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Categorical aspects of complemented subsets

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Contents

1 Introduction

Regarding the fundamental differences between set and category theory, the translation of ideas and concepts from Bishop set theory into category theory is an especially interesting topic. Bishop's theory of sets underlying Bishop-style constructive mathematics, or constructive analysis to be precise, was first developed in chapter 3 of Bishop's seminal book *Foundations of Constructive Analysis* and in chapter 3 of the book *Constructive Analysis*, where he co-authored with Bridges. Bishop's aim was to create a constructive framework for advanced mathematics to be done within. However, set theory was only briefly treated in both books. Bishop set theory (BST), developed in [9], is Petrakis' account of an informal, constructive theory of totalities and assignment routines, serving as a reconstruction of Bishop's theory of sets and functions. BST makes a clear distinction between sets and classes, e.g. the powerset of a set X is treated as a class.

While set theory is based on sets and functions, category theory is a formalization of mathematical structures. Categories are an abstract representation of mathematical concepts, consisting of objects and arrows. Our aim is to appropriately translate set-theoretic terms, within Bishop-style constructive mathematics, to categorical notions.

Bishop describes a subset of a set X as a pair consisting of a set A and an embedding of A into X. Bishop's definition of a subset of a set is related to the notion of a subobject of an object in category theory, as described by Awodey on p.89 in his book *Category Theory*, where a subobject of an object x is defined as a monomorphism into x. To avoid the use of negation in basic set theory, which may cause some problems in constructive analysis, Bishop introduced the notion of a complemented (sub)set, which avoids the negative definition of the complement of a subset. This gives us a positively defined notion of disjointness of subsets. In BST a complemented subset of a set $(X, =_X, \neq_X)$ with an equality and inequality is a pair of subsets of X that are disjoint with respect to $\neq_X$. In terms of Bishop-style constructive mathematics, an equality $=_X$ and inequality $\neq_X$, or apartness relation, on a set X were defined through some formula. In set theory, a relation between two sets X and Y is defined as a subset R of their cartesian product $X \times Y$, which one can translate into the theory of categories. In order to translate the notion of a subset and complemented subset to the language of categories, we need to capture equality and apartness categorically. To do so we first need to define what a relation is within category theory. Following Klein [4] we present the categorical notion of a binary relation.

Following Petrakis' work [8] with the categorical translations of equality and apartness we can then translate the notion of a complemented subset into category theory as the notion of a complemented subobject with respect to a given pair of equality and apartness of some object x in a category $\mathcal{C}$. This allows us to define the category of complemented subobjects of an object in a category, which can be seen as the categorical translation of the category of complemented subsets.

Before we can work on translating the set-theoretic concepts to category theory we will first need to revisit Bishop's set theory and introduce the basic notions. This thesis has the following structure:

(i) Following Petrakis' work [9] we introduce the basic notions of Bishop set theory. Especially focusing on complemented subsets, we prove various properties.

(ii) After having introduced complemented subsets and partial functions, we give a full proof of the existence of class functions between the the class of complemented subsets $\mathcal{P}^{\rrbracket\llbracket}(X)$ and the class of strongly extensional partial functions $\mathscr{F}^{se}(X, \mathbb{2})$. This proposition was first proved in [7].

(iii) We present the basic notions of category theory based on Awodey's book [1] and turn to the categorical aspects of complemented subsets. Following Petrakis' work [6] we present the category of complemented subsets $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$ and show its full embedding into the the Chu category $\mathbf{Chu}(\mathbf{Set}, X \times X)$.

(iv) Based on [4], we present the categorical notion of a binary relation between two objects x and y. Based on [8] we use this notion to define an equality relation, where we distinguish local equality from global equality and show that every global equality implies a local equality. We also define the notion of a local apartness relation.

(v) Finally, we define the category of complemented subobjects of an object of a category.

The notion of a global apartness relation is not treated in this thesis, however we give a brief outlook on which further research still needs to be done and what open task could be worked on in the future.

2 Complemented subsets

Before we can talk about the categorical aspects of complemented subsets, we will first present the fundamental notions of Bishop set theory. In this chapter we will follow [9] to introduce the basic elements of BST and prove some properties about subsets of a set, partial functions and complemented subsets.

2.1 Sets and functions

We have a primitive notion of a *totality.* Any totality X is defined through a membership condition $x \in X$, i.e. by describing what must be done to construct an *element* of X. For a totality X, we introduce the notion of an *equality* $x =_X y$ defined for any $x, y \in X$, satisfying the properties of an equivalence relation, i.e. reflexive, symmetric and transitive.

Definition 2.1.1. For a totality X and an equality $=_X$ we call the pair $(X, =_X)$ a *set.* We will only write X, if the equality $=_X$ is clear from the context.

We denote by $\mathbb{V}_0$ the totality of sets, which contains the primitive set $\mathbb{N}$ and all defined sets. $\mathbb{V}_0$ itself is not a set but a class. We clearly distinguish classes from sets. A *class* is a totality defined through a membership condition in which a quantification over $\mathbb{V}_0$ occurs. Particularly, the powerset $\mathcal{P}(X)$ of a set X, the totality $\mathcal{P}^{\rrbracket\llbracket}(X)$ of complemented subsets of a set X and the totality $\mathscr{F}(X, Y)$ of partial functions from a set X to a set Y are classes.

Definition 2.1.2. If X, Y are sets, their *product* $X \times Y$ is the totality defined by

$$z \in X \times Y :\Leftrightarrow \exists_{x \in X} \exists_{y \in Y} \big(z := (x, y)\big)$$

with the equality

$$z =_{X \times Y} z' :\Leftrightarrow (x, y) =_{X \times Y} (x', y') :\Leftrightarrow x =_X x' \ \& \ y =_Y y'$$

Definition 2.1.3. A bounded formula on a set X is called an *extensional property* on X, if

$$\forall_{x,y \in X} \big([x =_X y \ \& \ P(x)] \Rightarrow P(y)\big).$$

Definition 2.1.4. Let X be a set. An *inequality* on X, or an *apartness relation* on X, is a relation $x \neq_X y$ such that the following conditions are satisfied:
(i) $\forall_{x,y \in X}(x =_X y \ \& \ x \neq_X y \Rightarrow \bot)$
(ii) $\forall_{x,y \in X}(x \neq_X y \Rightarrow y \neq_X x)$
(iii) $\forall_{x,y \in X}\big(x \neq_X y \Rightarrow \forall_{z \in X}(z \neq_X x \vee z \neq_X y)\big)$
We write $(X, =_X, \neq_X)$ to denote the equality-inequality structure of a set X, and for simplicity we refer to the set $(X, =_X, \neq_X)$.

Remark 2.1.5. Let $(X, =_X, \neq_X)$ and $(Y, =_Y, \neq_Y)$ be sets.
(i) An inequality relation $x \neq_X y$ is extensional on $X \times X$.
(ii) The canonical inequality on $X \times Y$ induced by $\neq_X$ and $\neq_Y$, which is defined by

$$(x,y) \neq_{X\times Y} (x',y') :\Leftrightarrow x \neq_X x' \ \vee \ y \neq_Y y',$$

for every (x,y) and $(x',y') \in X \times Y$, is an inequality on $X \times Y$.

Proof. (i) Let $x \neq_X y$ and let $x', y' \in X$ with $x = x'$ and $y = y'$. By property (iii) in the definition of an inequality, we have that $x \neq x'$ or $x' \neq_X y$. Since the former is excluded definitionally, it holds that $x' \neq_X y$. Again by (iii), $x' \neq_X y$ od $y' \neq_X y$, which is again excluded. Hence $x' \neq_X y'$.
(ii) We show that the three conditions for an inequality are satisfied. Let $(x,y),(x',y') \in X \times Y$ with $(x,y) \neq_{X\times Y} (x',y')$. If $(x,y) =_{X\times Y} (x',y')$, then we have $x =_X x \ \wedge \ x \neq_X x' \Rightarrow \perp$ or $y =_Y y' \wedge y \neq_Y y' \Rightarrow \perp$. Obviously $(x,y) \neq_{X\times Y} (x',y') \Rightarrow (x',y') \neq_{X\times Y} (x,y)$. If $(z,z') \in X \times Y$, then $z \neq_X x \vee z \neq_X x'$ and $z' \neq_Y y \vee z' \neq_Y y'$. Hence $(z,z') \neq_{X\times Y} (x,y)$ or $(z,z') \neq_{X\times Y} (x',y')$. So $\neq_{X\times Y}$ is an inequality. $\square$

Definition 2.1.6.
(i) Let X, Y be totalities. A *non-dependent assignement routine* $f : X \rightsquigarrow Y$ from X to Y assigns to each element $x \in X$ an element $f(x) := y \in Y$.
(ii) If $(X, =_X)$ and $(Y, =_Y)$ are sets, a *function* $f : X \to Y$ is an assignement routine from X to Y that respects equality, i.e.

$$\forall_{x,x' \in X}\big(x =_X x' \Rightarrow f(x) =_Y f(x')\big).$$

(iii) A function $f : X \to Y$ is called an *embedding*, in symbols $f : X \hookrightarrow Y$, if

$$\forall_{x,x' \in X}\big(f(x) =_Y f(x') \Rightarrow x =_X x'\big).$$

If X is a set, the *identity map* id_X on X is the function $\mathrm{id}_X : X \rightsquigarrow X$, defined by $\mathrm{id}_X(x) := x$, for every $x \in X$. We denote the totality of all functions from X to Y by $\mathbb{F}(X,Y)$.

