \mathrm{eval}_{b,c} \circ (g \times 1_b) : d \times b \to c \tag{18}$$

and also call $\bar{g}$ the transpose of g.

Since

$$\hat{\bar{g}} : d \to c^b$$

is unique by (17), we have that

$$\hat{\bar{g}} = g$$

and for any $f : d \times b \to c$, by (17) there is a unique arrow $\hat{f} : d \to c^b$, then

$$\bar{\hat{f}} := \text{eval}_{b,c} \circ (\hat{f} \times 1_b) = d \times b \to c \tag{19}$$

is also unique and therefore

$$\bar{\hat{f}} = f.$$

Briefly, transposition of transposition is the identity.

Thus, the operation $\hat{}$

$$(f : d \times b \to c) \longmapsto (\hat{f} : d \to c^b)$$

provides an inverse to the operation $\bar{}$

$$(g : d \to c^b) \longmapsto (\bar{g} : d \times b \to c).$$

Since for any $f : d \times b \to c$ and $g : d \to c^b$

$$\begin{aligned} \bar{} \circ \hat{}(f) &= \bar{\hat{f}} = f = 1_{\text{Hom}_{\mathbf{C}}(d, c^b)}(f), \\ \hat{} \circ \bar{}(g) &= \hat{\bar{g}} = g = 1_{\text{Hom}_{\mathbf{C}}(d \times b, c)}(g) \end{aligned}$$

we have

$$\text{Hom}_{\mathbf{C}}(d \times b, c) \cong \text{Hom}_{\mathbf{C}}(d, c^b). \tag{20}$$

Definition 2.24. A category $\mathbf{C}$ is called *cartesian closed*, if it has a terminal object, products and exponential.

Example 2.25. The category **Set** of all sets, with functions as arrows is cartesian closed, since

- the terminal object is the singleton set,
- the product $A \times B$ is the cartesian product of A and B,
- the exponential C^B is the set of all functions from B to C and the evaluation function $\text{eval}_{B,C} : C^B \times B \to C$ is defined by $\text{eval}_{B,C}(g, b) = g(b)$ for any $g : B \to C$ and $b \in B$. This evaluation function has the following universal

mapping property (UMP): given any set A and any function $f : A \times B \to C$ there exist a unique function $\tilde{f} : A \to C^B$ defined by $\tilde{f}(a)(b) = f(a, b)$ for all $a \in A$ and $b \in B$ such that $\mathrm{eval}_{B,C} \circ (\tilde{f} \times 1_B) = f$. This holds because $\mathrm{eval}_{B,C}(\tilde{f}(a), b) = \tilde{f}(a)(b) = f(a, b)$.

2.8 Limits and colimits

Definition 2.26. Let **J** and **C** be categories. A *diagram of type* **J** in **C** is a functor

$$D : \mathbf{J} \to \mathbf{C}.$$

We write the objects in the "index category" **J** lower case, i, j,... and the values of the functor $D : \mathbf{J} \to \mathbf{C}$ in the form D_i, D_j, etc.
A *cone* $(c, (c_j)_{j \in \mathbf{J}})$ to a diagram D consists of an object c in **C** and a family of arrows in **C**,

$$c_j : c \to D_j$$

one for each object $j \in \mathbf{J}$, such that for each arrow $\alpha : i \to j$ in **J**, the following triangle commutes:

$$\begin{tikzcd}
c \arrow[r, "c_j"] \arrow[d, "c_i"'] \arrow[dr, phantom, "\#" near start] & D_j \\
D_i \arrow[ur, "D_\alpha"'] &
\end{tikzcd}$$

i.e.,

$$D_\alpha \circ c_i = c_j.$$

A *morphism of cones*

$$\vartheta : (c, c_j) \to (c', c'_j)$$

is an arrow ϑ in **C** making each triangle,

$$\begin{tikzcd}
c \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] \arrow[dr, phantom, "\#" near end, shift left=3] & c' \arrow[d, "c'_j"] \\
& D_j
\end{tikzcd}$$

commute. That is, such that $c_j = c'_j \circ \vartheta$ for all $j \in \mathbf{J}$. Thus, we have an evident category

$$\mathbf{Cone}(D)$$

of cones to D.

Definition 2.27. Let $\mathbf{J}$ and $\mathbf{C}$ be categories and $D : \mathbf{J} \to \mathbf{C}$ a functor. The category of cones, written,

$$\mathbf{Cone}(\mathrm{D})$$

is defined as follows.

(i) **Objects**: cones $(c, (c_j)_{j\in\mathbf{J}})$ to a diagram $D : \mathbf{J} \to \mathbf{C}$

(ii) **Arrows**: morphism of cones

(iii) If $(c, (c_j : c \to D_j)_{j\in\mathbf{J}})$, $(c', (c'_j : c' \to D_j)_{j\in\mathbf{J}})$ cones to D and if $\vartheta : (c, (c_j)_{j\in\mathbf{J}}) \to (c', (c'_j)_{j\in\mathbf{J}})$, $\varphi : (c', (c'_j)_{j\in\mathbf{J}}) \to (c'', (c''_j)_{j\in\mathbf{J}})$ arrows in $\mathbf{Cone}$(D), then

$$\varphi \circ \vartheta : (c, (c_j)_{j\in\mathbf{J}}) \to (c'', (c''_j)_{j\in\mathbf{J}})$$

is also an arrow in $\mathbf{Cone}$(D), since

$$\varphi \circ \vartheta : c \to c''$$

is an arrow in C and

$$c''_j \circ (\varphi \circ \vartheta) = (c''_j \circ \varphi) \circ \vartheta \tag{21}$$
$$= c'_j \circ \vartheta \tag{22}$$
$$= c_j. \tag{23}$$

(iv) For every cone $(c, (c_j : c \to D_j)_{j\in\mathbf{J}})$ in $\mathbf{Cone}$(D), there is an identity arrow

$$1_c : (c, (c_j)_{j\in\mathbf{J}}) \to (c, (c_j)_{j\in\mathbf{J}})$$

in $\mathbf{Cone}$(D), since $1_c : c \to c$ is the identity arrow in $\mathbf{J}$ and

$$c_j \circ 1_c = c_j. \tag{24}$$

In (21) we used the associativity of the category $\mathbf{C}$ and in equations (22), (23) we used that φ, ϑ are arrows in $\mathbf{Cone}$(D). In equation (24) we used that 1_c is identity arrow in $\mathbf{C}$ We need to show that the following conditions are satisfied:

(a) Unit: If $\vartheta : (c, (c_j)_{j\in\mathbf{J}}) \to (c'', (c''_j)_{j\in\mathbf{J}})$ in **Cone**(D), then

$$\begin{aligned} c'_j \circ (\vartheta \circ 1_c) &= (c'_j \circ \vartheta) \circ 1_c \\ &= c_j \circ 1_c \\ &= c_j \end{aligned}$$

and

$$\begin{aligned} c'_j \circ (1_{c'} \circ \vartheta) &= (c'_j \circ 1_{c'}) \circ \vartheta \\ &= c'_j \circ \vartheta \\ &= c_j. \end{aligned}$$

(b) Associativity: If $\vartheta : (c, (c_j)_{j\in\mathbf{J}}) \to (c', (c'_j)_{j\in\mathbf{J}}),\ \varphi : (c', (c'_j)_{j\in\mathbf{J}}) \to (c'', (c''_j)_{j\in\mathbf{J}}),\ \psi : (c'', (c''_j)_{j\in\mathbf{J}}) \to (c''', (c'''_j)_{j\in\mathbf{J}})$

$$\begin{aligned} c'''_j \circ (\psi \circ (\varphi \circ \vartheta)) &= c'''_j \circ ((\psi \circ \varphi) \circ \vartheta) && (25) \\ &= (c'''_j \circ (\psi \circ \varphi)) \circ \vartheta && (26) \\ &= c'_j \circ \vartheta && (27) \\ &= c_j && (28) \end{aligned}$$

and

$$\begin{aligned} c'''_j \circ ((\psi \circ \varphi) \circ \vartheta) &= c'''_j \circ (\psi \circ (\varphi \circ \vartheta)) && (29) \\ &= (c'''_j \circ \psi) \circ (\varphi \circ \vartheta) \\ &= c''_j \circ (\varphi \circ \vartheta) && (30) \\ &= c_j. && (31) \end{aligned}$$

Therefore,

$$(\psi \circ (\varphi \circ \vartheta)) = ((\psi \circ \varphi) \circ \vartheta).$$

In equations (27), (30), (28), (31) we used that $\psi \circ \varphi,\ \varphi \circ \vartheta,\ \vartheta$ and φ are arrows in **Cone**(D). In equations (25), (26) , (29) and (25) we used the associativity in **C**.

Definition 2.28. A *limit* for a diagram $D : \mathbf{J} \to \mathbf{C}$ is a terminal object in **Cone**(D). We often denote a limit in the form

$$p_i : \varprojlim_{j\in\mathbf{J}} D_j \to D_i.$$

The definition of a limit therefore says: given any cone $(c, (c_j)_{j \in \mathbf{J}})$ to D, there is a unique arrow $u : c \to \varprojlim_{j \in \mathbf{J}} D_j$ such that for all j in $\mathbf{J}$,

$$p_j \circ u = c_j.$$

Thus, the limiting cone $(\varprojlim_{j \in \mathbf{J}} D_j, (p_j)_{j \in \mathbf{J}})$ can be thought of as the "closest" cone to the diagram D, and indeed any other cone $(c, (c_j)_{j \in \mathbf{J}})$ comes from it just by composing with an arrow at the vertex, namely $u : C \to \varprojlim_{j \in \mathbf{J}} D_j$.

$$\begin{tikzcd}
c \arrow[rrr, dashed, "u"] \arrow[dr, "c_i"'] \arrow[drrrr, "c_j"] & & & \varprojlim_{j \in \mathbf{J}} D_j \arrow[dll, "p_i"'] \arrow[dr, "p_i"] & \\
& D_i \arrow[rrr, "D_\alpha"'] & & & D_j
\end{tikzcd}$$

A *finite limit* is a limit for a diagram on a finite index category $\mathbf{J}$.

Let us consider an examples of a limit.

Example 2.29. Take $\mathbf{J}$ to be the following category:

$$\begin{tikzcd}
i \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & j
\end{tikzcd}$$

A diagram D of type $\mathbf{J}$ in $\mathbf{C}$ looks like

$$\begin{tikzcd}
D_i \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_j
\end{tikzcd}$$

and a cone $(c, (c_j)_{j \in \mathbf{J}})$ to the diagram D consists of an object c in $\mathbf{C}$ and arrows

$$c_i : c \to D_i \quad \text{and} \quad c_j : c \to D_j$$

in $\mathbf{C}$, such that for $\alpha, \beta : i \to j$ in $\mathbf{J}$ the following triangle commutes:

$$\begin{tikzcd}
c \arrow[r, "c_j"] \arrow[d, "c_i"'] & D_j \\
D_i \arrow[ur, shift left, "D_\alpha"] \arrow[ur, shift right, "D_\beta"'] &
\end{tikzcd}$$

i.e.

$$D_\alpha \circ c_i = c_j \quad \text{and} \quad D_\beta \circ c_i = c_j. \tag{32}$$

Thus,

$$D_\alpha \circ c_i = D_\beta \circ c_i.$$

A limit of a diagram $D : \mathbf{J} \to \mathbf{C}$ is a terminal object in **Cone**(D). So, a limit is a cone

$$(\varprojlim_{j\in\mathbf{J}} D_j, (p_j)_{j\in\mathbf{J}}),$$

consisting of an object $\varprojlim_{j\in\mathbf{J}} D_j$ in $\mathbf{C}$ and arrows $p_i : \varprojlim_{j\in\mathbf{J}} D_j \to D_i$, $p_j : \varprojlim_{j\in\mathbf{J}} D_j \to D_j$ in $\mathbf{C}$ such that

$$D_\alpha \circ p_i = p_j \quad \text{and} \quad D_\beta \circ p_i = p_j, \tag{33}$$

that has the following property: given any cone $(c, (c_j)_{j\in\mathbf{J}})$ to D, there is a unique $u : c \to \varprojlim_{j\in\mathbf{J}} D_j$ such that

$$p_i \circ u = c_i \quad \text{and} \quad p_j \circ u = c_j. \tag{34}$$

We can show, that a limit for D is an equalizer of D_α and D_β. An equalizer of $D_\alpha,\ D_\beta$ consists of an object eq in $\mathbf{C}$ and an arrow $p_i : eq \to D_i$ in $\mathbf{C}$, universal such that

$$D_\alpha \circ p_i = D_\beta \circ p_i. \tag{35}$$

That is, given any $c_i : c \to D_i$ with

$$D_\alpha \circ c_i = D_\beta \circ c_i, \tag{36}$$

there is a unique $u : c \to eq$ with $p_i \circ u = c_i$. $\mathbf{C}$ is a category and therefore there exist composite arrows

$$D_\alpha \circ p_i : eq \to p_j \quad \text{and} \quad D_\beta \circ p_i : eq \to p_j.$$

Because of (35) and the fact that $P_j : eq \to D_j$. is an arrow in $\mathbf{C}$, we have

$$p_j = D_\alpha \circ p_i \text{ and } p_j = D_\beta \circ p_i. \tag{37}$$

Therefore, (33) is satisfied in the definition of a limit. So, eq together with arrows $p_i : c \to D_i,\ p_j : c \to D_j$, satisfying (37) is a cone, and therefore an object in the category **Cone**(D). Because of (37) and since

$$p_j \circ u = D_\alpha \circ p_j \circ u \tag{38}$$
$$= D_\alpha \circ c_i$$
$$= c_j, \tag{39}$$

it follows that also the property (34) of a limit is satisfied. In (38) we used (37) and in (39) we use the property (32) of the given cone $(c, (c_j)_{j\in\mathbf{J}})$. Because of (34), $(c, (p_i, p_j)_{j\in\mathbf{J}})$ is a cone. So, $(eq, (p_j)_{j\in\mathbf{J}})$ is a terminal object in **Cone**(D) and therefore $eq \cong \varprojlim_{j\in\mathbf{J}} D_j$. A limit for $D : \mathbf{C} \to \mathbf{C}$ in this example is therefore an equalizer for $D_\alpha,\ D_\beta$.

$$\begin{tikzcd}
c \arrow[rr, dashed, "u"] \arrow[dr, "c_i"'] \arrow[drrr, "c_j"] & & eq \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\
& D_i \arrow[rr, shift left, "D_\alpha"] \arrow[rr, shift right, "D_\beta"'] & & D_j
\end{tikzcd}$$

2.8.1 Colimits

The dual to the idea of limits is the notion of colimits.

Definition 2.30. Let $\mathbf{J}$ and $\mathbf{C}$ be categories and $D : \mathbf{J} \to \mathbf{C}$ a functor.
A *cocone* $(c, (c_j)_{j\in\mathbf{J}})$ *from the base* D consists of an object c in $\mathbf{C}$ and a family of arrows in $\mathbf{C}$

$$(c_j : D_j \to c)_{j\in\mathbf{J}},$$

such that for all $\alpha : i \to j$ in $\mathbf{J}$,

$$c_j \circ D(\alpha) = c_i.$$

A *morphism of cocones* $f : (c, (c_j)) \to (c', (c'_j))$ is an arrow $f : c \to c'$ in $\mathbf{C}$ such that

$$f \circ c_j = c'_j$$

for all $j \in \mathbf{J}$.

This is illustrated in simplified form below:

$$\begin{tikzcd}
\mathbf{J} \arrow[dd, "D"'] & i \arrow[rr, "\alpha"] & & j & \\
& D_i \arrow[rr, "D(\alpha)"] \arrow[dr, "c_i"'] \arrow[drrr, "c'_1"] & & D_j \arrow[dl, "c_j"] \arrow[dr, "c'_j"] & \\
\mathbf{C} & & C \arrow[rr, "f"'] & & C'
\end{tikzcd}$$

Definition 2.31. The category of cocones, written,

$$\mathbf{Cocone}(\mathrm{D})$$

denote the category which has following data:

(i) **Objects**: cocones from the base D,

(ii) **Arrows**: morphism of cocones.

Definition 2.32. A *colimit* for a diagram $D : \mathbf{J} \to \mathbf{C}$ is an initial object in the category $\mathbf{Cocone}(\mathrm{D})$. An initial cocone is a cocone that maps uniquely to any other cocone from D. We write such a colimit in the form

$$p_i : D_i \to \varinjlim_{j \in \mathbf{J}} D_j.$$

The definition of a colimit explicitly says: given any cocone $(c, (c_j)_{j \in \mathbf{J}})$ from the base D, there is a unique arrow $u : \varinjlim_{j \in \mathbf{J}} D_j \to c$ such that for all j in $\mathbf{J}$,

$$u \circ p_j = c_j.$$

$$\begin{tikzcd}
D_i \arrow[rr, "D_\alpha"] \arrow[dr, "c_i"'] \arrow[drrr, "p_i"] & & D_j \arrow[dl, "c_j"] \arrow[dr, "p_j"] & \\
& c & & \varinjlim_{j \in \mathbf{J}} D_j \arrow[ll, dashed, "u"']
\end{tikzcd}$$

2.8.2 Preservation of limits

Definition 2.33. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to *preserve limits of type* $\mathbf{J}$ if, whenever $p_j : L \to D_j$ is a limit for a diagram $D : \mathbf{J} \to \mathbf{C}$; the cone

$$F_{p_j} : F(L) \to F(D_j)$$

is then a limit for the diagram $F(D) : \mathbf{J} \to \mathbf{D}$. Briefly,

$$F(\varprojlim_{j \in \mathbf{J}} D_j) \cong \varprojlim_{j \in \mathbf{J}} F(D_j).$$

A functor that preserves all limits is said to be *continuous*.

Proposition 2.34. A category has all finite limits iff it has finite products and equalizers.

Proof. This proof is not given here. The proof can be found in chapter 5 in *Category Theory* by Awodey. □

Proposition 2.35. The representable functors $\mathrm{Hom}(c, -) : \mathbf{C} \to \mathbf{Set}$ preserve all finite limits.

Proof. Since limits in $\mathbf{C}$ can be constructed from products and equalizers, it suffices to show that $\mathrm{Hom}(c, -)$ preserves products and equalizers.

- Consider a terminal object 1 in $\mathbf{C}$, then for any $c \in \mathrm{C}_0$

  $$\alpha : c \to 1$$

  is unique. Therefore,

  $$\mathrm{Hom}(c, 1) = \{\alpha\}$$

  is a singleton set, and according to example 2.19 $\mathrm{Hom}(c, 1)$ is a terminal object. Thus,

  $$\mathrm{Hom}(c, 1) \cong 1.$$

- Consider a binary product $x \times y$ in $\mathbf{C}$, there is a unique arrow $f : c \to x \times y$ with

  $$pr_x \circ f = c_1 \quad \text{and} \quad pr_y \circ f = c_2. \tag{40}$$

$$
\begin{tikzcd}
\mathbf{C} \arrow[ddd, "\mathrm{Hom}(c,-)"'] & & c \arrow[dl, "c_1"'] \arrow[d, dashed, "f"] \arrow[dr, "c_2"] & \\
& x & x \times y \arrow[l, "pr_x"] \arrow[r, "pr_y"'] & y \\
& & \mathrm{Hom}(c,c) \arrow[dl, "\mathrm{Hom}(c,c_1)"'] \arrow[d, dashed, "\mathrm{Hom}(c,f)"] \arrow[dr, "\mathrm{Hom}(c,c_2)"] & \\
\mathbf{Set} & \mathrm{Hom}(c,x) & \mathrm{Hom}(c,x \times y) \arrow[l, "\mathrm{Hom}(c,pr_x)"] \arrow[r, "\mathrm{Hom}(c,pr_y)"'] & \mathrm{Hom}(c,y)
\end{tikzcd}
$$

Since f is unique we have that

$$\mathrm{Hom}(c, x \times y) = \{f\}$$

is a singleton set. Therefore, according to example 2.19, $\mathrm{Hom}(c, x \times y)$ is a terminal object. So, $\mathrm{Hom}(c, f)$ is an unique arrow. For any $k : c \to c$, we have that

$$
\begin{aligned}
([\mathrm{Hom}(c, pr_x)]_1 \circ [\mathrm{Hom}(c, f)]_1)(k) &= [\mathrm{Hom}(c, pr_x)]_1([\mathrm{Hom}(c, f)]_1(k)) \\
&= [\mathrm{Hom}(c, pr_x)]_1(f \circ k) && (41) \\
&= pr_x \circ (f \circ k) \\
&= (pr_x \circ f) \circ k && (42) \\
&= c_1 \circ k && (43) \\
&= [\mathrm{Hom}(c, c_1)]_1(k).
\end{aligned}
$$

Thus,

$$[\mathrm{Hom}(c, pr_x)]_1 \circ [\mathrm{Hom}(c, f)]_1 = [\mathrm{Hom}(c, c_1)]_1.$$

In equation (41) we used (11), in (42) we used the associativity in $\mathbf{C}$ and in equation (43) we used (40). With similar arguments we have that

$$[\mathrm{Hom}(c, pr_y)]_1 \circ [\mathrm{Hom}(c, f)]_1 = [\mathrm{Hom}(c, c_2)]_1.$$

Therefore, $\mathrm{Hom}(c, x \times y)$ is a Product of $\mathrm{Hom}(c, x)$ and $\mathrm{Hom}(c, y)$ and we can write

$$\mathrm{Hom}(c, x \times y) \cong \mathrm{Hom}(c, x) \times \mathrm{Hom}(c, y).$$

For arbitrary products $\prod_{i \in I} x_i$, one has analogously

$$\mathrm{Hom}(c, \prod_{i \in I} x_i) \cong \prod_{i \in I} \mathrm{Hom}(c, x_i).$$

- Given an equalizer of f and g in $\mathbf{C}$,

$$\begin{tikzcd}
eq \arrow[r, "e"] & x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y \\
c \arrow[u, dashed, "u"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

consider the resulting diagram:

$$\begin{tikzcd}
\mathrm{Hom}(c, eq) \arrow[r, "{[\mathrm{Hom}(c,e)]_1}"] & \mathrm{Hom}(c, x) \arrow[r, shift left, "{[\mathrm{Hom}(c,f)]_1}"] \arrow[r, shift right, "{[\mathrm{Hom}(c,g)]_1}"'] & \mathrm{Hom}(c, y) \\
\mathrm{Hom}(c, c) \arrow[u, dashed, "{[\mathrm{Hom}(c,u)]_1}"] \arrow[ur, "{[\mathrm{Hom}(c,h)]_1}"'] & &
\end{tikzcd}$$

To show that this is an equalizer in **Set**, let $h : c \to x \in Hom(c, x)$ with

$$[\mathrm{Hom}(c, f)]_1(h) = [\mathrm{Hom}(c, g)]_1(h).$$

Then by (11), we have that

$$f \circ h = g \circ h.$$

So, by definition of an equalizer there is a unique $u : c \to eq$ such that

$$e \circ u = h. \tag{44}$$

We have that

$$[\mathrm{Hom}(c, f)]_1 \circ [\mathrm{Hom}(c, e)]_1 = [\mathrm{Hom}(c, g)]_1 \circ [\mathrm{Hom}(c, e)]_1,$$

since

$$
\begin{aligned}
([\mathrm{Hom}(c,f)]_1 \circ [\mathrm{Hom}(c,e)]_1)(u) &= [\mathrm{Hom}(c,f)]_1([\mathrm{Hom}(c,e)]_1(u)) \\
&= [\mathrm{Hom}(c,f)]_1(e \circ u) \\
&= f \circ (e \circ u) \\
&= (f \circ e) \circ u \\
&= (g \circ e) \circ u \qquad (45) \\
&= g \circ (e \circ u) \\
&= [\mathrm{Hom}(c,g)]_1(e \circ u) \\
&= [\mathrm{Hom}(c,g)]_1([\mathrm{Hom}(c,e)]_1(u)) \\
&= ([\mathrm{Hom}(c,g)]_1 \circ [\mathrm{Hom}(c,e)]_1)(u).
\end{aligned}
$$

In equation (45) we used the property of the equalizer of f and g in $\mathbf{C}$. There is given an arrow $[\mathrm{Hom}(c,h)]_1 : \mathrm{Hom}(c,c) \to \mathrm{Hom}(c,x)$ with

$$[\mathrm{Hom}(c,f)]_1 \circ [\mathrm{Hom}(c,h)]_1 = [\mathrm{Hom}(c,g)]_1 \circ [\mathrm{Hom}(c,h)]_1,$$

since for any $k : c \to c$ we have that

$$
\begin{aligned}
[\mathrm{Hom}(c,f)]_1([\mathrm{Hom}(c,h)]_1(k)) &= [\mathrm{Hom}(c,f)]_1(h \circ k) \\
&= f \circ (h \circ k) \\
&= (f \circ h) \circ k \\
&= (g \circ h) \circ k \\
&= g \circ (h \circ k) \\
&= [\mathrm{Hom}(c,g)]_1([\mathrm{Hom}(c,h)]_1(k)).
\end{aligned}
$$

Since $u : c \to eq$ is unique, we have that

$$\mathrm{Hom}(c,eq) = \{u\}$$

is a singleton set. Therefore, according to example 2.19, $\mathrm{Hom}(c,eq)$ is a terminal object. Since $\mathrm{Hom}(c,eq)$ is terminal, $[\mathrm{Hom}(c,u)]_1$ is unique.

For any $k : c \to c$, we have

$$\begin{aligned}
([\mathrm{Hom}(c,e)]_1 \circ [\mathrm{Hom}(c,u)]_1)(k) &= [\mathrm{Hom}(c,e)]_1([\mathrm{Hom}(c,u)]_1(k)) \\
&= [\mathrm{Hom}(c,e)]_1(u \circ k) \\
&= e \circ (u \circ k) \\
&= (e \circ u) \circ k \\
&= h \circ k \qquad (46) \\
&= [\mathrm{Hom}(c,h)]_1(k).
\end{aligned}$$

In equation (46) we used (44). So, there is an unique arrow $[\mathrm{Hom}(c,u)]_1 : \mathrm{Hom}(c,c) \to \mathrm{Hom}(c,eq)$ with

$$[\mathrm{Hom}(c,u)]_1 \circ [\mathrm{Hom}(c,h)]_1 = [\mathrm{Hom}(c,e)]_1.$$

Therefore, $[\mathrm{Hom}(c,e)]_1 : \mathrm{Hom}(c,eq) \to \mathrm{Hom}(c,x)$ is indeed the equalizer of $\mathrm{Hom}(f,x)$ and $\mathrm{Hom}(g,x)$. $\square$

In general, it can be shown that the representable functors $\mathrm{Hom}(c,-) : \mathbf{C} \to \mathbf{Set}$ preserve all limits.

3 The Yoneda lemma

One of the central results dealing with category theory is the Yoneda lemma.

Definition 3.1 (Embeddings)**.** A functor $F : \mathbf{C} \to \mathbf{D}$ is called an *embedding* if it is full, faithful and injective on objects.

Definition 3.2. [The Yoneda Embedding] The *Yoneda embedding* is the functor[3] $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$, taking $a \in \mathrm{C}_0$ to the contravariant representable functor,

$$y_0(a) = y_a = \mathrm{Hom}(-, a) : \mathbf{C}^{op} \to \mathbf{Set}$$

and taking $f : a \to b$ in C_1 to the natural transformation,

$$y_1(f) = \mathrm{Hom}(-, f) : \underbrace{\mathrm{Hom}(-, a)}_{y_a} \Rightarrow \underbrace{\mathrm{Hom}(-, b)}_{y_b}.$$

For any $c \in \mathrm{C}_0$:

$$\begin{aligned} [y_1(f)]_c : \mathrm{Hom}(c, a) &\to \mathrm{Hom}(c, b) \\ (g : c \to a) &\mapsto (f \circ g : c \to b) \end{aligned}$$

Thus,

$$[y_1(f)]_c\,(g) = f \circ g. \tag{47}$$

It is not immediately obvious, that the Yoneda embedding is actually an embedding, but we will show in section 4.1 that this is the case.

Now we will show that y is indeed a covariant functor. Therefore we still need to show the properties 2.3 (a) and (b).