Definition 2.1.7. Let $(X, =_X, \neq_X)$ and $(Y, =_Y, \neq_Y)$ be sets. A function $f : X \to Y$ is called *strongly extensional*, if

$$\forall_{x,x' \in X}\big(f(x) \neq_Y f(x') \Rightarrow x \neq_X x'\big).$$

2.2 Subsets of a set

Definition 2.2.1. Let X be a set. A *subset* of X is a pair (A, i_A^X), where A is a set and $i_A^X : A \hookrightarrow X$ is an embedding of A into X. If (A, i_A^X) and (B, i_B^X) are subsets of X, then A is a subset of B, in symbols $(A, i_A^X) \subseteq (B, i_B^X)$, if there is a function $f : A \to B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "i_A^X"'] & & B \arrow[dl, hook', "i_B^X"] \\ & X & \end{tikzcd}$$

In this case we use the notion $f : A \subseteq B$. Usualy we write A instead of (A, i_A^X). The totalitiy of the subsets of X is the powerset $\mathcal{P}(X)$ of X and it is equipped with the equality

$$(A, i_A^X) =_{\mathcal{P}(X)} (B, i_B^X) :\Leftrightarrow A \subseteq B \ \& \ B \subseteq A$$

If $f : A \subseteq B$ and $g : B \subseteq A$, we write $(f,g) : A =_{\mathcal{P}(X)} B$.

Definition 2.2.2. If (A, i_A^X), $(B, i_B^X) \subseteq X$, their *union* $A \cup B$ is the totality defined by

$$z \in A \cup B :\Leftrightarrow z \in A \ \vee \ z \in B,$$

equipped with the non-dependent assignement routine $i_{A\cup B}^X : A \cup B \rightsquigarrow X$

$$i_{A\cup B}^X(z) := \begin{cases} i_A^X(z) & , z \in A \\ i_B^X(z) & , z \in B \end{cases}$$

Definition 2.2.3. If $(A, i_A^X), (B, i_B^X) \subseteq X$, their *intersection* $A \cap B$ is the totality defined by seperation on $A \times B$ as follows:

$$A \cap B := \big\{(a, b) \in A \times B \mid i_A^X(a) =_X i_B^X(b)\big\}.$$

Let the non-dependent assignement routine $i_{A\cap B}^X : A \cap B \rightsquigarrow X$, defined by $i_{A\cap B}^X(a, b) := i_A^X(a)$, for every $(a, b) \in A \cap B$. If (a, b) and (a', b') are in $A \cap B$, let

$$(a, b) =_{A\cap B} (a', b') :\Leftrightarrow i_{A\cap B}^X(a, b) =_X i_{A\cap B}^X(a', b') :\Leftrightarrow i_A^X(a) =_X i_A^X(a').$$

We write $A \between B$ to denote that the intersection $A \cap B$ is *inhabited*, i.e. $\exists_{x\in A\cap B}(x =_{A\cap B} x)$.

Proposition 2.2.4. Let A, B and C be subsets of the set X.
(i) $A \cup B =_{\mathcal{P}(X)} B \cup A$ and $A \cap B =_{\mathcal{P}(X)} B \cap A$.
(ii) $A \cup (B \cup C) =_{\mathcal{P}(X)} (A \cup B) \cup C$ and $A \cap (B \cap C) =_{\mathcal{P}(X)} (A \cap B) \cap C$.
(iii) $A \cap (B \cup C) =_{\mathcal{P}(X)} (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) =_{\mathcal{P}(X)} (A \cup B) \cap (A \cup C)$.

Proof. The proofs of (i) and (ii) are straightforward. We will only show the first statement of (iii), the other one follows similarly.

$$\begin{aligned} A \cap (B \cup C) &:= \big\{(a, x) \in A \times (B \cup C) \mid i_A^X(a) =_X i_{B\cup C}^X(z)\big\} \\ &:= \big\{(a, b) \in A \times B \ \vee \ (a, c) \in A \times C \mid i_A^X(a) =_X i_B^X(b) \ \vee \ i_A^X(a) =_X i_C^X(c)\big\} \\ &=_{\mathcal{P}(X)} \big\{(a, b) \in A \times B \mid i_A^X(a) =_X i_B^X(b)\big\} \cup \big\{(a, c) \in A \times C \mid i_A^X(a) =_X i_C^X(c)\big\} \\ &:= (A \cap B) \cup (A \cap C) \end{aligned}$$

$\square$

Definition 2.2.5. Let X, Y be sets, $(A, i_A^X), (C, i_C^X) \subseteq X$, $e : (A, i_A^X) \subseteq (C, i_C^X)$, $f : C \to Y$ and $(B, i_B^X) \subseteq Y$. The *restriction* $f_{|A}$ of f to A is the function $f_A := f \circ e$

$$\begin{tikzcd} A \arrow[r, hook, "e"] \arrow[rr, bend right, "f_{|A}"'] & C \arrow[r, "f"] & Y. \end{tikzcd}$$

The *image* $f(A)$ of A under f is the pair $f(A) := (A, f_A)$, where A is equipped with the equality $a =_{f(A)} a' :\Leftrightarrow f_{|A}(a) =_Y f_{|A}(a')$, for every $a, a' \in A$. We denote $f(A) := \{f(a) \mid a \in A\}$. The *pre-image* $f^{-1}(B)$ of B under f is the set

$$f^{-1}(B) := \big\{(c, b) \in C \times B \mid f(c) =_Y i_B^Y(b)\big\}.$$

Let $i_{f^{-1}(B)} : f^{-1} \hookrightarrow C$, defined by $i_{f^{-1}(B)}(c, b) := c$, for every $(c, b) \in f^{-1}(B)$. The equality of the extensional subset $f^{-1}(B)$ of $C \times B$ is inherited from the equality of $C \times B$.

Proposition 2.2.6. Let X, Y be sets, A, B subsets of X, C, D subsets of Y and $f : X \to Y$. Then
(i) $f^{-1}(C \cup D) =_{\mathcal{P}(X)} f^{-1}(C) \cup f^{-1}(D)$.
(ii) $f^{-1}(C \cap D) =_{\mathcal{P}(X)} f^{-1}(C) \cap f^{-1}(D)$.
(iii) $f(A \cup B) =_{\mathcal{P}(Y)} f(A) \cup f(B)$.
(iv) $f(A \cap B) =_{\mathcal{P}(Y)} f(A) \cap f(B)$.
(v) $A \subseteq f^{-1}\big(f(A)\big)$.
(vi) $f\big(f^{-1}(C) \cap A\big) =_{\mathcal{P}(Y)} C \cap f(A)$ and $f\big(f^{-1}(C)\big) =_{\mathcal{P}(Y)} C \cap f(X)$.

Proof.

$$\begin{aligned}
\text{(i) } f^{-1}(C \cup D) &:= \big\{(x, y) \in X \times (C \cup D) \mid f(x) =_Y i^Y_{C \cup D}(y)\big\} \\
&:= \big\{(x, y) \in X \times C \ \vee \ (x, y) \in X \times D \mid f(x) =_Y i^Y_C(y) \ \vee \ f(x) =_Y i^Y_D(y)\big\} \\
&=_{\mathcal{P}(X)} \big\{(x, y) \in X \times C \mid f(x) =_Y i^Y_C(y)\big\} \cup \big\{(x, y) \in X \times D \mid f(x) =_Y i^Y_D(y)\big\} \\
&:= f^{-1}(C) \cup f^{-1}(D)
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } f^{-1}(C \cap D) &:= \big\{(x, y) \in X \times (C \cap D) \mid f(x) =_Y i^Y_{C \cap D}(y)\big\} \\
&:= \big\{(x, (c, d)) \in X \times (C \times D) \mid f(x) =_Y i^Y_{C \cap D}(c, d) := i^Y_C(c) \text{ and } i^Y_C(c) =_Y i^Y_D(d)\big\} \\
&=_{\mathcal{P}(X)} \big\{(x, c) \in X \times C \mid f(x) =_Y i^Y_C(y)\big\} \cap \big\{(x, d) \in X \times D \mid f(x) =_Y i^Y_D(y)\big\} \\
&:= f^{-1}(C) \cap f^{-1}(D)
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } f(A \cup B) &:= \{f(x) \mid x \in A \cup B\} \\
&:= \{f(x) \mid x \in A \ \vee \ x \in B\} \\
&=_{\mathcal{P}(Y)} \{f(x) \mid x \in A\} \cup \{f(x) \mid x \in B\} \\
&:= f(A) \cup f(B)
\end{aligned}$$

$$\begin{aligned}
\text{(iv) } f(A \cap B) &:= \{f(x) \mid x \in A \cap B\} \\
&:= \{f(x) \mid x = (a, b) \in A \times B\} \\
&=_{\mathcal{P}(Y)} \{f(a) \mid a \in A\} \cap \{f(b) \mid b \in B\} \\
&:= f(A) \cap f(B)
\end{aligned}$$

(v) Let $a \in A$, then $f(a) \in f(A)$. Since

$$f^{-1}\big(f(A)\big) := \big\{\big(x, f(a)\big) \in X \times f(A) \mid f(x) = f(a)\big\},$$

we have $\big(a, f(a)\big) \in f^{-1}\big(f(A)\big)$ for every $a \in A$.
So $A \subseteq f^{-1}(f(A))$.

$$\begin{aligned}
\text{(vi) } f\big(f^{-1}(C) \cap A\big) &:= f\Big(\big\{\big((x, c), a\big) \in (X \times C) \times A \mid f(x) =_Y i^Y_C(c),\ i^X_{f^{-1}(C)}(x, c) := x =_X i^X_A(a)\big\}\Big) \\
&=_{\mathcal{P}(Y)} \big\{\big((f(x), c), f(a)\big) \in (f(X) \times C) \times f(A) \mid f(x) =_Y i^Y_C(c),\ f(x) =_Y f(a)\big\} \\
&=_{\mathcal{P}(Y)} \big\{\big(c, f(a)\big) \in C \times f(A) \mid f(a) =_Y i^Y_C(c)\big\} \\
&:= C \cap f(A)
\end{aligned}$$

The second equation is a special case of the one that we just proved. Since $f^{-1}(C) \subseteq X$, we have $f\big(f^{-1}(C) \cap X\big) = f\big(f^{-1}(C)\big) =_{\mathcal{P}(Y)} C \cap f(X)$.