Proof. We need to show that for any $a \in C_0$,

$$y_1(1_a) = 1_{\underbrace{\mathrm{Hom}(a, -)}_{y_0(a)}}$$

and that if $f : a \to b$ and $h : b \to b'$, then

$$y_1(h \circ f) = y_1(h) \circ y_1(f).$$

Taking an argument $g : c \to a$, we have for any $c \in C_0$,

[3] In the literature one can find different equivalent notations for $Fun(\mathbf{C}^{op}, \mathbf{Set}) = \mathbf{Set}^{\mathbf{C}^{op}} = [\mathbf{C}^{op}, \mathbf{Set}]$.

$$
\begin{aligned}
\left[y_1(1_a)\right]_c(g) &= 1_a \circ g \\
&= g \qquad (48)\\
&= 1_{\mathrm{Hom}(c,a)}(g) \\
&= 1_{[\mathrm{Hom}(-,a)]_c}(g)
\end{aligned}
$$

and if $f : a \to b$, $h : b \to b'$ we get

$$
\begin{aligned}
\left[y_1(h \circ f)\right]_c(g) &= (h \circ f) \circ g \\
&= h \circ (f \circ g) \qquad (49)\\
&= h \circ (\left[y_1(f)\right]_c(g)] \qquad (50)\\
&= \left[y_1(h)\right]_c(\left[y_1(f)\right]_c(g)) \\
&= \left[y_1(h)\right]_c \circ \left[y_1(f)\right]_c(g). \qquad (51)
\end{aligned}
$$

Since $c \in C_0$ is arbitrary we showed the desired properties.
In equations (48), (49) we used 2.1(a), (b), in equation (50) we used (47) and in equation (51) we used that $y_1(h) \in Set_1$ is a function. $\square$

Next we show that $y_1(f)$ is a is a natural transformation.

Proof. Therefore, we need to show that for any $h : c \to c'$ we have

$$
\underbrace{\left[\mathrm{Hom}(-,b)\right]_1(h)}_{\left[\mathrm{Hom}(h,b)\right]_1} \circ \left[y_1(f)\right]_{c'} = \left[y_1(f)\right]_c \circ \underbrace{\left[\mathrm{Hom}(-,a)\right]_1(h)}_{\left[\mathrm{Hom}(h,a)\right]_1}. \qquad (52)
$$

We proof equation (52) by taking any arrow $g' : c' \to a$, then

$$
\begin{aligned}
\left[\mathrm{Hom}(h,b)\right]_1(\left[y_1(f)\right]_{c'}(g')) &= \left[\mathrm{Hom}(h,b)\right]_1(f \circ g') \\
&= (f \circ g') \circ h \\
&= f \circ (g' \circ h) \\
&= (\left[y_1(f)\right]_c(g' \circ h) \\
&= (\left[y_1(f)\right]_c(\left[\mathrm{Hom}(h,a)\right]_1(g')).
\end{aligned}
$$
$\square$

3.1 The contravariant Yoneda lemma

If $\mathbf{C}$ is a locally small category then we already know from example 2.5 that each $a \in \mathrm{C}_0$ gives rise to a natural functor to **Set**

$$
\overbrace{\mathrm{Hom}(-,a)}^{y_a} : \mathbf{C}^{op} \to \mathbf{Set},
$$

called contravariant representable functor. Now we are asking, if $F : \mathbf{C}^{op} \to \mathbf{Set}$ is another functor from $\mathbf{C}^{op}$ to $\mathbf{Set}$, what are the "maps" $y_a \Rightarrow F$? So we are asking what natural transformations

$$\begin{tikzcd} \mathbf{C}^{op} \arrow[rr, bend left, "y_a"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\Downarrow"] & & \mathbf{Set} \end{tikzcd}$$

there are? We call the set of such natural transformations

$$\mathrm{Hom}(y_a, F).$$

Given the same preconditions, is there any other way to describe this set? [Leinster, 2014, chapter 4.2] The following lemma gives an answer to this question and is called the Yoneda lemma.

Lemma 3.3 (Yoneda)**.** Let $\mathbf{C}$ be locally small. For any object a in $\mathbf{C}$ and functor F: $\mathbf{C}^{op} \to \mathbf{Set}$, there is an isomorphism

$$e_a^F : \mathrm{Hom}(y_a, F) \;\cong\; F_0(a)$$

where,

$$\mathrm{Hom}(y_a, F) = [Fun(\mathbf{C}^{op}, \mathbf{Set})]_1 \,(y_a, F) = \{\eta : y_a \Rightarrow F\}$$

such that e_a^F is natural in F and in a.

$\mathrm{Hom}(y_a, F)$ and $F_0(a)$ are not only isomorphic for every a and F, but also, this isomorphism is in particular natural in F and in a. What naturality in F and in a means, is defined next in 3.4.

$$\mathrm{Hom}(y_a, \vartheta)$$

Definition 3.4. If $F, G : \mathbf{C}^{op} \to \mathbf{Set}$ functors.

- e_a^F is *Natural in F* means that, given $\vartheta : F \Rightarrow G$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(y_a, F) \arrow[r, "e_a^F"] \arrow[d, "\mathrm{Hom}(y_a,\vartheta)"'] & F_0(a) \arrow[d, "\vartheta_a"] \\ \mathrm{Hom}(y_a, G) \arrow[r, "e_a^G"'] & G_0(a) \end{tikzcd}$$

  where,

$$\begin{aligned} \mathrm{Hom}(y_a, \vartheta) : \mathrm{Hom}(y_a, F) &\to \mathrm{Hom}(y_a, G) \\ (\eta : y_a \Rightarrow F) &\mapsto (\vartheta \circ \eta : y_a \Rightarrow G) \end{aligned}$$

$$\begin{tikzcd} y_a \arrow[r, Rightarrow, "\eta"] \arrow[rr, Rightarrow, bend right, "\vartheta \circ \eta"'] & F \arrow[r, Rightarrow, "\vartheta"] & G \end{tikzcd}$$

Thus,

$$[\mathrm{Hom}(y_a, \vartheta)]\,(\eta) := \vartheta \circ \eta.$$

- e^F_a is *Natural in a* means that, given any $h : a \to b$ in C_1, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(y_a, F) \arrow[rr, "e^F_a"] & & F_0(a) \\
\mathrm{Hom}(y_b, F) \arrow[u, "\mathrm{Hom}(y_1(h),F)"] \arrow[rr, "e^F_b"'] & & F_0(b) \arrow[u, "F_1(h)"']
\end{tikzcd}$$

where,

$$\begin{aligned}\mathrm{Hom}(y_1(h), F) : \mathrm{Hom}(y_b, F) &\to \mathrm{Hom}(y_a, G)\\ (\eta : y_b \Rightarrow F) &\mapsto (\eta \circ (y_1(h)) : y_a \Rightarrow F)\end{aligned}$$

$$\begin{tikzcd}
y_a \arrow[r, Rightarrow, "y_1(h)"] \arrow[rr, Rightarrow, bend right, "\eta\, \circ\, (y_1(h))"'] & y_b \arrow[r, Rightarrow, "\eta"] & F
\end{tikzcd}$$

Thus,

$$[\mathrm{Hom}(y_1(h), F)]\,(\eta) := \eta \circ (y_1(h)). \tag{53}$$

We will proof the Yoneda lemma in the following 3 steps:

I) We define e^F_a and show that e^F_a is a bijection

II) We show e^F_a natural in F

III) We show e^F_a natural in a

Proof. I) First we define e^F_a and show that e^F_a is a bijection:

To define the isomorphism,

$$e^F_a : \mathrm{Hom}(y_a, F) \;\cong\; F_0(a)$$

take $\eta : y_a \Rightarrow F$ and let

$$e^F_a(\eta) := \eta_a(1_a),$$

where $\eta_a : y_a(a) \to F_0(a)$ and therefore $\eta_a(1_a) \in F_0(a)$.

Now we show that e_a^F is a bijection:

$$\begin{aligned} j_a^F : F_0(a) &\to \mathrm{Hom}(y_a, F) \\ x &\mapsto \eta^x : y_a \Rightarrow F \\ \eta^x_c &: \mathrm{Hom}(c,a) \to F_0(c) \end{aligned} \tag{54}$$

For any $g : c \to a$,

$$\begin{aligned} \left[j_a^F(x)\right]_c(g) &= \eta^x_c(g) \\ &:= [F_1(g)](x). \end{aligned} \tag{55}$$

j_a^F is well defined: if $h : c \to d$

$$\begin{tikzcd} \mathrm{Hom}(d,a) \arrow[r, "{[\mathrm{Hom}(-,a)]_1(h)}"] \arrow[d, "\eta^x_d"'] & \mathrm{Hom}(c,a) \arrow[d, "\eta^x_c"] \\ F_0(d) \arrow[r, "F_1(h)"'] & F_0(c) \end{tikzcd}$$

Let $g : d \to a$,

$$\begin{aligned} \eta^x_c([\mathrm{Hom}(h,a)]_1(g)) &= \eta^x_c(g \circ h) \\ &= [F_1(g \circ h)](x) \\ &= [F_1(h) \circ F_1(g)](x) \qquad (56) \\ &= [F_1(h)]([F_1(g)](x)) \qquad (57) \\ &= [F_1(h)](\eta^x_d(g)). \end{aligned}$$

In equation (56) we used (6) and in equation (57) we used that
$F_1(h), F_1(h) \in \mathrm{Set}_1$ are functions.

Now, we show that

$$\begin{tikzcd} F_0(a) \arrow[r, "j_a^F"] \arrow[rr, bend right, "e_a^F \circ j_a^F"'] & \mathrm{Hom}(y_a, F) \arrow[r, "e_a^F"] \arrow[rr, bend left, "j_a^F \circ e_a^F"] & F_0(a) \arrow[r, "j_a^F"] & \mathrm{Hom}(y_a, F) \end{tikzcd}$$

$$e_a^F \circ f_a^F = 1_{F_0(a)} \tag{58}$$

and

$$j_a^F \circ e_a^F = 1_{\mathrm{Hom}(y_a,F)}. \tag{59}$$

We show (58), by taking any $x \in F_0(a)$ and $\eta^x : y_a \Rightarrow F$, so we have

$$\begin{aligned}
e_a^F(j_a^F(x)) &= e_a^F(\eta^x) \\
&= (\eta_a^x)(1_a) \\
&= [F_1(1_a)](x) \qquad (60)\\
&= 1_{F_0(a)}(x) \qquad (61)\\
&= \mathrm{id}_{F_0(a)}(x)
\end{aligned}$$

and we show (59), by taking any $\eta : y_a \Rightarrow F$ and $c \in C_0$, then we have

$$\begin{aligned}
\left[j_a^F(e_a^F(\eta))\right]_c &= \left[j_a^F(\eta_a(1_a))\right]_c \qquad (62)\\
&= \eta_c^{\eta_a(1_a)} \\
&= \eta_c. \qquad (63)
\end{aligned}$$

Since $c \in C_0$ was chosen arbitrarily, we have

$$\begin{aligned}
j_a^F(e_a^F(\eta)) &= \eta \\
&= 1_{\mathrm{Hom}(y_a,F)}(\eta).
\end{aligned}$$

In equations (60), (62) we used (1), we used (5) in equation (61) and the equation above (66) gives us (63).

Let $\eta : y_a \Rightarrow F$, then

$$\begin{aligned}
\eta_c^{\eta_a(1_a)}(g) &= [F_1(g)]]\,(\eta_a(1_a)) \\
&= (F_1(g) \circ \eta_a)(1_a) \\
&= (\eta_c \circ [\mathrm{Hom}(g,a)]_1)(1_a) \qquad (64)\\
&= \eta_c([\mathrm{Hom}(g,a)]_1\,(1_a)) \qquad (65)\\
&= \eta_c(1_a \circ g) \\
&= \eta_c(g). \qquad (66)
\end{aligned}$$

In equation (64) we used that η_c, $[\mathrm{Hom}(-,a)]_1$ are functions and in equation (65) we used (67).

If $\eta : y_a \Rightarrow F$, then by definition of a natural transformation holds that for any $g : c \to a$ following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(a,a) \arrow[r, "{[\mathrm{Hom}(-,a)]_1(g)}"] \arrow[d, "\eta_a"'] & \mathrm{Hom}(c,a) \arrow[d, "\eta_c"] \\
F_0(a) \arrow[r, "F_1(g)"'] & F_0(c)
\end{tikzcd}$$

Thus,

$$\eta_c \circ ([\mathrm{Hom}(-,a)]_1\,(g)) = F_1(g) \circ \eta_a. \tag{67}$$

II) We need to show, that e_a^F is natural in F:
Let $F, F' : \mathbf{C}^{op} \to \mathbf{Set}$ and $\phi : F \Rightarrow F'$. We need to show, that the following diagram commutes i.e.,

$$\begin{tikzcd}
\mathrm{Hom}(y_a,F) \arrow[r, "e_a^F"] \arrow[d, "{\mathrm{Hom}(y_a,\phi)}"'] & F_0(a) \arrow[d, "\phi_a"] \\
\mathrm{Hom}(y_a,F') \arrow[r, "e_a^{F'}"'] & F_0'(a)
\end{tikzcd}$$

$$\phi_a \circ e_a^F = e_a^{F'} \circ [\mathrm{Hom}(y_a,\phi)]\,.$$

Let $\eta : y_a \Rightarrow F$ and $\phi \circ \eta : y_a \Rightarrow F'$, then

$$\begin{aligned}
\phi_a \circ e_a^F(\eta) &= \phi_a(e_a^F(\eta)) && (68)\\
&= \phi_a(\eta_a(1_a)) \\
&= \phi_a \circ \eta_a(1_a) \\
&= (\phi \circ \eta)_a(1_a) && (69)\\
&= e_a^{F'}(\phi \circ \eta) \\
&= e_a^{F'}([\mathrm{Hom}(y_a,\phi)]\,(\eta)) && (70)\\
&= e_a^{F'} \circ [\mathrm{Hom}(y_a,\phi)]\,(\eta). && (71)
\end{aligned}$$

In equations (68), (71) we used that ϕ, e_a^F, y_a are functions, in equation (69) we used (8) and in equation (70) we used (53).

III) In the last part of the proof we show, that e_a^F is natural in a: we need to show that for any $f : a' \to a \in \mathrm{C}_1$ the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Hom}(y_{a'},F) \arrow[r, "e_{a'}^F"] & F_0(a') \\
\mathrm{Hom}(y_a,F) \arrow[r, "e_a^F"'] \arrow[u, "{\mathrm{Hom}(y_1(f),F)}"] & F_0(a), \arrow[u, "F_1(f)"']
\end{tikzcd}$$

i.e.,

$$e^F_{a'} \circ \mathrm{Hom}(y_1(f), F) = F_1(f) \circ e^F_a.$$

If $f : a' \to a,\ \eta : y_a \Rightarrow F$ and $y_1(f) : y_{a'} \Rightarrow y_a$ then we have

$$\begin{aligned}
e^F_{a'} \circ \mathrm{Hom}(y_1(f), F)(\vartheta) &= e^F_{a'}(\mathrm{Hom}(y_1(f), F)(\vartheta)) \\
&= e^F_{a'}(\vartheta \circ y_1(f)) \\
&= [\vartheta \circ y_1(f)]_{a'}(1_{a'}) \\
&= \vartheta_{a'} \circ [y_1(f)]_{a'}(1_{a'}) \\
&= \vartheta_{a'}([y_1(f)]_{a'}(1_{a'})) && (72) \\
&= \vartheta_{a'}(f \circ 1_{a'}) \\
&= \vartheta_{a'}(f) \\
&= \vartheta_{a'}(1_a \circ f) \\
&= \vartheta_{a'}([\mathrm{Hom}(f, a)]_1(1_a)) && (73) \\
&= F_1(f)(\vartheta_a(1_a)) && (74) \\
&= F_1(f)(e^F_a(\vartheta)) \\
&= F_1(f) \circ e^F_a(\vartheta). && (75)
\end{aligned}$$

In equations (72), (75) we used that $y_1(f), \vartheta$ and $F_1(f), e^F_a$ are functions. In equation (73) we used (14) and in equation (74) we used (76).

If $\vartheta : y_a \Rightarrow F$, by definition of a natural transformation the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Hom}(a, a) \arrow[r, "{[\mathrm{Hom}(-,a)]_1(f)}"] \arrow[d, "\vartheta_a"'] & \mathrm{Hom}(a', a) \arrow[d, "\vartheta_{a'}"] \\
F_0(a) \arrow[r, "F_1(f)"'] & F_0(a')
\end{tikzcd}$$

that means

$$F_1(f) \circ \vartheta_a = \vartheta_{a'} \circ [\mathrm{Hom}(-, a)]_1(f). \quad (76)$$

□

Remark 3.5. If X, Y are sets, we define

$$\mathbb{F}(X, Y) = \{f \mid f : X \to Y\}.$$

If X and Y are sets, then

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\} \text{ and } \mathcal{P}(X \times Y) \text{ are sets.}$$

Since $\mathbb{F}(X,Y) \subseteq \mathcal{P}(X \times Y)$, we have that

$$\mathbb{F}(X,Y) \text{ is a set.} \tag{77}$$

Given functors $F, G : \mathbf{C}^{op} \to \text{Set}$, a natural transformation $\eta : F \Rightarrow G$ is family or arrows in **Set** of the form

$$(\eta_a : F_0(a) \to G_0(a))_{a \in \mathrm{C}_0},$$

satisfying (7). We have that

$$\eta_a \in \mathbb{F}(F_0(a), G_0(a)).$$

Since $F_0(a), G_0(a)$ are sets, then by (77) $\{\ \mathbb{F}(F_0(a), G_0(a)) \mid a \in \mathrm{C}_0\}$ is a set, and

$$\underbrace{\mathrm{Hom}(F,G)}_{\{\eta : F \Rightarrow G\}} \subseteq \bigcup_{a \in \mathrm{C}_0} \mathbb{F}(F_0(a), G_0(a)). \tag{78}$$

For functors $F,\ y_a : \mathbf{C}^{op} \to \text{Set}$ and any $a \in \mathrm{C}_0$, notice that:

- If $\mathbf{C}$ is small, that means C_0 is a set and $\mathrm{C}_1(a,b) = \{f \in \mathrm{C}_1 \mid \mathrm{dom}(f) = a, \mathrm{cod}(f) = b\}$ is a set, then by (77) we have that

$$\mathbb{F}(y_a, F_0(a))$$

is a set and since C_0 is a set,

$$\bigcup_{a \in \mathrm{C}_0} \mathbb{F}(y_a, F_0(a))$$

is a set. Hence, $\mathrm{Hom}(y_a, F)$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ is a set, because of (78).

- If $\mathbf{C}$ is locally small, then C_0 is not a set, therefore

$$\bigcup_{a \in \mathrm{C}_0} \mathbb{F}(y_a, F_0(a))$$

is not a set. In this case, we can apply the Yoneda lemma, that tells us that $\mathrm{Hom}(y_a, F)$ is always a set. In particular, the Yoneda Lemma shows that the natural transformations between the functors y_a and F form a set, since the class of natural transformations between y_a and F is bijectively related to a set, namely F(a), and therefore form itself a set.

- If $\mathbf{C}$ is not locally small, then $y \to \mathbf{Set}^{\mathbf{C}^{op}}$ will not even be defined, so the

Yoneda lemma does not apply.

3.2 The covariant Yoneda lemma

Lemma 3.6 (Yoneda)**.** Let $\mathbf{C}$ be locally small. For any object $a \in C_0$ and functor F: $\mathbf{C} \to \mathbf{Set}$ there is an isomorphism

$$
{}^{F}_{a}e : \mathrm{Hom}({}_{a}y, F) \;\cong\; F_0(a)
$$
$$
\mathrm{Hom}({}_{a}y, F) = [Fun(\mathbf{C}^{op}, \mathbf{Set})]_1\,({}_{a}y, F) = \{\eta : {}_{a}y \Rightarrow F\}
$$

such that ${}^{F}_{a}e$ is natural in F and in a.

We proof the covaraint Yoneda lemma in the same 5 steps similar to the proof of the contravariant Yoneda lemma.

Proof. i) Let C be a locally small category. We begin the proof by showing that that $y : \mathbf{C}^{op} \to \mathbf{Set}^{\mathbf{C}}$ is a contravariant functor, taking any object $a \in C_0$ to the covariant representable functor,

$$
y_0(a) = {}_{a}y = \mathrm{Hom}(a, -) : \mathbf{C} \to \mathbf{Set}
$$

and taking any $f : a \to b$ in C_1 to the natural transformation,

$$
y_1(f) = \mathrm{Hom}(f, -) : \underbrace{\mathrm{Hom}(b, -)}_{{}_{b}y} \Rightarrow \underbrace{\mathrm{Hom}(a, -)}_{{}_{a}y}
$$

i.e. for any $c \in C_0$,

$$
\begin{aligned}
[y_1(f)]_c : \mathrm{Hom}(b, c) &\to \mathrm{Hom}(a, c) \\
(g : b \to c) &\mapsto (g \circ f : a \to c).
\end{aligned}
$$

Thus,

$$
[y_1(f)]_c\,(g) = g \circ f. \tag{79}
$$

Therefore, we need to show that

$$
y_1(1_a) = 1_{\mathrm{Hom}(a,-)}
$$

and that

$$
y_1(g \circ f) = y_1(f) \circ y_1(g).
$$

For any $c \in C_0$, taking an argument $g : c \to a$, we have

$$\begin{aligned}
[y_1(1_a)]_c(g) &= g \circ 1_a \\
&= g \\
&= 1_{\mathrm{Hom}(a,c)}(g) \\
&= [1_{[\mathrm{Hom}(a,-)]}]_c(g)
\end{aligned}$$

and if $f : a \to b$, $h : b \to b'$ we get

$$\begin{aligned}
[y_1(h \circ f)]_c(g) &= g \circ (h \circ f) \\
&= (g \circ h) \circ f \\
&= [y_1(f)]_c(g \circ h) \\
&= [y_1(f)]_c([y_1(h)]_c(g)) \\
&= [y_1(f)]_c \circ [y_1(h)]_c(g).
\end{aligned}$$

Since $c \in \mathbf{C}_0$ is arbitrary we showed the desired properties.

ii) $y_1(f)$ is a natural transformation similar to equation (52).

iii) The isomorphism is given by ${}^{F}_{a}e := \eta(1_a)$.

iv) ${}^{F}_{a}e$ is natural in F.

v) ${}^{F}_{a}e$ is natural in a.

The proof of iv) and v) is similar to the proof of iv) and v) of the covariant Yoneda lemma. Therefore the proof will not be repeated at this point. $\square$

4 Applications of the Yoneda lemma

4.1 The Yoneda theorem

The next theorem tells us that y is an embedding, which we will prove with the help of the Yoneda lemma.

Theorem 4.1 (Yoneda theorem)**.** For $\mathbf{C}$ locally small, $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ is injective on objects, full and faithful.

Proof. First we show that $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ is injective on objects:
we need to show, that for $y_a, y_b : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ following holds:

$$y_a = y_b \Rightarrow a = b,$$

where $\quad (y_a)_0\,(a) = \mathrm{Hom}(a, a) \;$ and $\; (y_b)_0\,(a) = \mathrm{Hom}(a, b)$.

$$\begin{aligned} \mathrm{Hom}(a, a) = \mathrm{Hom}(a, b) &\Rightarrow \Big(1_a \in \mathrm{Hom}(a, a) \Rightarrow 1_a \in \mathrm{Hom}(a, b)\Big) \\ &\Leftrightarrow \; a = \mathrm{dom}(1_a) = b \\ &\Rightarrow \; a = b \end{aligned}$$

Next, we show that $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ is full and faithful:

If $a, b \in C_0$, we show that

$$\begin{aligned} y_{ab} : \; & \mathrm{Hom}_{\mathbf{C}}(a, b) \to \mathrm{Hom}(y_a, y_b) \\ & (f : a \to b) \mapsto (y_1(f) : y_a \Rightarrow y_b) \end{aligned}$$

Thus, for any $c \in C_0$ and $g : c \to a$:

$$[y_{ab}(f)]_c\,(g) = [y_1(f)]_c\,(g) = f \circ g.$$

We know from the Yoneda lemma that

$$j_a^{y_b} : \underbrace{y_b(a)}_{\mathrm{Hom}(a,b)} \to \mathrm{Hom}(y_a, y_b)$$

is a bijection, and we can show that $y_{ab} = j_a^{y_b}$:

$$\begin{aligned} [j_a^{y_b}(f)]_c\,(g) &= \eta^f(g) \\ &= [\mathrm{Hom}(c, f)]_1\,(g) \\ &= f \circ g \\ &= [y_1(f)]_c\,(g) \\ &= [y_{ab}(f)]_c\,(g) \end{aligned}$$

$$\Rightarrow [j_a^{y_b}(f)]_c = [y_{ab}(f)]_c$$
$$\Rightarrow j_a^{y_b}(f) = y_{ab}(f)$$
$$\Rightarrow j_a^{y_b} = y_{ab}.$$

We showed that y_{ab} is injective and surjective and therefore y is full and faithful.

□

Theorem 4.2 (Co-Yoneda theorem)**.** For a locally small category $\mathbf{C}$, the contravariant functor $y : \mathbf{C}^{op} \to \mathbf{Sets}^{\mathbf{C}}$ is an embedding.

Proof. The co-Yoneda theorem can be proven analogously to the Yoneda theorem in 4.1. □

Corollary 4.3. If F: $\mathbf{C} \to \mathbf{D}$ is full and faithful, then for any objects a, b in $\mathbf{C}$

$$F_0(a) \cong F_0(b) \;\Leftrightarrow\; a \cong b.$$

Proof. Suppose $F_0(a) \cong F_0(b)$ i.e., for any arrow $p : F_0(a) \to F_0(b)$ there exist an arrow $q : F_0(b) \to F_0(a),$ such that

$$p \circ q = 1_{F_0(b)} \text{ and } q \circ p = 1_{F_0(a)}.$$

Because F is full, there is an arrow $f : a \to b$ so that $F_1(f) = p$. Similarly, there is an arrow $g : b \to a$ so that $F_1(g) = q$. Then

$$\begin{aligned} 1_{F_0(a)} &= q \circ p \\ &= F_1(g) \circ F_1(f) \\ &= F(g \circ f). \end{aligned} \tag{80}$$

But also

$$F_1(1_a) = 1_{F_0(a)}. \tag{81}$$

Since F is faithful,

$$F_1(g \circ f) = F_1(1_a) \text{ implies } g \circ f = 1_a.$$

In equation (80) we used the property (4) and in (81) we used the property (3).

A similar argument shows that $g \circ f = \mathrm{id}_b$ and therefore f is an isomorphism. The other direction follows because F is a functor: if a and b are isomorphic, then so are $F_0(a)$ and $F_0(b)$. □

4.2 The Yoneda principle

Corollary 4.4 (Yoneda principle)**.** Given objects a and b in any locally small category $\mathbf{C}$,

$$y_a \cong y_b \;\Leftrightarrow\; a \cong b.$$

Remark 4.5. Since functors always preserve isomorphism, the power of this statement lies in the implication

$$y_a \cong y_b \;\Rightarrow\; a \cong b.$$

In other words, if $\mathrm{Hom}(x, a) \cong \mathrm{Hom}(x, b)$ naturaly in x, then $a \cong b$.