$\square$

Proposition 2.2.7. Let $(A, i_A^X), (B, i_B^X), (A', i_A'^X), (B', i_B'^X) \subseteq X$, such that $A =_{\mathcal{P}(X)} A'$ and $B =_{\mathcal{P}(X)} B'$. Let also $(C, i_C^Y), (C', i_C'^Y), (D, i_D^Y) \subseteq Y$, such that $C =_{\mathcal{P}(X)} C'$ and let $f : X \to Y$.
(i) $A \cap B =_{\mathcal{P}(X)} A' \cap B'$ and $A \cup B =_{\mathcal{P}(X)} A' \cup B$.
(ii) $f(A) =_{\mathcal{P}(Y)} f(A')$ and $f^{-1}(C) =_{\mathcal{P}(X)} f^{-1}(C')$.
(iii) $(A \times C, i_A^X \times i_C^Y) \subseteq X \times Y$, where the map $i_A^X \times i_C^Y : A \times C \hookrightarrow X \times Y$ is defined by

$$(i_A^X \times i_C^Y)(a, c) := \big(i_A^X(a), i_C^Y(c)\big), \qquad (a, c) \in A \times C.$$

(iv) $A \times C =_{\mathcal{P}(X \times Y)} A' \times C'$.
(v) $A \times (C \cup D) =_{\mathcal{P}(X \times Y)} (A \times C) \cup (A \times D)$.
(vi) $A \times (S \cap D) =_{\mathcal{P}(X \times Y)} (A \times C) \cap (A \cap D)$.

Proof. We will show (i)-(iv). Let $A =_{\mathcal{P}(X)} A'$ and $B =_{\mathcal{P}(X)} B'$, i.e. $A \subseteq A'$ & $A' \subseteq A$ and $B \subseteq B'$ & $B' \subseteq B$.

(i) $A \cup B = \{z \mid z \in A \vee z \in B\} =_{\mathcal{P}(X)} \{z \mid z \in A' \vee z \in B'\} := A' \cup B'$

$$\begin{aligned} A \cap B &:= \big\{(a, b) \in A \times B \mid i_A^X(a) =_X i_B^X(b)\big\} \\ &=_{\mathcal{P}(X)} \big\{(a, b) \in A' \times B' \mid i_A'^X(a) =_X i_B'^Y(b)\big\} \\ &:= A' \cap B' \end{aligned}$$

(ii) $f(A) :=\big\{f(a) \mid a \in A\} =_{\mathcal{P}(Y)} \{f(a) \mid a \in A'\big\} := f(A')$

$$\begin{aligned} f^{-1}(C) &:= \big\{(x, c) \in X \times C \mid f(x) =_Y i_C^X(b)\big\} \\ &=_{\mathcal{P}(X)} \big\{(x, c) \in X \times C' \mid f(x) =_Y i_{C'}^X(c)\big\} \\ &:= f^{-1}(C') \end{aligned}$$

(iii) Let $(a, c) \in A \times C$. Since $A \subseteq X$, $i_A^X : A \hookrightarrow X$ and $C \subseteq Y$, $i_C^Y : C \hookrightarrow Y$, we have

$$(i_A^X \times i_C^Y)(a, c) := \big(i_A^x(a), i_C^Y(c)\big) \in X \times Y, \quad (a, c) \in A \times C$$

(iv) $A \times C = \big\{(a, c) \mid a \in A,\ c \in C\big\} =_{\mathcal{P}(X \times Y)} \big\{(a, c) \mid a \in A',\ c \in C'\big\} := A' \times C'$ $\square$

2.3 Partial functions

Definition 2.3.1. Let X, Y be sets. A *partial function from* X *to* Y is a triplet (A, i_A^X, f_A^Y), where $(A, i_A^X) \subseteq X$ and $f_A^Y \in \mathbb{F}(A, Y)$. We will sometimes use the notation f_A^Y instead of the triplet (A, i_A^X, f_A^Y) and we write $f_A^Y : X \rightharpoonup Y$. If (A, i_A^X, f_A^Y) and (B, i_B^X, f_B^Y) are partial functions from X to Y, we call f_A^Y a *subfunction* of f_B^Y, in symbols $(A, i_A^X, f_A^Y) \leq (B, i_B^X, f_B^Y)$, or $f_A^Y \leq f_B^Y$, if there is $e_{AB} : A \to B$ such that the following inner diagrams commute

$$\begin{tikzcd} A \arrow[rr, "e_{AB}"] \arrow[dr, hook, "i_A^X"'] \arrow[ddr, bend right, "f_A^Y"'] & & B \arrow[dl, hook', "i_B^X"] \arrow[ddl, bend left, "f_b^Y"] \\ & X & \\ & Y & \end{tikzcd}$$

In this case we use the notation $e_{AB} : f_A^Y \leq f_B^Y$. The *totality of partial functions from X to Y* is the partial function space $\mathscr{F}(X,Y)$ and it is equipped with the equality

$$(A, i_A^X, f_A^Y) =_{\mathscr{F}(X,Y)} (B, i_B^X, f_B^Y) :\Leftrightarrow f_A^Y \leq f_B^Y \ \& \ f_B^Y \leq f_A^Y.$$

If $e_{AB} : f_A^Y \leq f_B^Y$ and $e_{BA} : f_B^Y \leq f_A^Y$, we write $(e_{AB}, e_{BA}) : f_A^Y =_{\mathscr{F}(X,Y)} f_B^Y$.

Since the membership condition for $\mathscr{F}(X,Y)$ requires quantification over $\mathbb{V}_0$, the totality $\mathscr{F}(X,Y)$ is not a set, but a class. We denote by $\mathscr{F}(X,\mathbb{2})$ the totality of partial functions from the set X to the Booleans $\mathbb{2} := \{0,1\}$.

Definition 2.3.2. Let the operation of multiplication on $\mathbb{2} := \{0,1\}$ be defined by $0 \cdot 1 := 1 \cdot 0 := 0 \cdot 0 := 0$ and $1 \cdot 1 := 1$. If $(A, i_A^X, f_A^{\mathbb{2}}), (B, i_B^X, g_B^{\mathbb{2}}) \in \mathscr{F}(X,\mathbb{2})$, let

$$f_A \cdot g_B := \big(A \cap B, i_{A\cap B}^X, (f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}\big),$$

where $(f_A \cdot g_B)_{A\cap B}^{\mathbb{2}} : A \cap B \to \mathbb{2}$ is defined, for every $(a,b) \in A \cap B$, by

$$(f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}(a,b) := f_A^{\mathbb{2}}(a) \cdot g_B^{\mathbb{2}}(b).$$

Remark 2.3.3. If $(a,b), (c,d) \in A \cap B$ and $(a,b) =_{A\cap B} (c,d)$, then $f_A^{\mathbb{2}}(a) =_{\mathbb{2}} f_A^{\mathbb{2}}(c)$ and $f_B^{\mathbb{2}}(b) =_{\mathbb{2}} f_B^{\mathbb{2}}(d)$. By the equality of the product on $A \cap B$, it follows

$$(f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}(a,b) := f_A^{\mathbb{2}}(a) \cdot g_B^{\mathbb{2}}(b) =_{\mathbb{2}} f_A^{\mathbb{2}}(c) \cdot g_B^{\mathbb{2}}(d) =: (f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}(c,d).$$

Hence $(f_A \cdot g_B)_{A\cap B}^{\mathbb{2}}$ is a function.

2.4 Complemented subsets

The notion of a complemented subset gives us a positively defined notion of disjointness of subsets of X. This allows us to avoid the negative definition of the complement of a set.

Definition 2.4.1. Let $(X, =_X, \neq_X)$ be a set and $(A, i_A^X), (B, i_B^X) \subseteq X$. A and B are *disjoint with respect to $\neq_X$*, in symbols $A]\![_{\neq_X} B$, if

$$A]\!] [\![_{\neq_X} B :\Leftrightarrow \forall_{a\in A}\forall_{b\in B}\big(i_A^X(a) \neq_X i_B^X(b)\big).$$

If $\neq_X$ is clear from the context, we write $A]\!] [\![B$.

Clearly, if $A]\!] [\![B$, then $A \cap B$ is not inhabited.

Definition 2.4.2. A *complemented subset* of a set $(X, =_X, \neq_X)$ is a pair $\mathbf{A} := (A^1, A^0)$, where $(A^1, i_{A^1}^X)$ and $(A^0, i_{A^0}^X)$ are subsets of X such that $A^1]\!] [\![A^0$. We call A^1 the *1-component* of $\mathbf{A}$ and A^0 the *0-component* of $\mathbf{A}$.
If $\mathtt{Dom}(\mathbf{A}) := A^1 \cup A^0$ is the *domain* of $\mathbf{A}$, the *indicator function*, or *characterictic function*, of $\mathbf{A}$ is the operation $\chi_{\mathbf{A}} : \mathtt{Dom}(\mathbf{A}) \rightsquigarrow \mathbb{2}$ defined by

$$\chi_{\mathbf{A}}(x) := \begin{cases} 1 & , x \in A^1 \\ 0 & , x \in A^0. \end{cases}$$

Let $x \in \mathbf{A} :\Leftrightarrow x \in A^1$ and $x \notin \mathbf{A} :\Leftrightarrow x \in A^0$. If $\mathbf{A}$, $\mathbf{B}$ are complemented subsets of X, let

$$\mathbf{A} \subseteq \mathbf{B} :\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 \subseteq A^0,$$
$$\mathbf{A} \subseteq_0 \mathbf{B} :\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 = A^0,$$
$$\mathbf{A} \subseteq_1 \mathbf{B} :\Leftrightarrow A^1 = B^1 \ \& \ B^0 \subseteq A^0.$$

Let $\mathcal{P}^{]\![}(X)$ be their *totality*, equipped with the equality $\mathbf{A} =_{\mathcal{P}^{]\![}(X)} \mathbf{B} :\Leftrightarrow \mathbf{A} \subseteq \mathbf{B} \ \& \ \mathbf{B} \subseteq \mathbf{A}$. A *map* $\boldsymbol{f} : \mathbf{A} \to \mathbf{B}$ from $\mathbf{A}$ to $\mathbf{B}$ is a pair (f^1, f^0), where $f^1 : A^1 \to B^1$ and $f^0 : A^0 \to B^0$.