Proof. Since the Yoneda embedding y is full and faithful by theorem 4.1, the proof follows immediately from corollary 4.3. □

Corollary 4.6. Given objects a, b and c in a cartesian closed category $\mathbf{C}$,

$$(a^b)^c \cong a^{(b\times c)}.$$

Proof. Because of corollary 4.4, it is sufficient to proof that the following isos are natural in x:

$$\begin{aligned}
\mathrm{Hom}(x, (a^b)^c) &\cong \mathrm{Hom}(x \times c, a^b) && (82)\\
&\cong \mathrm{Hom}((x \times c) \times b, a) && (83)\\
&\cong \mathrm{Hom}(x \times (c \times b), a) && (84)\\
&\cong \mathrm{Hom}(x \times (b \times c), a) && (85)\\
&\cong \mathrm{Hom}(x, a^{b\times c}). && (86)
\end{aligned}$$

First we show that lines (82)-(86) are indeed isomorphic. The isomorphisms in (82), (83) and (86) follow from (20). Now, we show the isomorphism in (84):

$$\begin{aligned}
h_x : \mathrm{Hom}((x \times c) \times b, a) &\to \mathrm{Hom}(x \times (c \times b), a)\\
(\, g : (x \times c) \times b \to a) &\mapsto (\, \widetilde{g} \, : x \times (c \times b) \to a)\\
h_x(g) = &\; \widetilde{g} \;, && (87)
\end{aligned}$$

where $\widetilde{g}$ is defined in the following way: given any $g : (x \times c) \times b \to a$

$$\begin{tikzcd}
 & x \times (c \times b) \arrow[dl, "(\;\widetilde{\;}\;)_x"'] \arrow[d, "\widetilde{g}"] \\
(x \times c) \times b \arrow[r, "g"'] & a
\end{tikzcd}$$

and an unique arrow

$$(\widetilde{\ })_x : \ x \times (c \times b) \to (x \times c) \times b,$$

we have that

$$g \circ (\widetilde{\ })_x = \widetilde{g}$$

is unique.

$$\begin{aligned} h_x^{-1} &: \mathrm{Hom}(x \times (c \times b), a) \to \mathrm{Hom}((x \times c) \times b, a) \\ (\widetilde{k} &: x \times (c \times b) \to a) \mapsto (k : (x \times c) \times b \to a) \\ & h_x^{-1}(\widetilde{k}) = k, \end{aligned} \tag{88}$$

where k is defined in the following way: given any $\widetilde{k} : x \times (c \times b) \to a$

$$\begin{tikzcd} & x \times (c \times b) \arrow[d, "\widetilde{k}"] \\ (x \times c) \times b \arrow[ur, "(\widetilde{\ })_x^{-1}"] \arrow[r, "k"'] & a \end{tikzcd}$$

and an unique arrow

$$(\widetilde{\ })_x^{-1} : (x \times c) \times b \to x \times (c \times b),$$

we have that

$$k = \widetilde{k} \circ (\widetilde{\ })_x^{-1}$$

is unique.

To show that h_x is an isomorphism, we show that

$$(h_x \circ h_x^{-1})(\widetilde{k}) = (h_x(h_x^{-1}((\widetilde{k})) \tag{89}$$
$$= h_x(k) \tag{90}$$
$$= \widetilde{k} \tag{91}$$
$$= 1_{\mathrm{Hom}(x \times (c \times b), a)}(\widetilde{k})$$

and

$$
\begin{aligned}
(h_x^{-1} \circ h_x)(g) &= h_x^{-1}(h_x(g)) && (92)\\
&= h_x^{-1}(\widetilde{g}) && (93)\\
&= g && (94)\\
&= 1_{\mathrm{Hom}((x \times c) \times b, a)}(g).
\end{aligned}
$$

In equations (89),(92) we used that $h_x, h_x^{-1} \in \mathrm{Set}_1$. Furthermore in equations (90),(94) we used (88) and in equations (91),(93) we used (87).

$$
\begin{aligned}
&\Rightarrow\ h_x \circ h_x^{-1} = 1_{\mathrm{Hom}(x \times (c \times b), a)} \ \text{ and } \ h_x^{-1} \circ h_x = 1_{\mathrm{Hom}((x \times c) \times b, a)}.\\
&\Rightarrow\ h_x \text{ is an iso.}\\
&\Rightarrow\ \mathrm{Hom}((x \times c) \times b, a) \cong \mathrm{Hom}(x \times (c \times b), a).
\end{aligned}
$$

Now, we show the isomorphism in (85). We define

$$
\begin{aligned}
t_x : \mathrm{Hom}(x \times (c \times b), a) &\to \mathrm{Hom}(x \times (b \times c), a)\\
(g : x \times (c \times b)) \to a) &\mapsto (g^* : x \times (b \times c) \to a)\\
t_x(g) &= g^*, && (95)
\end{aligned}
$$

where g^* is defined in the following way: given any $g : x \times (c \times b) \to a$,

$$
\begin{tikzcd}
 & x \times (b \times c) \arrow[dl, "(\ ^*\ )_x"'] \arrow[d, "g^*"] \\
x \times (c \times b) \arrow[r, "g"'] & a
\end{tikzcd}
$$

and an unique

$$(\ ^*\ )_x : x \times (b \times c) \to x \times (c \times b),$$

we have that

$$g^* = g \circ (\ ^*\ )_x$$

is unique.

$$
\begin{aligned}
t_x^{-1} : \mathrm{Hom}(x \times (b \times c), a) &\to \mathrm{Hom}(x \times (c \times b), a) \\
(k^* : x \times (b \times c) \to a) &\mapsto (k : x \times (c \times b) \to a) \\
t_x^{-1}(k^*) &= k, \qquad (96)
\end{aligned}
$$

where k is defined in the following way: given any $k^* : x \times (b \times c) \to a$,

$$
\begin{tikzcd}
 & x \times (b \times c) \arrow[d, "k^*"] \\
x \times (c \times b) \arrow[ur, "(\ ^*\ )_x^{-1}"] \arrow[r, "k"'] & a
\end{tikzcd}
$$

and an unique arrow

$$(\ ^*\ )_x^{-1} : x \times (c \times b) \to x \times (b \times c),$$

we have that

$$k = k^* \circ (\ ^*\ )_x^{-1}$$

is unique.

To show that t_x is an isomorphism, we show that

$$
\begin{aligned}
(t_x \circ t_x^{-1})(k^*) &= t_x(t_x^{-1}(k^*)) && (97) \\
&= t_x(k) && (98) \\
&= k^* && (99) \\
&= 1_{\mathrm{Hom}(x \times (b \times c), a)}(k^*)
\end{aligned}
$$

and

$$
\begin{aligned}
(t_x^{-1} \circ t_x)(g) &= t_x^{-1}(t_x(g)) && (100) \\
&= t_x^{-1}(g^*) && (101) \\
&= g && (102) \\
&= 1_{\mathrm{Hom}(x \times (c \times b), a)}(g).
\end{aligned}
$$

In equations (97), (100) we used that $t_x, t_x^{-1} \in \mathrm{Set}_1$. Furthermore in equations

(99), (101) we used (95) and in (98),(102) we used (96).

$$\Rightarrow\ t_x \circ t_x^{-1} = 1_{\mathrm{Hom}(x\times(b\times c),a)} \text{ and } t_x^{-1}\circ t_x = 1_{\mathrm{Hom}(x\times(c\times b),a)}.$$
$$\Rightarrow\ t_x \text{ is an iso.}$$
$$\Rightarrow\ \mathrm{Hom}(x\times(c\times b),a) \cong \mathrm{Hom}(x\times(b\times c),a).$$

So, we showed that lines (82)-(86) are isomorphic. It is also necessary to check that these isomorphisms are natural in x. Therefore, in (82) we need to show that, $\mathrm{Hom}(-,(a^b)^c) \Rightarrow \mathrm{Hom}(-\times c,(a^b))$ is a natural transformation. That means, we need to show that for every $f : x' \to x$ in C_1 the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Hom}(x,(a^b)^c) \arrow[r, "(\,\bar{}\,)_x"] \arrow[d, "{[\mathrm{Hom}(f,(a^b)^c)]_1}"'] & \mathrm{Hom}(x\times c, a^b) \arrow[d, "{[\mathrm{Hom}(f\times 1_c,(a^b)^c)]_1}"] \\
\mathrm{Hom}(x',(a^b)^c) \arrow[r, "(\,\bar{}\,)_{x'}"'] & \mathrm{Hom}(x'\times c, a^b)
\end{tikzcd}$$

i.e.

$$[\mathrm{Hom}(f\times 1_c,(a^b)^c)]_1 \ \circ\ (\bar{\ })_x \ = \ (\bar{\ })_{x'} \ \circ\ [\mathrm{Hom}(f,(a^b)^c)]_1,$$

where the symbol $\bar{\ }$ denotes the transposition defined in (18).
Similar to (14) any arrow $f : x' \to x$ in C_1 induces a function

$$\begin{aligned}
[\mathrm{Hom}(f,(a^b)^c)]_1 : \mathrm{Hom}(x,(a^b)^c) &\to \mathrm{Hom}(x',(a^b)^c) \\
(g : x \to (a^b)^c) &\mapsto (g\circ f : x' \to (a^b)^c)
\end{aligned}$$

Thus,

$$[\mathrm{Hom}(f,(a^b)^c)]_1(g) = g\circ f, \tag{103}$$

and $f\times 1_c : x'\times c \to x\times c$ in C_1 induces a function

$$\begin{aligned}
\mathrm{Hom}(f\times 1_c,(a^b)^c)]_1 : \mathrm{Hom}(x\times c,a^b) &\to \mathrm{Hom}(x'\times c,a^b) \\
(\bar{g} : x\times c \to a^b) &\mapsto (\bar{g}\circ(f\times 1_c) : x'\times c \to a^b)
\end{aligned}$$
$$[\mathrm{Hom}(f\times 1_c,(a^b)^c)]_1(\bar{g}) = \bar{g}\circ(f\times 1_c). \tag{104}$$

Therefore we need to show that,

$$[\mathrm{Hom}(f\times 1_c,(a^b)^c)]_1(\bar{g})_x = (\overline{[\mathrm{Hom}(f,(a^b)^c)]_1(g)})_{x'}.$$

Because of (103), (104) this is equivalent to

$$\bar{g} \circ (f \times 1_c) = \overline{g \circ f}.$$

The situation is as follows,

$$\begin{tikzcd}
(a^b)^c & & (a^b)^c \times c \arrow[rr, "eval_{c,a^b}"] & & a^b \\
x \arrow[u, "f"'] & & x \times c \arrow[u, "g \times 1_c"] \arrow[urr, "\bar{g}"'] & & \\
x' \arrow[u, "g"'] \arrow[uu, bend left, "g \circ f"] & & x' \times c. \arrow[u, "f \times 1_c"] \arrow[uu, bend left, "(g \circ f) \times 1_c"] \arrow[uurr, bend right, "\overline{g \circ f}"'] & &
\end{tikzcd}$$

From the statement in (19) follows that $\overline{g \circ f}$ is an unique arrow and since $\mathbf{C}$ is a category the composition

$$\bar{g} \circ (f \times 1_c) = \overline{g \circ f}.$$

So we showed that $\mathrm{Hom}(-, (a^b)^c) \Rightarrow \mathrm{Hom}(- \times c, (a^b))$ is a natural transformation.

In a similar way, we can show that $\mathrm{Hom}(- \times c, a^b) \Rightarrow \mathrm{Hom}((- \times x) \times b, a)$ and $\mathrm{Hom}((- \times (b \times c), a) \Rightarrow \mathrm{Hom}(-, a^{b \times c})$ is a natural transformation . Therefore, the isomorphisms in (83) and (86) are natural in x.

In equation (84) we need to show that, $\mathrm{Hom}((- \times c) \times b, a) \Rightarrow \mathrm{Hom}(- \times (c \times b), a)$ are natural transformations. Suppose we have $f : x' \to x$. We need to show, that the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Hom}((x \times c) \times b, a) \arrow[r, "h_x"] \arrow[d, "{[\mathrm{Hom}((f \times 1_c) \times 1_b, a)]_1}"'] & \mathrm{Hom}(x \times (c \times b), a) \arrow[d, "{[\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1}"] \\
\mathrm{Hom}((x' \times c) \times b, a) \arrow[r, "h_{x'}"'] & \mathrm{Hom}(x' \times (c \times b), a)
\end{tikzcd}$$

i.e.

$$h_{x'} \circ [\mathrm{Hom}((f \times 1_c) \times 1_b, a)]_1 = [\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1 \circ h_x. \qquad (105)$$

Similar to (14) any arrow $(f \times 1_c) \times 1_b : (x' \times c) \times b \to (x \times c) \times b$ in C_1 induces a function

$$\begin{aligned}[\mathrm{Hom}((f\times 1_c)\times 1_b, a)]_1 : \mathrm{Hom}((x\times c)\times b, a) &\to \mathrm{Hom}((x'\times c)\times b, a)\\ (g:(x\times c)\times b\to a) &\mapsto (g\circ((f\times 1_c)\times 1_b):(x'\times c)\times b\to a)\\ [\mathrm{Hom}((f\times 1_c)\times 1_b, a)]_1(g) &= g\circ((f\times 1_c)\times 1_b) \qquad (106)\end{aligned}$$

and any arrow $f\times(1_c\times 1_b) : x'\times(c\times b)\to x\times(c\times b)$ induces a function

$$\begin{aligned}[\mathrm{Hom}(f\times(1_c\times 1_b), a)]_1 : \mathrm{Hom}(x\times(c\times b), a) &\to \mathrm{Hom}(x'\times(c\times b), a)\\ (\widetilde{g} : x\times(c\times b)\to a) &\mapsto (\widetilde{g}\circ(f\times(1_c\times 1_b)) : x'\times(c\times b)\to a)\\ [\mathrm{Hom}(f\times(1_c\times 1_b), a)]_1(\widetilde{g}) &= \widetilde{g}\circ(f\times(1_c\times 1_b)). \qquad (107)\end{aligned}$$

Now, we show equation (105): given any $g:(x\times c)\times b\to a$

$$\begin{aligned}(h_{x'}\circ[\mathrm{Hom}((f\times 1_c)\times 1_b, a)]_1)(g) &= h_{x'}([\mathrm{Hom}((f\times 1_c)\times 1_b, a)]_1(g)) && (108)\\ &= h_{x'}(g\circ((f\times 1_c)\times 1_b)) && (109)\\ &= \widetilde{g\circ((f\times 1_c)\times 1_b)} && (110)\\ &= \widetilde{g}\circ\widetilde{((f\times 1_c)\times 1_b)} && (111)\\ &= \widetilde{g}\circ(f\times(1_c\times 1_b)) && (112)\\ &= [\mathrm{Hom}(f\times(1_c\times 1_b), a)]_1(\widetilde{g}) && (113)\\ &= [\mathrm{Hom}(f\times(1_c\times 1_b), a)]_1(h_x(g)) && (114)\\ &= ([\mathrm{Hom}(f\times(1_c\times 1_b), a)]_1\circ h_x)(g) && (115)\end{aligned}$$

In equation (109) we used (106), in equations (110), (114) we used (87), in (111) we used that

$$\widetilde{g\circ((f\times 1_c)\times 1_b)} : x'\times(c\times b)\to a$$

is unique. Furthermore, in equation (112) we used the uniqueness of the arrow

$$\widetilde{(f\times 1_c)\times 1_b} : x'\times(c\times b)\to x\times(c\times b).$$

and in (113) we used (107). In equations (108), (115) we used that h_x, $[\mathrm{Hom}((f\times 1_c)\times 1_b, a)]_1 \in \mathrm{Set}_1$.