Remark 2.4.3.
i) Clearly $\mathbf{A} =_{\mathcal{P}^{]\![}(X)} \mathbf{B} :\Leftrightarrow A^1 =_{\mathcal{P}^{]\![}(X)} B^1 \ \& \ A^0 =_{\mathcal{P}^{]\![}(X)} B^0$.
ii) Since the membership condition for $\mathcal{P}^{]\![}(X)$ requires quantification over $\mathbb{V}_0$, the totality $\mathcal{P}^{]\![}(X)$ is a class.
iii) The operation $\chi_{\mathbf{A}}$ is a partial function in $\mathscr{F}(X, \mathbb{2})$. Let $z, w \in A^1 \cup A^0$, such that $z =_{A^1 \cup A^0} w$, i.e.

$$\left.\begin{matrix} i^X_{A^1}(z) & , z \in A^1 \\ i^X_{A^0}(z) & , z \in A^0 \end{matrix}\right\} := i^X_{A^1 \cup A^0}(z) =_X i^X_{A^1 \cup A^0}(w) := \begin{cases} i^X_{A^1}(w) & , w \in A^1 \\ i^X_{A^0}(w) & , w \in A^0. \end{cases}$$

Let $z \in A^1$. If $w \in A^0$, then

$$i^X_{A^1}(z) := i^X_{A^1 \cup A^0}(z) =_X i_{A^1 \cup A^0}(w) := i^X_{A^0}(w),$$

therefore $(z, w) \in A^1 \cap A^0$, but this is a contradicton to $A^1]\![A^0$. Hence $w \in A^1$ and

$$\chi_{\mathbf{A}}(z) = \chi_{\mathbf{A}}(w).$$

If $z \in A^0$, we work in a similar way.

Definition 2.4.4. If $\mathbf{A}, \mathbf{B} \in \mathcal{P}^{]\![}(X)$ and $\mathbf{C} \in \mathcal{P}^{]\![}(Y)$, we define

$$\mathbf{A} \cup \mathbf{B} := (A^1 \cup B^1, A^0 \cap B^0),$$
$$\mathbf{A} \cap \mathbf{B} := (A^1 \cap B^1, A^0 \cup B^0),$$
$$-\mathbf{A} := (A^0, A^1),$$
$$\mathbf{A} - \mathbf{B} := (A^1 \cap B^0, A^0 \cup B^1),$$
$$\mathbf{A} \times \mathbf{C} := \big(A^1 \times C^1, [A^0 \times Y] \cup [X \times C^0]\big).$$

Proposition 2.4.5. If $\mathbf{A}, \mathbf{B} \in \mathcal{P}^{]\![}(X)$ and $\mathbf{C} \in \mathcal{P}^{]\![}(Y)$, then $\mathbf{A} \cup \mathbf{B}$, $\mathbf{A} \cap \mathbf{B}$, $-\mathbf{A}$ and $\mathbf{A} - \mathbf{B}$ are in $\mathcal{P}^{]\![}(X)$ and $\mathbf{A} \times \mathbf{C}$ is in $\mathcal{P}^{]\![}(X \times Y)$.

Proof. We only show $\mathbf{A} \cup \mathbf{B} \in \mathcal{P}^{]\![}(X)$ and $\mathbf{A} \times \mathbf{C} \in \mathcal{P}^{]\![}(X \times Y)$, the others follow in a similar way. Let $\mathbf{A}, \mathbf{B} \in \mathcal{P}^{]\![}(X)$. If $a_1 \in A^1$, $a_0 \in A^0$, $b_1 \in B^1$ and $b_0 \in B^0$, then

$$i^X_{A^1}(a_1) \neq_X i^X_{A^0}(a_0) \ \text{ and } \ i^X_{B^1}(b_1) \neq_X i^X_{B^0}(b_0).$$

By definition

$$i^X_{A^1 \cup B^1}(z) := \begin{cases} i^X_{A^1}(z) & , \text{if } \ z \in A^1 \\ i^X_{B^1}(z) & , \text{if } \ z \in B^1 \end{cases}$$

for $z \in A^1 \cup B^1$ and

$$i^X_{A^0 \cap B^0}(a_0, b_0) := i^X_{A^0}(a_0) := i^X_{B^0}(b_0).$$

Hence

$$i^X_{A^1\cup B^1}(z) \neq_X i^X_{A^0\cap B^0}(a_0, b_0)$$

for all $z \in A^1 \cup B^1$ and $(a_0, b_0) \in A^0 \cap B^0$, so $\mathbf{A} \cup \mathbf{B} \in \mathcal{P}^{\rrbracket\llbracket}(X)$. Now let $\mathbf{C} \in \mathcal{P}^{\rrbracket\llbracket}(Y)$. If $c_1 \in C^1$ and $c_0 \in C^0$, then

$$i^Y_{C^1}(c_1) \neq_Y i^Y_{C^0}(c_0).$$

For $z \in [A^0 \times Y] \cup [X \times C^0]$

$$i^{X\times Y}_{[A^0\times Y]\cup[X\times C^0]}(z) := \begin{cases} i^{X\times Y}_{A^0\times Y}(z) & ,\text{if } \ z \in A^0 \times Y \\ i^{X\times Y}_{X\times C^0}(z) & ,\text{if } \ z \in X \times C^0. \end{cases}$$

Let $z = (a_0, y) \in A^0 \times Y$, since $i^X_{A^1}(a_1) \neq_X i^X_{A^0}(a_0)$, it holds that $i^{X\times Y}_{A^1\times C^1}(a_1, c_1) \neq_{X\times Y} i^{X\times Y}_{A^0\times Y}(a_0, y)$. Now let $z = (x, c_0) \in X \times C^0$, since $i^Y_{C^1}(c_1) \neq_Y i^Y_{C^0}(c_0)$, it holds that $i^{X\times Y}_{A^1\times C^1}(a_1, c_1) \neq_{X\times Y} i^{X\times Y}_{X\times C^0}(x, c_0)$. Hence, the following holds

$$i^{X\times Y}_{A^1\times C^1}(a_1, c_1) \neq_{X\times Y} i^{X\times Y}_{[A^0\times Y]\cup[X\times C^0]}(z)$$

for every $(a_1, c_1) \in A^1 \times C^1$ and $z \in$, so $\mathbf{A} \times \mathbf{C} \in \mathcal{P}^{\rrbracket\llbracket}(X \times Y)$. $\square$

Proposition 2.4.6. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be in $\mathcal{P}^{\rrbracket\llbracket}(X)$. Then the following hold:
(i) $-(-\mathbf{A}) := \mathbf{A}$.
(ii) $-(\mathbf{A} \cup \mathbf{B}) := (-\mathbf{A}) \cap (-\mathbf{B})$.
(iii) $-(\mathbf{A} \cap \mathbf{B}) := (-\mathbf{A}) \cup (-\mathbf{B})$.
(iv) $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$.
(v) $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$.
(vi) $\mathbf{A} - \mathbf{B} := \mathbf{A} \cap (-\mathbf{B})$.
(vii) $\mathbf{A} \subseteq \mathbf{B} \Leftrightarrow (\mathbf{A} \cap \mathbf{B}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \mathbf{A}$.
(viii) $\mathbf{A} \subseteq \mathbf{B} \Leftrightarrow -\mathbf{B} \subseteq -\mathbf{A}$.
(ix) If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{C}$, then $\mathbf{A} \subseteq \mathbf{C}$.

Proof. All the statements follow directly by the definitions. Let $\mathbf{A} := (A^1, A^0), \mathbf{B} := (B^1, B^0)$ and $\mathbf{C} := (C^1, C^0)$. Then

$$\text{(i)}\ -(-\mathbf{A}) := -(A^0, A^1) := (A^1, A^0) := \mathbf{A}.$$

$$\text{(ii)}\ (-\mathbf{A}) \cap (-\mathbf{B}) := \big((A^0, A^1) \cap (B^0, B^1)\big) := (A^0 \cap B^0, A^1 \cup B^1) := -(A^1 \cup B^1, A^0 \cap B^0) := -(\mathbf{A} \cup \mathbf{B}).$$

$$\text{(iii)}\ (-\mathbf{A}) \cup (-\mathbf{B}) := \big((A^0, A^1) \cup (B^0, B^1)\big) := (A^0 \cup B^0, A^1 \cap B^1) := -(A^1 \cap B^1, A^0 \cup B^0) := -(\mathbf{A} \cap \mathbf{B}).$$

(iv) $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) := (A^1, A^0) \cup \big((B^1, B^0) \cap (C^1, C^0)\big)$
$:= (A^1, A^0) \cup (B^1 \cap C^1, B^0 \cup C^0)$
$:= \big(A^1 \cup (B^1 \cap C^1), A^0 \cap (B^0 \cap C^0)\big)$
$=_{\mathcal{P}^{\mathbb{II}}(X)} ((A^1 \cup B^1) \cap (A^1 \cup C^1), (A^0 \cap B^0) \cup (A^0 \cap C^0))$
$:= (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$.

(v) $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) := ((A^1, A^0) \cap ((B^1, B^0) \cup (C^1, C^0))$
$:= (A^1, A^0) \cap (B^1 \cup C^1, B^0 \cap C^0)$
$:= (A^1 \cap (B^1 \cap C^1), A^0 \cup (B^0 \cap C^0))$
$=_{\mathcal{P}^{\mathbb{II}}(X)} \big((A^1 \cap B^1) \cup (A^1 \cap C^1), (A^0 \cup B^0) \cap (A^0 \cup C^0)\big)$
$:= (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$.

(vi) $\mathbf{A} - \mathbf{B} := (A^1 \cap B^0, A^0 \cup B^1)$
$:= (A^1, A^0) \cap (B^0, B^1)$
$:= \mathbf{A} \cap (-\mathbf{B})$.

(vii) $(\mathbf{A} \cap \mathbf{B}) =_{\mathcal{P}^{\mathbb{II}}(X)} \mathbf{A} \Leftrightarrow (A^1 \cap B^1, A^0 \cup B^0) =_{\mathcal{P}^{\mathbb{II}}(X)} (A^1, A^0)$
$\Leftrightarrow A^1 \cap B^1 =_{\mathcal{P}^{\mathbb{II}}(X)} A^1 \ \& \ A^0 \cup B^0 =_{\mathcal{P}^{\mathbb{II}}(X)} A^0$
$\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 \subseteq A^0$
$\Leftrightarrow \mathbf{A} \subseteq \mathbf{B}$.

(viii) $\mathbf{A} \subseteq \mathbf{B} :\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 \subseteq A^0$
$\Leftrightarrow B^0 \subseteq A^0 \ \& \ A^1 \subseteq B^1$
$\Leftrightarrow (B^0, B^1) \subseteq (A^0, A^1)$
$\Leftrightarrow -(B^1, B^0) \subseteq -(A^1, A^0)$
$:\Leftrightarrow -\mathbf{B} \subseteq -\mathbf{A}$.

(xi) Let $\mathbf{A} \subseteq \mathbf{B}$ & $\mathbf{B} \subseteq \mathbf{C}$, i.e. $A^1 \subseteq B^1$ & $B^0 \subseteq A^0$ and $B^1 \subseteq C^1$ & $C^0 \subseteq B^0$.
Therefore, we have $A^1 \subseteq C^1$ & $C^0 \subseteq A^0 :\Leftrightarrow \mathbf{A} \subseteq \mathbf{C}$. $\square$

Proposition 2.4.7. Let $\mathbf{A} \in \mathcal{P}^{\mathbb{II}}(X)$ and $\mathbf{B}, \mathbf{C} \in \mathcal{P}^{\mathbb{II}}Y)$. Then
(i) $\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) =_{\mathcal{P}^{\mathbb{II}}(X \times Y)} (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$.
(ii) $\mathbf{A} \times (\mathbf{B} \cap \mathbf{C}) =_{\mathcal{P}^{\mathbb{II}}(X \times Y)} (\mathbf{A} \times \mathbf{B}) \cap (\mathbf{A} \times \mathbf{C})$.

Proof. We have that

(i) $\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) := (A^1, A^0) \times (B^1 \cup C^1, B^0 \cap C^0)$
$:= \big(A^1 \times (B^1 \cup C^1), [A^0 \times Y] \cup [X \times (B^0 \cap C^0)]\big)$
$=_{\mathcal{P}^{\mathbb{II}}(X)} \big((A^1 \times B^1) \cup (A^1 \times C^1), [A^0 \times Y] \cup [(X \times B^0) \cap (X \times C^0)]\big)$
$=_{\mathcal{P}^{\mathbb{II}}(X)} \big((A^1 \times B^1) \cup (A^1 \times C^1), [(A^0 \times Y) \cup (X \times B^0)] \cap [(A^0 \times Y) \cup (X \times C^0)]\big)$
$=_{\mathcal{P}^{\mathbb{II}}(X)} \big(A^1 \times B^1, [A^0 \times Y] \cup [X \times B^0]\big) \cup \big(A^1 \times C^1, [A^0 \times Y] \cup [X \times C^0]\big)$
$:= (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$.

(ii) $\mathbf{A} \times (\mathbf{B} \cap \mathbf{C}) := (A^1, A^0) \times (B^1 \cap C^1, B^0 \cup C^0)$

$$
\begin{aligned}
&:= \big(A^1 \times (B^1 \cap C^1), [A^0 \times Y] \cup [X \times (B^0 \cup C^0)]\big) \\
&=_{\mathcal{P}^{\rrbracket\llbracket}(X)} \big((A^1 \times B^1) \cap (A^1 \times C^1), [A^0 \times Y] \cup [(X \times B^0) \cup (X \times C^0)]\big) \\
&=_{\mathcal{P}^{\rrbracket\llbracket}(X)} \big((A^1 \times B^1) \cap (A^1 \times C^1), [(A^0 \times Y) \cup (X \times B^0)] \cup [(A^0 \times Y) \cup (X \times C^0)]\big) \\
&=_{\mathcal{P}^{\rrbracket\llbracket}(X)} \big(A^1 \times B^1, [A^0 \times Y] \cup [X \times B^0]\big) \cap \big(A^1 \times C^1, [A^0 \times Y] \cup [X \times C^0]\big) \\
&:= (\mathbf{A} \times \mathbf{B}) \cap (\mathbf{A} \times \mathbf{C}). \quad \square
\end{aligned}
$$

Remark 2.4.8. Let $(X, =_X, \neq_X^f)$ and $(Y, =_Y, \neq_Y)$ be sets. If $f : X \to Y$, let $x_1 \neq_X^f x_2 :\Leftrightarrow f(x_1) \neq_Y f(x_2)$, for every $x_1, x_2 \in X$.

Proposition 2.4.9. Let the sets $(X, =_X, \neq_X^f)$ and $(Y, =_Y, \neq_Y)$, where $f : X \to Y$. Let also $\mathbf{A}{:=}(A^1, A^0)$ and $\mathbf{B} := (B^1, B^0)$ in $\mathcal{P}^{\rrbracket\llbracket}(X)$.
(i) $f^{-1}(\mathbf{A}) := \big(f^{-1}(A^1), f^{-1}(A^0)\big) \in \mathcal{P}^{\rrbracket\llbracket}(X)$.
(ii) $f^{-1}(\mathbf{A} \cup \mathbf{B}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} f^{-1}(\mathbf{A}) \cup f^{-1}(\mathbf{B})$.
(iii) $f^{-1}(\mathbf{A} \cap \mathbf{B}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} f^{-1}(\mathbf{A}) \cap f^{-1}(\mathbf{B})$.
(iv) $f^{-1}(-\mathbf{A}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} -f^{-1}(\mathbf{A})$.
(v) $f^{-1}(\mathbf{A} - \mathbf{B}) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} f^{-1}(\mathbf{A}) - f^{-1}(\mathbf{B})$.

Proof. (i) By the definition we have that

$$
\begin{aligned}
f^{-1}(A^1) &:= \big\{(x, a_1) \in X \times A^1 \mid f(x) =_Y i^X_{A^1}(a_1)\big\}, \quad i^X_{f^{-1}(A^1)}(x, a_1) := x, \\
f^{-1}(A^0) &:= \big\{(x, a_0) \in X \times A^0 \mid f(x) =_Y i^X_{A^0}(a_0)\big\}, \quad i^X_{f^{-1}(A^0)}(x, a_0) := x.
\end{aligned}
$$

Let $(x, a_1) \in f^{-1}(A^1)$ and $(y, a_0) \in f^{-1}(A^0)$. Since $i^X_{A^1}(a_1) \neq_Y i^X_{A^0}(a_0)$ and by the extensionality of $\neq_Y$ it holds that

$$
i^X_{f^{-1}(A^1)}(x, a_1) \neq_X^f i^X_{f^{-1}(A^0)}(y, a_0) :\Leftrightarrow x \neq_X^f y :\Leftrightarrow f(x) \neq_Y f(y) \Leftrightarrow i^X_{A^1}(a_1) \neq_Y i^X_{A^0}(a_0).
$$

Hence $f^{-1}(\mathbf{A}) \in \mathcal{P}^{\rrbracket\llbracket}(X)$. Next we have

(ii) $f^{-1}(\mathbf{A} \cup \mathbf{B}) := f^{-1}(A^1 \cup B^1, A^0 \cap B^0)$

$$
\begin{aligned}
&\overset{(i)}{:=} \big(f^{-1}(A^1 \cup B^1), f^{-1}(A^0 \cap B^0)\big) \\
&=_{\mathcal{P}^{\rrbracket\llbracket}(X)} \big(f^{-1}(A^1) \cup f^{-1}(B^1), f^{-1}(A^0) \cap f^{-1}(B^0)\big) \\
&:= f^{-1}(\mathbf{A}) \cup f^{-1}(\mathbf{B}).
\end{aligned}
$$

(iii) $f^{-1}(\mathbf{A} \cap \mathbf{B}) := f^{-1}(A^1 \cap B^1, A^0 \cup B^0)$

$$
\begin{aligned}
&\overset{(i)}{:=} \big(f^{-1}(A^1 \cap B^1), f^{-1}(A^0 \cup B^0)\big) \\
&=_{\mathcal{P}^{\rrbracket\llbracket}(X)} \big(f^{-1}(A^1) \cap f^{-1}(B^1), f^{-1}(A^0) \cup f^{-1}(B^0)\big) \\
&:= f^{-1}(\mathbf{A}) \cap f^{-1}(\mathbf{B}).
\end{aligned}
$$

(iv) $f^{-1}(-\mathbf{A}) := f^{-1}(A^0, A^1)$

$$
\begin{aligned}
&\overset{(i)}{:=} \big(f^{-1}(A^0), f^{-1}(A^1)\big) \\
&= -\big(f^{-1}(A^1), f^{-1}(A^0)\big) \\
&:= -f^{-1}(A^1, A^0) \\
&:= -f^{-1}(\mathbf{A}).
\end{aligned}
$$

(v) $f^{-1}(\mathbf{A}-\mathbf{B}) := f^{-1}(A^1 \cap B^0, A^0 \cup B^1)$
$$\begin{aligned} &:= \big(f^{-1}(A^1 \cap B^0), f^{-1}(A^0 \cup B^1)\big) \\ &= \big(f^{-1}(A^1) \cap f^{-1}(B^0), f^{-1}(A^0) \cup f^{-1}(B^1)\big) \\ &:= f^{-1}(\mathbf{A}) - f^{-1}(\mathbf{B}). \quad \square \end{aligned}$$

The following definition gives alternative operations between complemented subsets.