Last we show in (83) that the iso, $\mathrm{Hom}(x\times(c\times b), a)\cong\mathrm{Hom}(x\times(b\times c), a)$, is natural in x. Therefore, we show that $\mathrm{Hom}(-\times(c\times b), a)\Rightarrow\mathrm{Hom}(-\times(b\times c), a)$ is a natural transformation. That means we need to show that for every $f : x'\to x$ in C_1 the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Hom}(x \times (c \times b), a) \arrow[r, "t_x"] \arrow[d, "{[\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1}"'] & \mathrm{Hom}(x \times (b \times c), a) \arrow[d, "{[\mathrm{Hom}(f \times (1_b \times 1_c), a)]_1}"] \\
\mathrm{Hom}(x' \times (c \times b), a) \arrow[r, "t_{x'}"'] & \mathrm{Hom}(x' \times (b \times c), a)
\end{tikzcd}$$

i.e.

$$t_{x'} \circ [\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1 = [\mathrm{Hom}(f \times (1_b \times 1_c), a)]_1 \circ t_x. \tag{116}$$

Similar to (14) any arrow $f \times (1_c \times 1_b) : x' \times (c \times b) \to x \times (c \times b)$ in $\mathbf{C}$ induces a function

$$[\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1 : \mathrm{Hom}(x \times (c \times b), a) \to \mathrm{Hom}(x' \times (c \times b), a),$$

so that for any $g : x \times (c \times b) \to a$

$$[\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1(g) = g \circ (f \times (1_c \times 1_b))$$

and any arrow $f \times (1_b \times 1_c) : x' \times (b \times c) \to x \times (b \times c)$

$$[\mathrm{Hom}(f \times (1_b \times 1_c), a)]_1 : \mathrm{Hom}(x \times (b \times c), a) \to \mathrm{Hom}(x' \times (b \times c), a)$$

for any $g^* : x \times (b \times c) \to a$

$$[\mathrm{Hom}(f \times (1_b \times 1_c), a)]_1(g^*) = g^* \circ (f \times (1_b \times 1_c).$$

Since the function t_x is already defined in (95), we can now show (116): given any $g : x \times (c \times b) \to a$ and with similar arguments to those in equations (108) - (115) we have that

$$\begin{aligned}
(t_{x'} \circ [\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1)(g) &= t_{x'}([\mathrm{Hom}(f \times (1_c \times 1_b), a)]_1(g)) \\
&= t_{x'}(g \circ (f \times (1_c \times 1_b))) \\
&= (g \circ (f \times (1_c \times 1_b)))^* \\
&= g^* \circ (f \times (1_c \times 1_b))^* \\
&= g^* \circ (f \times (1_b \times 1_c)) \\
&= [\mathrm{Hom}(f \times (1_b \times 1_c), a)]_1(g^*) \\
&= [\mathrm{Hom}(f \times (1_b \times 1_c), a)]_1(t_x(g)). \\
&= ([\mathrm{Hom}(f \times (1_b \times 1_c), a)]_1 \circ t_x)(g) \qquad \square
\end{aligned}$$

4.3 Limits and colimits in categories of diagrams

Definition 4.7. A category $\mathcal{E}$ is said to be *complete* if it has all small limits; that is, for any small category J and functor $F : J \to E$, there is a limit $L = \varprojlim_{j \in \mathbf{J}} F_j$ in $\mathcal{E}$ and a "cone" $\eta : \Delta L \to F$ in $\mathcal{E}^J$, universal among arrows from constant functors ΔE. Here, the constant functor $\Delta : \mathcal{E} \to \mathcal{E}^J$ is the transposed projection $\mathcal{E} \times J \to \mathcal{E}$.

Proposition 4.8. For any locally small category $\mathbf{C}$, the functor category $\mathbf{Set}^{\mathbf{C}^{op}}$ is complete. Moreover, for every object $c \in \mathbf{C}$, the evaluation functor

$$\mathrm{ev}_c : \mathbf{Set}^{\mathbf{C}^{op}} \to \mathbf{Set}$$

preserves all limits.

Proof. Suppose we have a small category $\mathbf{J}$ and a functor $F : \mathbf{J} \to \mathbf{Set}^{\mathbf{C}^{op}}$. The limit of F, if it exists, is an object in $\mathbf{Set}^{\mathbf{C}^{op}}$, hence is a functor,

$$(\varprojlim_{j \in \mathbf{J}} F_j) : \mathbf{C}^{op} \to \mathbf{Set}.$$

By the Yoneda lemma, if we had such a functor, then for each object $c \in \mathrm{C}_0$ we would have a natural isomorphism,

$$(\varprojlim_{j \in \mathbf{J}} F_j)(c) \cong \mathrm{Hom}(y_c, \varprojlim_{j \in \mathbf{J}} F_j).$$

But then it would be the case that

$$\begin{aligned} \mathrm{Hom}(y_c, \varprojlim_{j \in \mathbf{J}} F_j) &\cong \varprojlim_{j \in \mathbf{J}} \mathrm{Hom}(y_c, F_j) && \text{in } \mathbf{Set} \\ &\cong \varprojlim_{j \in \mathbf{J}} (F_j(c)) && \text{in } \mathbf{Set}, \end{aligned}$$

where we used in the first isomorphism that representable functors preserve limits (proposition 2.35), and in the second isomorphism we used the Yoneda lemma again. Thus, we are led to define the limit $\varprojlim_{j \in \mathbf{J}} F_j$ to be

$$(\varprojlim_{j \in \mathbf{J}} F_j)(c) = \varprojlim_{j \in \mathbf{J}} (F_j(c)) \tag{117}$$

that is, the *pointwise limit* of the functors F_j. The functor $\varprojlim_{j \in \mathbf{J}} F_j$ acts on $\mathbf{C}$-arrows in the following way: for any $f : c \to c'$ in $\mathbf{C}$ we define

$$(\varprojlim_{j \in \mathbf{J}} F_j)(f) = \varprojlim_{j \in \mathbf{J}} (F_j(f)), \tag{118}$$

such that the following diagram commutes,

$$\begin{tikzcd}
\varprojlim_{j \in \mathbf{J}}(F_j(c')) \arrow[rr, "\varprojlim_{j \in \mathbf{J}}(F_j(f))"] \arrow[d, "{[\vartheta_j]_{c'}}"'] \arrow[drr, phantom, "\#"] & & \varprojlim_{j \in \mathbf{J}}(F_j(c)) \arrow[d, "{[\vartheta_j]_c}"] \\
F_j(c') \arrow[rr, "F_j(f)"'] & & F_j(c)
\end{tikzcd}$$

i.e.

$$[\vartheta_j]_c \circ \varprojlim_{j \in \mathbf{J}}(F_j(f)) = F_j(f) \circ [\vartheta_j]_{c'}. \tag{119}$$

Defining the functor $\varprojlim_{j \in \mathbf{J}} F_j$ as above, we can prove that

$$(\varprojlim_{j \in \mathbf{J}} F_j, (\vartheta_j : \varprojlim_{j \in \mathbf{J}} F_j \to F_j)_{j \in \mathbf{J}})$$

is indeed a limit in $\mathbf{Set}^{\mathbf{C}^{op}}$.

$$\begin{tikzcd}
\mathbf{J} \arrow[dd, "F"] & i \arrow[rrr, "\alpha"] & & & j \\
& z \arrow[rr, dashed, Rightarrow, "q^*"] \arrow[dr, Rightarrow, "q_i"'] \arrow[drrr, Rightarrow, "q_j"] & & \varprojlim_{j \in \mathbf{J}} F_j \arrow[dl, Rightarrow, "\vartheta_i"] \arrow[dr, Rightarrow, "\vartheta_j"] & \\
\mathbf{Set}^{\mathbf{C}^{op}} & & F_i \arrow[rr, Rightarrow, "F_\alpha"'] & & F_j
\end{tikzcd}$$

Therefore, we need to show

i) $(\varprojlim_{j \in \mathbf{J}} F_j, (\vartheta_j)_{j \in \mathbf{J}}) \in \mathbf{Cone}_0(F)$, i.e. $F_\alpha \circ \vartheta_i = \vartheta_j$,

ii) $q^* : z \Rightarrow \varprojlim_{j \in \mathbf{J}} F_j$ is an arrow in $\mathbf{Set}^{\mathbf{C}^{op}}$,

iii) given any cone $(z, (q_j : z \to F_j)_{j \in J})$, we have $\vartheta_j \circ q^* = q_j$ for all $j \in \mathbf{J}$,

iv) $q^* : z \Rightarrow \varprojlim_{j \in \mathbf{J}} F_j$ is unique.

To show i) we need to show, that for arbitrary $c \in \mathrm{C}_0$

$$[F_\alpha \circ \vartheta_i]_c = [\vartheta_j]_c. \tag{120}$$

Because of (8), this is equivalent to

$$[F_\alpha]_c \circ [\vartheta_i]_c = [\vartheta_j]_c. \tag{121}$$

Therefore, we need to show that the following diagram in **Set** commutes,

$$\begin{tikzcd}
& (\varprojlim_{j\in\mathbf{J}} F_j)(c) \arrow[dl, "{[\vartheta_i]_c}"'] \arrow[dr, "{[\vartheta_j]_c}"] \arrow[d, phantom, "\#"] & \\
F_i(c) \arrow[rr, "{[F_\alpha]_c}"'] & & F_j(c).
\end{tikzcd}$$

Since

$$(\varprojlim_{j\in\mathbf{J}} F_j)(c) = \varprojlim_{j\in\mathbf{J}}(F_j(c)),$$

and

$$\Big(\varprojlim_{j\in\mathbf{J}}(F_j(c))\ ,\ ([\vartheta_j]_c : \varprojlim_{j\in\mathbf{J}} F_j(c) \to F_j(c))_{j\in\mathbf{J}}\Big)$$

is a limit in **Set**, equation (121) follows by definition of a limit.
For ii), we show that $q^* : z \Rightarrow \varprojlim_{j\in\mathbf{J}} F_j$ is arrow in $\mathbf{Set}^{\mathbf{C}^{op}}$, i.e. q^* is a natural transformation. Therefore, we need to show, that there is a family of arrows in **Set** of the form $[q^*]_c : z(c) \to \varprojlim_{j\in\mathbf{J}} F_j(c)$, such that for every c in C_0, and every $f : c \to c'$ in C_1 the following diagram commutes

$$\begin{tikzcd}
z(c') \arrow[r, "z(f)"] \arrow[d, "{[q^*]_{c'}}"'] & z(c) \arrow[d, "{[q^*]_c}"] \\
(\varprojlim_{j\in\mathbf{J}} F_j)(c') \arrow[r, "(\varprojlim_{j\in\mathbf{J}} F_j)(f)"'] & (\varprojlim_{j\in\mathbf{J}} F_j)(c).
\end{tikzcd}$$

We have that

$$\begin{aligned}
[\vartheta_j]_c \circ \varprojlim_{j\in\mathbf{J}}(F_j(f)) \circ [q^*]_{c'} &= F_j(f) \circ [\vartheta_j]_{c'} \circ [q^*]_{c'} && (122)\\
&= F_j(f) \circ [q_j]_{c'} && (123)\\
&= [q_j]_c \circ z(f) && (124)\\
&= [\vartheta_j]_c \circ [q^*]_c \circ z(f). && (125)
\end{aligned}$$

Therefore

$$[q^*]_c \circ z(f) = \overbrace{\varprojlim_{j\in\mathbf{J}}(F_j(f))}^{(\varprojlim_{j\in\mathbf{J}} F_j)(f)} \circ [q^*]_{c'}.$$

In equation (122) we used (119), equations (123) and (125) follows because $(\varprojlim_{j\in\mathbf{J}}(F_j(c)), ([\vartheta_j]_c)_{j\in\mathbf{J}})$ is a limit in **Set** . In equation (124) we used that $q_j : z \Rightarrow F_j$ is a natural transformation, i.e. the following diagram commutes,

$$\begin{tikzcd}
z(c') \arrow[r, "z(f)"] \arrow[d, "{[q_j]_{c'}}"'] \arrow[dr, phantom, "\#"] & z(c) \arrow[d, "{[q_j]_c}"] \\
F_j(c') \arrow[r, "F_j(f)"'] & F_j(c).
\end{tikzcd}$$

For iii), let $(z, (q_j : z \to F_j)_{j\in\mathbf{J}})$ be a cone to the diagram F in $\mathbf{Set}^{\mathbf{C}^{op}}$. For each $c \in \mathrm{C}_0$, we have a cone

$$\Big(z(c)\ ,\ ([q_j]_c : z(c) \to F_j(c))_{j\in\mathbf{J}}\Big)$$

in **Set**.

$$\begin{tikzcd}
z(c) \arrow[rr, dashed, "{[q^*]_c}"] \arrow[dr, "{[q_i]_c}"'] \arrow[drrr, "{[q_j]_c}" near start] & & \varprojlim_{j\in\mathbf{J}}(F_j(c)) \arrow[dl, "{[\vartheta_i]_c}"] \arrow[dr, "{[\vartheta_j]_c}"] & \\
& F_i(c) \arrow[rr, "{[F\alpha]_c}"'] & & F_j(c)
\end{tikzcd}$$

Since $(\varprojlim_{j\in\mathbf{J}}(F_j(c)), ([\vartheta_j]_c)_{j\in\mathbf{J}})$ is a limit in **Set**, there is a unique arrow

$$[q^*]_c : z(c) \to \varprojlim_{j\in\mathbf{J}}(F_j(c))$$

in **Set** such that for all j in **J**

$$[\vartheta_j]_c \circ [q^*]_c = [q_j]_c.$$

Therefore, for any $j \in \mathbf{J}$ we have

$$\vartheta_j \circ q^* = q_j.$$

For iv), we need to show the uniqueness of $q^* : z \Rightarrow \varprojlim_{j\in\mathbf{J}} F_j$. Given any $\tau : z \Rightarrow \varprojlim_{j\in\mathbf{J}} F_j$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ such that for any $j \in \mathbf{J}$

$$\vartheta_j \circ \tau = q_j,$$

we have that

$$\vartheta_j \circ \tau = q_j = \vartheta_j \circ q^*. \tag{126}$$

In equation (126) we used iii). Therefore,

$$\tau = q^*.$$

Finally, the preservation of limits by evaluation functors is stated by (118), since

$$\begin{aligned}\mathrm{ev}_c(\varprojlim_{j\in\mathbf{J}} F_j) &= (\varprojlim_{j\in\mathbf{J}} F_j)(c)\\ &= \varprojlim_{j\in\mathbf{J}}(F_j(c))\\ &= \varprojlim_{j\in\mathbf{J}}(\mathrm{ev}_c(F_j)).\end{aligned}$$

□

Theorem 4.9 (Density theorem)**.** For any small category $\mathbf{C}$, every object P in the functor category $\mathbf{Set}^{\mathbf{C}^{op}}$ is a colimit of representable functors,

$$\varinjlim_{j\in J}\ y(c_j)\ \cong P.$$

More precisely, there is a canonical choice of an index category I and a functor $\pi : \mathrm{I} \to \mathbf{C}$ such that there is a natural isomorphism $\varinjlim_{j\in J} y \circ \pi \cong P$.