Definition 2.4.10. If $\mathbf{A}, \mathbf{B} \in \mathcal{P}^{][}(X)$ and $\mathbf{C} \in \mathcal{P}^{][}(Y)$, let

$$\mathbf{A} \vee \mathbf{B} := \big([A^1 \cap B^1] \cup [A^1 \cap B^0] \cup [A^0 \cap B^1], A^0 \cap B^0\big),$$
$$\mathbf{A} \wedge \mathbf{B} := \big(A^1 \cap B^1, [A^1 \cap B^0] \cup [A^0 \cap B^1] \cup [A^0 \cap B^0]\big),$$
$$\mathbf{A} \ominus \mathbf{B} := \mathbf{A} \wedge (-\mathbf{B}),$$
$$\mathbf{A} \otimes \mathbf{C} := \big(A^1 \times C^1, [A^1 \times C^0] \cup [A^0 \times C^1] \cup [A^0 \times C^0]\big).$$

Remark 2.4.11. If $\mathbf{A}, \mathbf{B} \in \mathcal{P}^{][}(X)$ and $\mathbf{C} \in \mathcal{P}^{][}(Y)$, then $\mathbf{A} \vee \mathbf{B}$, $\mathbf{A} \wedge \mathbf{B}$, $\mathbf{A} \ominus \mathbf{B}$ are in $\mathcal{P}^{][}(X)$ and $\mathbf{A} \otimes \mathbf{C}$ is in $\mathcal{P}^{][}(X \times Y)$.

Proof. The proofs are straightforward to show. We only show the first and last membership. By definition $\mathbf{A} \vee \mathbf{B} := \big([A^1 \cap B^1] \cup [A^1 \cap B^0] \cup [A^0 \cap B^1], A^0 \cap B^0\big) =: \big((A \vee B)^1, (A \vee B)^0\big)$.
We have $\big((A \vee B)^1, i^X_{(A \vee B)^1}\big)$ and $\big((A \vee B)^0, i^X_{(A \vee B)^0}\big)$, where

$$i^X_{(A\vee B)^1}(z) := \begin{cases} i^X_{A^1 \cap B^1}(a,b) & \text{,if } z := (a,b) \in A^1 \cap B^1 \\ i^X_{A^1 \cap B^0}(a,b) & \text{,if } z := (a,b) \in A^1 \cap B^0 \\ i^X_{A^0 \cap B^1}(a,b) & \text{,if } z := (a,b) \in A^0 \cap B^1 \end{cases} := \begin{cases} i^X_{A^1}(a) & \text{,if } (a,b) \in (A^1 \cap B^1) \cup (A^1 \cap B^0) \\ i^X_{A^0}(a) & \text{,if } a \in A^0 \cap B^1 \end{cases}$$

and

$$i^X_{(A\vee B)^0}(z) := i^X_{A^0 \cap B^0}(a,b) := i^X_{A^0}(a) \quad \text{,for every } z := (a,b) \in A^0 \cap B^0.$$

Let $(a_1, b_1) \in (A \vee B)^1$, $(a_0, b_0) \in A^0 \cap B^0$, then we get the required inequality

$$i^X_{(A\vee B)^1}(a_1, b_1) \neq_X i^X_{A^0 \cap B^0}(a_0, b_0).$$

Hence $(A \vee B)^1][(A \vee B)^0$ and therefore $\mathbf{A} \vee \mathbf{B} \in \mathcal{P}^{][}(X)$. Now let $(a_1, c_1) \in A^1 \times C^1$ and $(a_0, c_0) \in A^0 \times C^0$, then

$$i_{A^1}(a_1) \neq_X i_{A^0}(a_0) \text{ and } i_{C^1}(c_1) \neq_Y i_{C^0}(c_0).$$

By definition

$$i^{X \times Y}_{A^1 \times C^1}(a_1, c_1) := (i^X_{A^1}(a_1), i^Y_{C^1}(c_1).$$

If $(a_1, c_0) \in A^1 \times C^0$, then $i^{X\times Y}_{A^1 \times C^0}(a_1, c_0) = (i^X_{A^1}(a_1), i^Y_{C^0}(c_0))$, if $(a_0, c_1) \in A^0 \times C^1$, then $i^{X\times Y}_{A^0 \times C^1}(a_0, c_1) = (i^X_{A^0}(a_0), i^Y_{C^1}(c_1))$ and if $(a_0, c_0) \in A^0 \times C^0$, then $i^{X\times Y}_{A^0 \times C^0}(a_0, c_0) = (i^X_{A^0}(a_0), i^Y_{C^0}(c_0))$.
In every one f these cases we have the inequality

$$i^{X\times Y}_{A^1 \times C^1}(a_1, c_1) \neq_{X \times Y} \begin{cases} i^{X\times Y}_{A^1 \times C^0}(a_1, c_0) \\ i^{X\times Y}_{A^0 \times C^1}(a_0, c_1) \\ i^{X\times Y}_{A^0 \times C^0}(a_0, c_0). \end{cases}$$

Hence $\mathbf{A} \otimes \mathbf{C} \in \mathcal{P}^{\rrbracket\llbracket}(X \times Y)$. $\square$

With the previous definitions the corresponding characteristic functions are expressed through the characteristic functions of $\mathbf{A}$ and $\mathbf{B}$, which is shown in the next proposition.

Proposition 2.4.12. If $\mathbf{A}$, $\mathbf{B}$ are complemented subsets of X, then $\mathbf{A} \vee \mathbf{B}$, $\mathbf{A} \wedge \mathbf{B}$, $\mathbf{A} \ominus \mathbf{B}$, $\mathbf{A} \otimes \mathbf{B}$ and $-\mathbf{A}$ are complemented subsets of X with chraracteristic functions
(i) $\chi_{\mathbf{A} \vee \mathbf{B}} =_{\mathscr{F}(X,\mathbb{2})} \chi_{\mathbf{A}} \vee \chi_{\mathbf{B}}$.
(ii) $\chi_{\mathbf{A} \wedge \mathbf{B}} =_{\mathscr{F}(X,\mathbb{2})} \chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}}$.
(iii) $\chi_{\mathbf{A} \ominus \mathbf{B}} =_{\mathscr{F}(X,\mathbb{2})} \chi_{\mathbf{A}}(1 - \chi_{\mathbf{B}})$.
(iv) $\chi_{\mathbf{A} \otimes \mathbf{B}}(x, y) =_{\mathscr{F}(X \times X,\mathbb{2})} \chi_{\mathbf{A}}(x) \cdot \chi_{\mathbf{B}}(y)$.
(v) $\chi_{-\mathbf{A}} =_{\mathscr{F}(X,\mathbb{2})} 1 - \chi_{\mathbf{A}}$.

Proof.
(i) The partial function $\chi_{\mathbf{A} \vee \mathbf{B}}$ is defined as the triplet

$$\chi_{\mathbf{A} \vee \mathbf{B}} := (\mathtt{Dom}(\mathbf{A} \vee \mathbf{B}), i^X_{\mathtt{Dom}(\mathbf{A} \vee \mathbf{B})}, (\chi_{\mathbf{A} \vee \mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A} \vee \mathbf{B})}).$$

If $\chi_{\mathbf{A}} : \mathtt{Dom}(\mathbf{A}) \to \mathbb{2}$ and $\chi_{\mathbf{B}} : \mathtt{Dom}(\mathbf{B}) \to \mathbb{2}$, we have the partial function

$$\chi_{\mathbf{A}} \vee \chi_{\mathbf{B}} := (\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B}), i^X_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}, (\chi_{\mathbf{A}} \vee \chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}),$$

where

$$(\chi_{\mathbf{A}} \vee \chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}(a, b) := \begin{cases} 1 & \text{, if } a \in A^1 \vee b \in B^1 \\ 0 & \text{, if } (a, b) \in A^0 \cap B^0 \end{cases}$$

for $(a, b) \in \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$.
Since $\mathtt{Dom}(\mathbf{A} \vee \mathbf{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$, and if $(f, g) : \mathtt{Dom}(\mathbf{A} \vee \mathbf{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$, the following inner diagrams commute

$$\begin{tikzcd}
\mathtt{Dom}(\mathbf{A} \vee \mathbf{B}) \arrow[rr, bend left, "f"] \arrow[dr, hook, "i^X_{\mathtt{Dom}(\mathbf{A} \vee \mathbf{B})}"'] \arrow[ddr, bend right, "\chi^{\mathbb{2}}_{\mathbf{A} \vee \mathbf{B}}"'] & & \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B}) \arrow[ll, bend left, "g"] \arrow[dl, "i^X_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}"] \arrow[ddl, bend left, "(\chi_{\mathbf{A}} \vee \chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}"] \\
& X & \\
& \mathbb{2} &
\end{tikzcd}$$

Since $(\chi_{\mathbf{A}} \vee \chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})} \circ f = \chi^{\mathbb{2}}_{\mathbf{A} \vee \mathbf{B}}$ and $\chi^{\mathbb{2}}_{\mathbf{A} \vee \mathbf{B}} \circ g = (\chi_{\mathbf{A}} \vee \chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}$ the outer diagrams commute,and therefore the two partial functions $\chi_{\mathbf{A} \vee \mathbf{B}}$ and $\chi_{\mathbf{A}} \vee \chi_{\mathbf{B}}$ are equal in $\mathscr{F}(X, \mathbb{2})$.
(ii) By the definition of multiplication of partial maps $\chi_{\mathbf{A}} : \mathtt{Dom}(\mathbf{A}) \to \mathbb{2}$ and $\chi_{\mathbf{B}} : \mathtt{Dom}(\mathbf{B}) \to \mathbb{2}$, we have the partial function

$$\chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}} := \big(\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B}), i^X_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}, (\chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}\big),$$
$$(\chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})}(u, v) := \chi_{\mathbf{A}}(u) \cdot \chi_{\mathbf{B}}(v)$$

for every $(u, v) \in \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$. The partial function $\chi_{\mathbf{A}\wedge\mathbf{B}}$ is the triplet

$$\chi_{\mathbf{A}\wedge\mathbf{B}} := \big(\mathtt{Dom}(\mathbf{A} \wedge \mathbf{B}), i^X_{\mathtt{Dom}(\mathbf{A}\wedge\mathbf{B})}, (\chi_{\mathbf{A}\wedge\mathbf{B}})^{2}_{\mathtt{Dom}(\mathbf{A}\wedge\mathbf{B})}\big).$$

Since $\mathtt{Dom}(\mathbf{A} \wedge \mathbf{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$ and if $(f, g) : \mathtt{Dom}(\mathbf{A} \wedge \mathbf{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$, then the following inner diagrams commute

$$\begin{tikzcd}
\mathtt{Dom}(\mathbf{A} \wedge \mathbf{B}) \arrow[rr, bend left, "f"] \arrow[dr, hook, bend right, "i^X_{\mathtt{Dom}(\mathbf{A}\wedge\mathbf{B})}"'] \arrow[ddr, bend right=60, "(\chi_{\mathbf{A}\wedge\mathbf{B}})^{2}_{\mathtt{Dom}(\mathbf{A}\wedge\mathbf{B})}"'] & & \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B}) \arrow[ll, bend left, "g"] \arrow[dl, hook', bend left, "i^X_{\mathtt{Dom}(\mathbf{A})\cap\mathtt{Dom}(\mathbf{B})}"] \arrow[ddl, bend left=60, "(\chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}})^{2}_{\mathtt{Dom}(\mathbf{A})\cap\mathtt{Dom}(\mathbf{B})}"] \\
& X & \\
& 2 &
\end{tikzcd}$$

Let $(a_1, b_1) \in A^1 \cap B^1$, then

$$(\chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}})(a_1, b_1) := \chi_{\mathbf{A}}(a_1) \cdot \chi_{\mathbf{B}}(b_1) = 1 \cdot 1 = 1 = \chi_{\mathbf{A}\wedge\mathbf{B}}(a_1, b_1).$$

Working similar for the other cases, we have commutativity of the above outer diagrams, hence $\chi_{\mathbf{A}\wedge\mathbf{B}} =_{\mathscr{F}(X,2)} \chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}}$.
(iii) The multiplication of the partial functions $\chi_{\mathbf{A}} : \mathtt{Dom}(\mathbf{A}) \to 2$ and $1 - \chi_{\mathbf{B}} : \mathtt{Dom}(\mathbf{B}) \to 2$ is the partial function

$$\chi_{\mathbf{A}}(1 - \chi_{\mathbf{B}}) := \big(\mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B}), i^X_{\mathtt{Dom}(\mathbf{A})\cap\mathtt{Dom}(\mathbf{B})}, \big(\chi_{\mathbf{A}}(1 - \chi_{\mathbf{B}})\big)^{2}_{\mathtt{Dom}(\mathbf{A})\cap\mathtt{Dom}(\mathbf{B})}\big),$$
$$\chi_{\mathbf{A}}(1 - \chi_{\mathbf{B}})(a, b) := \chi_{\mathbf{A}}(a) \cdot \big(1 - \chi_{\mathbf{B}}(b)\big)$$

for every $(a, b) \in \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$. The partial function $\chi_{\mathbf{A}\ominus\mathbf{B}}$ is defined as

$$\chi_{\mathbf{A}\ominus\mathbf{B}} := \big(\mathtt{Dom}(\mathbf{A} \ominus \mathbf{B}), i^X_{\mathtt{Dom}(\mathbf{A}\ominus\mathbf{B})}, (\chi_{\mathbf{A}\ominus\mathbf{B}})^{2}_{\mathtt{Dom}(\mathbf{A}\ominus\mathbf{B})}\big).$$

Since $\mathtt{Dom}(\mathbf{A} \ominus \mathbf{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$, and if $(f, g) : \mathtt{Dom}(\mathbf{A} \ominus \mathbf{B}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B})$, the following inner diagrams commute

$$\begin{tikzcd}
\mathtt{Dom}(\mathbf{A} \ominus \mathbf{B}) \arrow[rr, bend left, "f"] \arrow[dr, hook, bend right, "i^X_{\mathtt{Dom}(\mathbf{A}\ominus\mathbf{B})}"'] \arrow[ddr, bend right=60, "(\chi_{\mathbf{A}\ominus\mathbf{B}})^{2}_{\mathtt{Dom}(\mathbf{A}\ominus\mathbf{B})}"'] & & \mathtt{Dom}(\mathbf{A}) \cap \mathtt{Dom}(\mathbf{B}) \arrow[ll, bend left, "g"] \arrow[dl, hook', bend left, "i^X_{\mathtt{Dom}(\mathbf{A})\cap\mathtt{Dom}(\mathbf{B})}"] \arrow[ddl, bend left=60, "\big(\chi_{\mathbf{A}}(1-\chi_{\mathbf{B}})\big)^{2}_{\mathtt{Dom}(\mathbf{A})\cap\mathtt{Dom}(\mathbf{B})}"] \\
& X & \\
& 2 &
\end{tikzcd}$$

Let $a \in A^1, b \in B^1$. By the definition of multiplication on 2, we have

$$\big(\chi_{\mathbf{A}}(1 - \chi_{\mathbf{B}})\big)\big(f(a, b)\big) = \big(\chi_{\mathbf{A}}(1 - \chi_{\mathbf{B}})\big)(a, b) = \chi_{\mathbf{A}}(a)\big(1 - \chi_{\mathbf{B}}(b)\big) = 1 \cdot (1 - 1) = 0 = \chi_{\mathbf{A}\ominus\mathbf{B}}(a, b).$$

Working similarly for the other cases, we see that the above outer diagrams commute and therefore $\chi_{\mathbf{A}\ominus\mathbf{B}} =_{\mathscr{F}(X,\mathbb{2})} \chi_{\mathbf{A}}(1-\chi_{\mathbf{B}})$.
(iv) $\chi_{\mathbf{A}}(x)\cdot\chi_{\mathbf{B}}(y)$ is the partial function

$$\chi_{\mathbf{A}}(x)\cdot\chi_{\mathbf{B}}(y) := \big(\mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B}), i^{X\times X}_{\mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B})}, \big(\chi_{\mathbf{A}}(x)\cdot\chi_{\mathbf{B}}(y)\big)^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B})}\big)$$

for every $(x,y)\in\mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B})$. The partial function $\chi_{\mathbf{A}\otimes\mathbf{B}}$ is the triplet

$$\chi_{\mathbf{A}\otimes\mathbf{B}} := \big(\mathtt{Dom}(\mathbf{A}\otimes\mathbf{B}), i^{X}_{\mathtt{Dom}(\mathbf{A}\otimes\mathbf{B})}, (\chi_{\mathbf{A}\otimes\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}\otimes\mathbf{B})}\big).$$

Since $\mathtt{Dom}(\mathbf{A}\otimes\mathbf{B}) =_{\mathcal{P}(X\times X)} \mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B})$ and if $(f,g) : \mathtt{Dom}(\mathbf{A}\otimes\mathbf{B}) =_{\mathcal{P}(X\times X)} \mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B})$, the following inner diagrams commute

$$\begin{tikzcd}
\mathtt{Dom}(\mathbf{A}\otimes\mathbf{B}) \arrow[rr, bend left, "f"] \arrow[dr, hook, "i^{X}_{\mathtt{Dom}(\mathbf{A}\otimes\mathbf{B})}"'] \arrow[ddr, bend right, "(\chi_{\mathbf{A}\otimes\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A}\otimes\mathbf{B})}"'] & & \mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B}) \arrow[ll, bend left, "g"] \arrow[dl, hook', "i^{X}_{\mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B})}"] \arrow[ddl, bend left, "(\chi_{\mathbf{A}}\cdot\chi_{\mathbf{B}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A})\times\mathtt{Dom}(\mathbf{B})}"] \\
& X & \\
& \mathbb{2} &
\end{tikzcd}$$

Let $(x,y)\in A^1\times B^1$, then $\chi_{\mathbf{A}\otimes\mathbf{B}}(x,y) := 1 = 1\cdot 1 =: \chi_{\mathbf{A}}(x)\cdot\chi_{\mathbf{B}}(y)$. In the other cases it holds that $\chi_{\mathbf{A}\otimes\mathbf{B}}(x,y) := 0 =: \chi_{\mathbf{A}}(x)\cdot\chi_{\mathbf{B}}(y)$. Therefore the above outer diagrams commute as well and we have $\chi_{\mathbf{A}\otimes\mathbf{B}}(x,y) =_{\mathscr{F}(X\times X,\mathbb{2})} \chi_{\mathbf{A}}(x)\cdot\chi_{\mathbf{B}}(y)$.
(v) The partial functions $\chi_{-\mathbf{A}}$ and $1-\chi_{\mathbf{A}}$ are defined as follows

$$\begin{aligned}
\chi_{-\mathbf{A}} &:= \big(\mathtt{Dom}(-\mathbf{A}), i^{X}_{\mathtt{Dom}(-\mathbf{A})}, (\chi_{-\mathbf{A}})^{\mathbb{2}}_{\mathtt{Dom}(-\mathbf{A})}\big),\\
1-\chi_{\mathbf{A}} &:= \big(\mathtt{Dom}(\mathbf{A}), i^{X}_{\mathtt{Dom}(\mathbf{A})}, (1-\chi_{\mathbf{A}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A})}\big).
\end{aligned}$$

Since $\mathtt{Dom}(-\mathbf{A}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A})$ and if $(f,g) : \mathtt{Dom}(-\mathbf{A}) =_{\mathcal{P}(X)} \mathtt{Dom}(\mathbf{A})$, the following inner diagram commutes

$$\begin{tikzcd}
\mathtt{Dom}(-\mathbf{A}) \arrow[rr, bend left, "f"] \arrow[dr, hook, "i^{X}_{\mathtt{Dom}(-\mathbf{A})}"'] \arrow[ddr, bend right, "(\chi_{-\mathbf{A}})^{\mathbb{2}}_{\mathtt{Dom}(-\mathbf{A})}"'] & & \mathtt{Dom}(\mathbf{A}) \arrow[ll, bend left, "g"] \arrow[dl, hook', "i^{X}_{\mathtt{Dom}(\mathbf{A})}"] \arrow[ddl, bend left, "(1-\chi_{\mathbf{A}})^{\mathbb{2}}_{\mathtt{Dom}(\mathbf{A})}"] \\
& X & \\
& \mathbb{2} &
\end{tikzcd}$$

If $a\in A^1$, then $1-\chi_{\mathbf{A}}(a) := 1-1 = 0 = \chi_{-\mathbf{A}}(a)$ and if $a\in A^0$, $1-\chi_{\mathbf{A}}(a) := 1-0 = 1 = \chi_{-\mathbf{A}}(a)$. Therefore the outer diagrams commute and it holds that $\chi_{-\mathbf{A}} =_{\mathscr{F}(X,\mathbb{2})} 1-\chi_{\mathbf{A}}$. $\square$

2.5 Complemented subsets and $\mathbb{2}$-valued partial functions

We denote by $\mathscr{F}^{se}(X, \mathbb{2})$ the class of all strongly extensional partial functions from X to $\mathbb{2}$. The next proposition shows that there are class functions between the two classes $\mathcal{P}^{\mathbb{II}}(X)$ and $\mathscr{F}^{se}(X, \mathbb{2})$, and that they are in fact inverse to each other. We follow [7], where the proof of the following proposition originally lies. Here we give a full proof and show that the two class functions are inverse to each other.