Proof. Given $P : \mathbf{C}^{op} \to \mathbf{Set}$, the index category I we need is the so-called category of elements of P, written,

$$\mathrm{Groth}(\mathbf{C},\ P)\ ^{4}$$

and defined as follows.

(i) **Objects** : pairs (a, x) where $a \in \mathbf{C}$ and $x \in P_0(a)$.

(ii) **Arrows** : an $f : (a, x) \to (b, y)$ is an arrow $f : a \to b$ in $\mathbf{C}$ such that

$$[P_1(f)](y) = x. \tag{127}$$

Actually, the arrows are triples of the form $(f, (a, x), (b, y))$ satisfying (127).

[4] The category $\mathrm{Groth}(\mathbf{C},\ P)$ is named after the mathematician Alexander Grothendieck. This category is sometimes also written as $\sum\limits_{C} P$ or $\int\limits_{C} P$.

To show that Groth($\mathbf{C}$, P) is indeed a category, we need to show:

(iii) if $g : (c, z) \to (a, x)$, $f : (a, x) \to (b, y)$ arrows in Groth($\mathbf{C}, P$), then

$$f \circ g : (c, z) \to (b, y)$$

is an arrow in Groth($\mathbf{C}, P$). Since $\mathbf{C}$ is a category, and $g : c \to a$, $f : a \to b$ are arrows in $\mathbf{C}$, then also

$$f \circ g : c \to b$$

is an arrow in $\mathbf{C}$, such that

$$\begin{aligned} [P_1(f \circ g)](y) &= [P_1(g) \circ P_1(f)](y) && (128) \\ &= [P_1(g)](P_1(f)(y)) && (129) \\ &= [P_1(g)](x) && (130) \\ &= z. && (131) \end{aligned}$$

In equation (128) we used (6), in equation (129) we used that $P(g), P(f)$ are functions Set_1, in equations (130) and (131) we used that f and g are arrows in Groth($\mathbf{C}, P$).

(iv) For every $(a, x) \in$ Groth($\mathbf{C}, P$), the identity arrow

$$1_a : (a, x) \to (a, x)$$

is an arrow in Groth($\mathbf{C}, P$). Since the identity arrow

$$1_a : a \to a$$

is an arrow in $\mathbf{C}$ and

$$\begin{aligned} [P_1(1_a)](x) &= [1_{P_0(a)}](x) && (132) \\ &= 1_{P_0(a)}(x) \\ &= x. && (133) \end{aligned}$$

In equation (132) we used (5) and equation (133) follows because $1_{P_0(a)}$ is the identity function on $P_0(a)$ and $x \in P_0(a)$.

Furthermore, we need to show that the following conditions are satisfied:

(a) Unit: If $f : (a, x) \to (b, y)$ and $1_b : (b, y) \to (b, y)$ in Groth$(\mathbf{C}, P)_1$, we have that $1_b \circ f : a \to b$ is an arrow in $\mathbf{C}$, such that for any $y \in P_0(b)$

$$
\begin{aligned}
[P_1(1_b \circ f)](y) &= [P_1(f) \circ P_1(1_b)](y) && (134)\\
&= [P_1(f)]([P_1(1_b)](y)) && (135)\\
&= [P_1(f)](y) && (136)\\
&= x. && (137)
\end{aligned}
$$

Therefore,

$$1_b \circ f = f.$$

In equation (134) we used (6), in (135) we used that $P_1(f), P_1(1_b) \in \text{Set}_1$, in equations (136), (137) we used that $1_b, f$ are arrows in $\text{Groth}(\mathbf{C}, P)$.

If $f : (a,x) \to (b,y),\ 1_a : (a,x) \to (a,x)$ in $\text{Groth}(\mathbf{C}, P)_1$ we need also to show that

$$f \circ 1_a = f.$$

This follows with similar arguments: since $f \circ 1_a : a \to b$ is an arrow in $\mathbf{C}$, such that for any $y \in P_0(b)$

$$
\begin{aligned}
[P_1(f \circ 1_a)](y) &= [P_1(1_a) \circ P_1(f)](y)\\
&= [P_1(1_a)]([P_1(f)](y))\\
&= [P_1(1_a)](x)\\
&= x.
\end{aligned}
$$

(b) Associativity: If $h : (d,s) \to (c,z),\ g : (c,z) \to (a,x),\ f : (a,x) \to (b,y)$ in $\text{Groth}(\mathbf{C}, P)_1$ we have that $f \circ (g \circ h) : d \to b$ is an arrow in $\mathbf{C}$, such that for any $y \in P_0(b)$

$$
\begin{aligned}
[P_1(f \circ (g \circ h))](y) &= [P(g \circ h) \circ P(f)](y) && (138)\\
&= [P_1(g \circ h)]([P_1(f)](y)) && (139)\\
&= [P_1(g \circ h)](x) && (140)\\
&= [P_1(h) \circ P_1(g)](x) && (141)\\
&= [P_1(h)]([P_1(g)](x)) && (142)\\
&= [P_1(h)](z) && (143)\\
&= d. && (144)
\end{aligned}
$$

In equations (138), (141) we used (6), in (139), (142) we used that

$P(g \circ h), P(f), P(g), P(h) \in \text{Set}_1$ and in equations (143), (144) and (140) we used that f, g and h are arrows in Groth($\mathbf{C}, P$).

With similar arguments we have that $(f \circ g) \circ h : d \to b$ is an arrow in $\mathbf{C}$, such that for any $y \in P_0(b)$

$$\begin{aligned}
[P_1((f \circ g) \circ h)](y) &= [P_1(h) \circ P_1(f \circ g)](y) \\
&= [P_1(h)]([P_1(f \circ g)](y)) \\
&= [P_1(h)]([P_1(g) \circ P_1(f)](y)) \\
&= [P_1(h)]([P_1(g)]([P_1(f)](y))) \\
&= [P_1(h)]([P_1(g)](x)) \\
&= [P_1(h)](z) \\
&= d.
\end{aligned}$$

Therefore for any $h : (d, s) \to (c, z),\ g : (c, z) \to (a, x)$,
$f : (a, x) \to (b, y)$ in Groth$(\mathbf{C}, P)_1$ we have that

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

So, we showed that Groth($\mathbf{C}, P$) is a category, and since $\mathbf{C}$ is small, Groth($\mathbf{C}, P$) is a small category. There is a "projection" functor

$$\pi : \text{Groth}(\mathbf{C}, P) \to \mathbf{C},$$

defined in following way:

(i) For any object (a, x) in Groth($\mathbf{C}, P$),

$$\pi_0(a, x) = a,$$

(ii) for any arrow $f : (a, x) \to (b, y)$ in Groth($\mathbf{C}, P$), written in the form of the triple $(f, (a, x), (b, y))$,

$$\pi_1((f, (a, x), (b, y))) = f. \tag{145}$$

In order to be a functor we require π to satisfy the following conditions:

(a) For any $(a, x) \in$ Groth($\mathbf{C}, P$)

$$\begin{aligned}
\pi_1((1_a, (a, x), (a, x))) &= 1_a \\
&= 1_{\pi_0(a,x)},
\end{aligned}$$

and that

(b) if $g : (c, z) \to (a, x),\ f : (a, x) \to (b, y)$, then

$$\begin{aligned}\pi_1(((f \circ g), (c, z), (b, y))) &= f \circ g \\ &= \pi_1((f, (a, x), (b, y))) \circ \pi_1((g, (c, z), (a, x))).\end{aligned}$$

Given a functor $P : \mathbf{C}^{op} \to \mathbf{Set}$, for the index Category I we choose the category $\mathrm{I} = \mathrm{Groth}(\mathbf{C}, P)$, that is defined above. Again we refer to $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ as the Yoneda embedding, defined in definition 3.2, and $\pi : \mathrm{Groth}(\mathbf{C}, P) \to \mathbf{C}$ denotes the "projection" functor, defined above. So, $y \circ \pi : \mathrm{Groth}(\mathbf{C}, P) \to \mathbf{Set}^{\mathbf{C}^{op}}$ is a diagram of type $\mathrm{Groth}(\mathbf{C}, P)$ in $\mathbf{Set}^{\mathbf{C}^{op}}$.

To proof proposition 4.9, we need to show that P is an initial object in the category $\mathbf{Cocone}(y \circ \pi)$. Therefore, we need to show that P is an object in $\mathbf{Cocone}(y \circ \pi)$ and that there is a unique arrow to any other cocone from the base $(y \circ \pi)$.

First, we show that $P : \mathbf{C}^{op} \to \mathbf{Set}$ is an object in $\mathbf{Cocone}(y \circ \pi)$, i.e. that P can be written as a cocone P^* to $y \circ \pi$.

To define the cocone P^* we use the following results:

(i) first we use that

$$\begin{aligned}(y \circ \pi)_{(a,x)} &= (y \circ \pi)(a, x) \\ &= (y(\pi(a, x)) \\ &= y(a) \\ &= y_a,\end{aligned}$$

(ii) then we use that

$$\begin{aligned}(y \circ \pi)((f, (a, x), (b, y))) &= y(\pi((f, (a, x), (b, y))) \\ &= y_1(f),\end{aligned}$$

where $y_1(f)$ is defined as in (3.2). For any $c \in \mathrm{C}_0$ and $g : c \to a$ in C_1,

$$[(y \circ \pi)((f, (a, x), (b, y))]_c(g) = [y(\pi((f, (a, x), (b, y)))]_c(g) \tag{146}$$
$$= [y_1(f)]_c(g) \tag{147}$$
$$= f \circ g. \tag{148}$$

In equation (148) we used (47), in (147) we used (145) and in (146) we used the functor composition.

A cocone P^* to $y \circ \pi : \mathrm{Groth}(\mathbf{C}, P) \to \mathbf{Set}^{\mathbf{C}^{op}}$ is, a pair

$$\begin{aligned} P^* &:= \Big(P \ , \ (\vartheta_{(a,x)} : (y \circ \pi)_{(a,x)} \Rightarrow F)_{(a,x) \in \mathrm{Groth}(\mathbf{C},P)_0} \Big) \\ &= \Big(P \ , \ (\vartheta_{(a,x)} : y_a \Rightarrow P)_{(a,x) \in \mathrm{Groth}(\mathbf{C},P)_0} \Big) \end{aligned}$$

that consists of an object $P : \mathbf{C}^{op} \to \mathbf{Set}$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ and a family of arrows in $\mathbf{Set}^{\mathbf{C}^{op}}$

$$\vartheta_{(a,x)} : y_a \Rightarrow P$$

one for each object $(a, x) \in \mathrm{Groth}(\mathbf{C}, P)_0$, such that for each arrow

$$f : (a, x) \to (b, y) \text{ in } \mathrm{Groth}(\mathbf{C}, P)_1$$

the following triangle commutes,

$$\begin{tikzcd} y_a \arrow[rr, Rightarrow, "y_1(f)"] \arrow[dr, Rightarrow, "\vartheta_{(a,x)}"'] & \# & y_b \arrow[dl, Rightarrow, "\vartheta_{(b,y)}"] \\ & P & \end{tikzcd}$$

i.e.

$$\vartheta_{(b,y)} \circ y_1(f) = \vartheta_{(a,x)}, \tag{149}$$

where $\vartheta_{(a,x)} : y_a \Rightarrow P$ is defined in the following way: since $x \in P_0(a)$ and because of (54) we define

$$\vartheta_{(a,x)} := \eta^x, \tag{150}$$

where

$$[\eta^x]_c : \mathrm{Hom}(c, a) \to P_0(c)$$

and because of (55), for any $g : c \to a$ in C_1, we have

$$[\eta^x]_c(g) = [P_1(g)](x).$$

Thus,

$$\begin{aligned} [\vartheta_{(a,x)}]_c(g) &= [\eta^x]_c(g) \qquad (151) \\ &= [P_1(g)](x). \qquad (152) \end{aligned}$$

To show equation (149), it is enough to show that

$$[\vartheta_{(b,y)}]_c \circ [y_1(f)]_c = [\vartheta_{(a,x)}]_c, \tag{153}$$

for arbitrary $c \in \mathrm{C}_0$.
Equation (153) follows since for any $g : c \to a$ in C_1, we have that

$$\begin{aligned}
([\vartheta_{(b,y)}]_c \circ [y_1(f)]_c)(g) &= [\vartheta_{(b,y)}]_c([y_1(f)]_c(g)) \\
&= [\vartheta_{(b,y)}]_c(f \circ g) && (154)\\
&= [\eta^y]_c(f \circ g) && (155)\\
&= [P_1(f \circ g)](y) && (156)\\
&= [P_1(g) \circ P_1(f)](y) && (157)\\
&= [P_1(g)]([P_1(f)](y)) && (158)\\
&= [P_1(g)](x) && (159)\\
&= [\eta^x]_c(g) && (160)\\
&= [\vartheta_{(b,x)}]_c(g). && (161)
\end{aligned}$$

In equation (154) we used (148), in equations (155), (161) we used (151), in equations (156), (160) we used (152). Furthermore, in equation (157) we used (6), in equation (158) we used that $P_1(f), P_1(g)$ are functions and in (159) we used that $f : (a, x) \to (b, y) \in \mathrm{Groth}(\mathbf{C}, P)_1$.

Now, we show the initiality of P^* in $\mathbf{Cocone}(y \circ \pi)$, i.e. given any other cocone $F^* = \left(F, (\varphi_{(a,x)} : y_a \Rightarrow F)_{(a,x) \in \mathrm{Groth}(\mathbf{C},P)_0}\right)$ in $\mathbf{Cocone}(y \circ \pi)$ there exist a unique arrow $\sigma : P^* \to F^*$ in $\mathbf{Cocone}(y \circ \pi)$. Therefore, we need to show that there is an unique arrow $\sigma : P \Rightarrow F$ in $\mathbf{Set}^{\mathbf{C}^{op}}$, such that for all $(a, x) \in \mathrm{Groth}(\mathbf{C}, P)$ the following triangle commutes,

$$\begin{tikzcd}
F & \\
 & y_a \arrow[ul, Rightarrow, "\varphi_{(a,x)}"'] \arrow[dl, Rightarrow, "\vartheta_{(a,x)}"] \arrow[l, phantom, "\#"] \\
P \arrow[uu, Rightarrow, "\sigma"] &
\end{tikzcd}$$

i.e.

$$\sigma \circ \vartheta_{(a,x)} = \varphi_{(a,x)},$$

for any object $(a, x) \in \mathrm{Groth}(\mathbf{C}, P)$.

So, for the initiality of P^* there are 3 conditions to show:

(i) $\sigma : P \Rightarrow F$ is an arrow in $\mathbf{Set}^{\mathbf{C}^{op}}$, i.e. σ is natural transformation from P to F,

(ii) $\sigma \circ \vartheta_{(a,x)} = \varphi_{(a,x)}$,

(iii) $\sigma : P \Rightarrow F$ is unique.

We need first to define

$$\sigma_c : P_0(c) \to F_0(c)$$

for any component $c \in \mathrm{C}_0$. We already know, that

$$\varphi_{(c,z)} : \underbrace{y_c}_{\mathrm{Hom}(-,c)} \Rightarrow F$$

and for any c $\in \mathrm{C}_0$ we have

$$[\varphi_{(c,z)}]_c : \mathrm{Hom}(c,c) \to F_0(c).$$

For $1_c \in \mathrm{Hom}(c,c)$, we have that

$$[\varphi_{(c,z)}]_c(1_c) \in F_0(c),$$

and for $z \in P_0(c)$

$$\sigma_c(z) \in F_0(c).$$

Therefore, we define

$$\sigma_c(z) := [\varphi_{(c,z)}]_c(1_c). \tag{162}$$

Now we show (i), i.e. $\sigma : P \Rightarrow F$ is a natural transformation. Therefore, we need to show that there is a family of arrows in **Set** of the form $\sigma_c : P_0(c) \to F_0(c)$, such that for every $c \in \mathrm{C}_0$, and every $h : c \to c'$ in C_1, the following diagram commutes,

$$\begin{tikzcd}
P_0(c') \arrow[rr, "P_1(h)"] \arrow[d, "\sigma_{c'}"'] \arrow[drr, phantom, "\#"] & & P_0(c) \arrow[d, "\sigma_c"] \\
F_0(c') \arrow[rr, "F_1(h)"'] & & F_0(c),
\end{tikzcd}$$

i.e.

$$\sigma_c \circ P_1(h) = F_1(h) \circ \sigma_{c'}. \tag{163}$$

To prove equation (163), we use the following properties:

(a) Given a cocone $F^* = \Big(F\ ,\ (\varphi_{(c,z)}\ :\ y_c\ \Rightarrow\ F)_{(c,z)\in \mathrm{Groth}(\mathbf{C},P)_0}\Big)$, for any $h : c \to c' \in \mathrm{C}_1$ the following triangle in $\mathbf{Set}^{\mathbf{C}^{op}}$ commutes,

$$\begin{tikzcd} y_c \arrow[rr, Rightarrow, "y_1(h)"] \arrow[dr, Rightarrow, "\varphi_{(c,z)}"'] \arrow[rr, phantom, "\#", shift right=4ex] & & y_{c'} \arrow[dl, Rightarrow, "\varphi_{(c',z')}"] \\ & F. & \end{tikzcd}$$

Therefore, for any $c \in C_0$, the following triangle in $\mathbf{Set}$ commutes

$$\begin{tikzcd} y_c(c) \arrow[rr, "{[y_1(h)]_c}"] \arrow[dr, "{[\varphi_{(c,z)}]_c}"'] \arrow[rr, phantom, "\#", shift right=4ex] & & y_{c'}(c) \arrow[dl, "{[\varphi_{(c',z')}]_c}"] \\ & F(c), & \end{tikzcd}$$

i.e.

$$[\varphi(c', z')]_c \circ [y_1(h)]_c = [\varphi_{(c,z)}]_c. \tag{164}$$

(b) Since $\varphi_{(c',z')} : y_{c'} \Rightarrow F$ is a natural transformation, for any $h : c \to c'$ in C_1, the following diagram in $\mathbf{Set}$ commutes,

$$\begin{tikzcd} y_{c'}(c') \arrow[r, "{[y_1(h)]_{c'}}"] \arrow[d, "{[\varphi_{(c',z')}]_{c'}}"'] \arrow[dr, phantom, "\#"] & y_{c'}(c) \arrow[d, "{[\varphi_{(c',z')}]_c}"] \\ F_0(c') \arrow[r, "F_1(h)"'] & F_0(c), \end{tikzcd}$$

i.e.

$$[\varphi_{(c',z')}]_c \circ [y_1(h)]_{c'} = F_1(h) \circ [\varphi_{(c',z')}]_{c'}. \tag{165}$$

Now, we prove equation (163). For any $z' \in P_0(c')$ and $h : c \to c'$, we have

$$\begin{aligned}
(F_1(h) \circ \sigma_{c'})(z') &= F_1(h)(\sigma_{c'}(z')) \\
&= F_1(h)([\varphi_{(c',z')}]_{c'}(1_{c'})) && (166)\\
&= [\varphi_{(c',z')}]_c([y_1(h)]_{c'}(1_{c'})) && (167)\\
&= [\varphi_{(c',z')}]_c(1_{c'} \circ h) && (168)\\
&= [\varphi_{(c',z')}]_c(h) \\
&= [\varphi_{(c',z')}]_c(h \circ 1_c) \\
&= [\varphi_{(c',z')}]_c([y_1(h)]_c(1_c)) && (169)\\
&= [\varphi_{(c,z)}]_c(1_c) && (170)\\
&= \sigma_c(z). && (171)
\end{aligned}$$

In equations (166) and (171) we used (162). In equation (167) we used (165). In equation (168) we used (79) and in equation (169) we used (47). In equation (170) we used (164).

Now, we show (ii) i.e.

$$\sigma \circ \vartheta_{(a,x)} = \varphi_{(a,x)}.$$

Because of (150), this is equivalent to

$$\sigma \circ \eta^x = \varphi_{(a,x)}. \tag{172}$$

To show equation (172) it is enough to show

$$[\sigma \circ \eta^x]_c = [\varphi_{(a,x)}]_c,$$

for arbitrary $c \in \mathrm{C}_0$. This follows, since for any $c \in \mathrm{C}_0$ and $g : c \to a$ in C_1

$$\begin{aligned}
[\sigma \circ \eta^x]_c(g) &= (\sigma_c \circ [\eta^x]_c)(g) && (173)\\
&= \sigma_c([\eta^x]_c)(g)) \\
&= \sigma_c([P_1(g)](x)) \\
&= \sigma_c(z) \\
&= [\varphi_{(c,z)}]_c(1_c) && (174)\\
&= ([\varphi_{(a,x)}]_c \circ [y_1(g)]_c)(1_c) && (175)\\
&= [\varphi_{(a,x)}]_c([y_1(g)]_c)(1_c)) \\
&= [\varphi_{(a,x)}]_c(g \circ 1_c) && (176)\\
&= [\varphi_{(a,x)}]_c(g).
\end{aligned}$$

In equation (173) we used (8), in equation (176) we used (47) and in equation (174) we used (162). Equation (175) follows from the fact that

$$\left(F\ ,\ (\varphi_{(c,z)} : y_c \Rightarrow F)_{(c,z)\in \mathrm{Groth}(\mathbf{C},P))_0}\right)$$

is a cocone, so for any $g : c \to a$ in C_1 the following triangle in $\mathbf{Set}^{\mathbf{C}^{op}}$ commutes

$$\begin{tikzcd}
y_c \arrow[rr, Rightarrow, "y_1(g)"] \arrow[dr, Rightarrow, "\varphi_{(c,z)}"'] & \# & y_a \arrow[dl, Rightarrow, "\varphi_{(a,x)}"] \\
& F, &
\end{tikzcd}$$

and therefore, for any $c \in \mathrm{C}_0$ the following triangle in $\mathbf{Set}$ commutes,

$$\begin{tikzcd}
y_c(c) \arrow[rr, "{[y_1(g)]_c}"] \arrow[dr, "{[\varphi_{(c,z)}]_c}"'] & \# & y_a(c) \arrow[dl, "{[\varphi_{(a,x)}]_c}"] \\
& F(c), &
\end{tikzcd}$$

i.e.

$$[\varphi(a,x)]_c \circ [y_1(g)]_c = [\varphi_{(c,z)}]_c. \tag{177}$$

In equation (175) we used (177).
Now, we show the uniqueness of σ in (iii). Given any $\tau : P^* \to F^*$ in $\mathbf{Cocone}(y \circ \pi)$ that is, an arrow $\tau : P \Rightarrow F$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ such that

$$\tau \circ \vartheta_{(a,x)} = \varphi_{(a,x)},$$

i.e. the following triangle commutes for any such τ, and with property (ii) it also commutes for σ,

$$\begin{tikzcd}
& y_a \arrow[dl, Rightarrow, "\vartheta_{(a,x)}"'] \arrow[dr, Rightarrow, "\varphi_{(a,x)}"] \arrow[d, phantom, "\#"] & \\
P \arrow[rr, Rightarrow, shift left, "\sigma"] \arrow[rr, Rightarrow, shift right, "\tau"'] & {} & F,
\end{tikzcd}$$

for all $(a, x) \in \mathrm{Groth}(\mathbf{C}, P)_0$. It follows that for any $c \in \mathrm{C}_0$ the following triangle in $\mathbf{Set}$ commutes,

$$\begin{tikzcd}
 & y_a(c) \arrow[dl, "{[\vartheta_{(a,x)}]_c}"'] \arrow[dr, "{[\varphi_{(a,x)}]_c}"] \arrow[d, phantom, "\#"] & \\
P(c) \arrow[rr, shift left, "\sigma_c"] \arrow[rr, shift right, "\tau_c"'] & & F(c),
\end{tikzcd}$$

for all $(a, x) \in \mathrm{Groth}(\mathbf{C}, P)_0$, i.e.,

$$\tau_c \circ [\vartheta_{(a,x)}]_c = [\varphi_{(a,x)}]_c \quad \text{and} \quad \sigma_c \circ [\vartheta_{(a,x)}]_c = [\varphi_{(a,x)}]_c.$$

Therefore,

$$\tau_c \circ [\vartheta_{(a,x)}]_c = \sigma_c \circ [\vartheta_{(a,x)}]_c. \tag{178}$$

We have that

$$\begin{aligned}
(\tau_c \circ [\vartheta_{(a,x)}]_c)(g) &= \tau_c([\vartheta_{(a,x)}]_c(g)) \\
&= \tau_c(\eta_c^x(g)) && (179)\\
&= \tau_c([P_1(g)](x)) && (180)\\
&= \tau_c(x). && (181)
\end{aligned}$$

In equation (179) we used (150), in (180) we used (152) and in equation (181) we used that $g : (c, z) \to (a, x)$ is arrow in $\mathrm{Groth}(\mathbf{C}, P)$. With similar arguments we have

$$\begin{aligned}
(\sigma_c \circ [\vartheta_{(a,x)}]_c)(g) &= \sigma_c([\vartheta_{(a,x)}]_c(g)) \\
&= \sigma_c(\eta_c^x(g)) \\
&= \sigma_c([P_1(g)](x)) \\
&= \sigma_c(x).
\end{aligned}$$

Because of (178), we have that

$$\tau_c(x) = \sigma_c(x).$$

Since $x \in P_0(a)$ was chosen arbitrary,

$$\tau_c = \sigma_c.$$

for arbitrary $c \in \mathrm{C}_0$. Therefore,

$$\tau = \sigma. \qquad \square$$

5 Conclusion

To summarize the central statement of the Yoneda lemma, we take up the question raised in the introduction: how can the set $\mathrm{Hom}(y_a, F) = \{y_a \Rightarrow F\}$ be described? The Yoneda lemma says that the natural transformations from y_a to F are in one-to-one correspondence with the elements of the set $F_0(a)$. In other words, $\mathrm{Hom}(y_a, F) \cong F_0(a)$. This correspondence turns out to be an isomorphism which is natural in a and F.

The Yoneda lemma has many applications. Selected applications of the Yoneda lemma were discussed in the context of this thesis, for example the Yoneda theorem in 4.1, which tells us that $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ is an embedding. The Yoneda theorem is proven with the help of the Yoneda lemma. Therefore, the Yoneda lemma allows the embedding of any locally small category $\mathbf{C}$ into the functor category $\mathbf{Set}^{\mathbf{C}^{op}}$.

From the Yoneda theorem and corollary 4.3 follows the Yoneda principle in 4.4, that states that if $\mathrm{Hom}(-, a)$, i.e., the arrows to a, and $\mathrm{Hom}(-, b)$, i.e., the arrows to b, are the same, then a and b are the same. This statement is very useful because one often understands $\mathrm{Hom}(-, a)$, $\mathrm{Hom}(-, b)$ better than the objects a, b themselves.

In the proof of proposition 4.8 the Yoneda lemma is included in defining the limit of the functors F_j in the functor category $\mathbf{Set}^{\mathbf{C}^{op}}$, that is defined pointwise. With this definition, we showed that the functor category $\mathbf{Set}^{\mathbf{C}^{op}}$ is complete and consequently that the evaluation functor from $\mathbf{Set}^{\mathbf{C}^{op}}$ to $\mathbf{Set}$ preserves all limits.

The last application discussed in this thesis is the density theorem in 4.9, that tells us that every functor $P : \mathbf{C}^{op} \to \mathbf{Set}$ is really “built up” from representable functors $y_a = \mathrm{Hom}(-, a)$. Formally, every such functor P is a colimit of certain representable functors in a canonical way.

More applications can be found in *Category theory in context* by Emily Riehl.

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