Proposition 2.5.1. If $(X, =_X, \neq_X)$ is a set with an inequality, let the proper class-assignement routines be defined as follows

$$\chi^X : \mathcal{P}^{\mathbb{II}}(X) \rightsquigarrow \mathscr{F}^{se}(X, \mathbb{2}), \quad \mathbf{A} \mapsto \chi^X(\mathbf{A}) =: \chi_{\mathbf{A}},$$
$$\chi_{\mathbf{A}} := \big(A^1 \cup A^0, i^X_{A^1 \cup A^0}, \chi^{\mathbb{2}}_{A^1 \cup A^0}\big),$$

$$\delta^X : \mathscr{F}^{se}(X, \mathbb{2}) \rightsquigarrow \mathcal{P}^{\mathbb{II}}(X), \quad f_A := (A, i^X_A, f^{\mathbb{2}}_A) \mapsto \delta^X(f_A),$$
$$\delta^X(f_A) := \Big(\delta^1_0(f^{\mathbb{2}}_A), (i^X_A)_{|\delta^1_0(f^{\mathbb{2}}_A)}, \delta^0_0(f^{\mathbb{2}}_A), (i^X_A)_{|\delta^0_0(f^{\mathbb{2}}_A)}\Big),$$

where

$$\delta^1_0(f^{\mathbb{2}}_A) := \{a \in A \mid f^{\mathbb{2}}_A =_{\mathbb{2}} 1\} =: [f^{\mathbb{2}}_A =_{\mathbb{2}} 1],$$
$$\delta^0_0(f^{\mathbb{2}}_A) := \{a \in A \mid f^{\mathbb{2}}_A =_{\mathbb{2}} 0\} =: [f^{\mathbb{2}}_A =_{\mathbb{2}} 0],$$

for every $\mathbf{A} := (A^1, i^X_{A^1}, A^0, i^X_{A^0}) \in \mathcal{P}^{\mathbb{II}}(X)$ and every $f^{\mathbb{2}}_A := (A, i^X_A, f^{\mathbb{2}}_A) \in \mathscr{F}^{se}(X, \mathbb{2})$. Then
(i) χ^X is a well-defined, proper class-function.
(ii) δ^X is a well-defined, proper class-function.
(iii) χ^X and δ^X are inverse to each other.

Proof.
(i) Let $\mathbf{A}, \mathbf{B} \in \mathcal{P}^{\mathbb{II}}(X)$, $\mathbf{A} = (A^1, i^X_{A^1}, A^0, i^X_{A^0})$ and $\mathbf{B} = (B^1, i^X_{B^1}, B^0, i^X_{B^0})$. To show that χ^X is well-defined, we only need to show that $\chi_{\mathbf{A}}$ is strongly extensional, i.e. $\chi^{\mathbb{2}}_{A^1 \cup A^0}$ is strongly extensional. Let $z, w \in A^1 \cup A^0$, such that

$$\chi^{\mathbb{2}}_{A^1 \cup A^0}(z) \neq_{\mathbb{2}} \chi^{\mathbb{2}}_{A^1 \cup A^0}(w).$$

Now suppose that $\chi^{\mathbb{2}}_{A^1 \cup A^0}(z) := 1$ and $\chi^{\mathbb{2}}_{A^1 \cup A^0}(w) := 0$, i.e. $z \in A^1$ and $w \in A^0$. By the definition of a complemented subset we get

$$i^X_{A^1}(z) \neq_X i^X_{A^0}(w) \Leftrightarrow: z \neq_{A^1 \cup A^0} w.$$

If $\chi^{\mathbb{2}}_{A^1 \cup A^0}(z) := 0$ and $\chi^{\mathbb{2}}_{A^1 \cup A^0}(w) := 1$, i.e. $z \in A^0$ and $w \in A^1$, we get

$$i^X_{A^0}(z) \neq_X i^X_{A^1}(w) \Leftrightarrow: z \neq_{A^1 \cup A^0} w.$$

Therefore χ^X is strongly extensional. We still need to show that

$$\mathbf{A} =_{\mathcal{P}^{\mathbb{II}}(X)} \mathbf{B} \Rightarrow \chi_{\mathbf{A}} =_{\mathscr{F}^{se}(X, \mathbb{2})} \chi_{\mathbf{B}}$$

in order for χ^X to be a proper class-function.

Let $(e_1, j_1) : A^1 =_{\mathcal{P}^{\rrbracket\llbracket}(X)} B^1$ and $(e_0, j_0) : A^0 =_{\mathcal{P}^{\rrbracket\llbracket}(X)} B^0$, then the following diagrams commute

$$\begin{tikzcd}
A^1 \arrow[rr, bend left, "e_1"] \arrow[dr, hook, "i^X_{A^1}"'] & & B^1 \arrow[ll, bend left, "j_1"] \arrow[dl, hook', "i^X_{B^1}"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^0 \arrow[rr, bend left, "e_0"] \arrow[dr, hook, "i^X_{A^0}"'] & & B^0 \arrow[ll, bend left, "j_0"] \arrow[dl, hook', "i^X_{B^0}"] \\
& X &
\end{tikzcd}$$

We define the functions $e : A^1 \cup A^0 \to B^1 \cup B^0$ and $j : B^1 \cup B^0 \to A^1 \cup A^0$ respectively by

$$e(z) := \begin{cases} e_1(z) & , z \in A^1 \\ e_0(z) & , z \in A^0 \end{cases} \quad , \quad j(w) := \begin{cases} j_1(w) & , w \in B^1 \\ j_0(w) & , w \in B^0. \end{cases}$$

Let $z \in A^1 \cup A^0$. If $z \in A^1$, then $i^X_{B^1 \cup B^0}\big(e(z)\big) := i^X_{B^1}\big(e_1(z)\big) = i^X_{A^1}(z) := i^X_{A^1 \cup A^0}(z)$ and if $z \in A^0$, we have $i^X_{B^1 \cup B^0}\big(e(z)\big) := i^X_{B^0}\big(e_0(z)\big) = i^X_{A^0}(z) := i^X_{A^1 \cup A^0}(z)$. Working similarly for $w \in B^1 \cup B^0$, the following diagram commutes

$$\begin{tikzcd}
A^1 \cup A^0 \arrow[rr, bend left, "e"] \arrow[dr, hook, bend right, "i^X_{A^1 \cup A^0}"'] \arrow[ddr, bend right=60, "\chi^{\mathbb{2}}_{A^1 \cup A^0}"'] & & B^1 \cup B^0 \arrow[ll, bend left, "j"] \arrow[dl, hook', bend left, "i^X_{B^1 \cup B^0}"] \arrow[ddl, bend left=60, "\chi^{\mathbb{2}}_{B^1 \cup B^0}"] \\
& X & \\
& \mathbb{2} &
\end{tikzcd}$$

and therefore $(e, j) : \chi_{\mathbf{A}} =_{\mathscr{F}^{se}(X,\mathbb{2})} \chi_{\mathbf{B}}$, meaning that χ^X is a proper class-function.

(ii) Let $f_A \in \mathscr{F}^{se}(X, \mathbb{2})$, $f_A := (A, i^X_A, f^{\mathbb{2}}_A)$. We first show that $\delta^X(f_A) \in \mathscr{P}^{\rrbracket\llbracket}(X)$.

Let $a \in \delta^1_0(f^{\mathbb{2}}_A)$ and $b \in \delta^0_0(f^{\mathbb{2}}_A)$. As

$$f^{\mathbb{2}}_A(a) =_{\mathbb{2}} 1 \neq_{\mathbb{2}} 0 =_{\mathbb{2}} f^{\mathbb{2}}_A(b),$$

by the strong extensionality of $f^{\mathbb{2}}_A$ and according to the definition of the canonical inequality of the subset (A, i^X_A), we get

$$a \neq_A b :\Leftrightarrow i^X_A(a) \neq_X i^X_A(b).$$

Now we want to show that

$$f_A =_{\mathscr{F}^{se}(X,\mathbb{2})} f_B \Rightarrow \delta^X(f_A) =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \delta^X(f_B).$$

If $(A, i_A^X, f_A^{\mathbb{2}}) =_{\mathscr{F}^{se}(X,\mathbb{2})} (B, i_B^X, f_B^{\mathbb{2}})$, then the following diagram commutes

$$\begin{tikzcd}
A \arrow[rr, bend left, "e_{AB}", "\#_1"'] \arrow[dr, bend right, "i_A^X"', "\#_3"'] \arrow[ddr, bend right=60, "f_A^{\mathbb{2}}"'] & & B \arrow[ll, bend left, "e_{BA}"', "\#_2"] \arrow[dl, bend left, "i_B^X", "\#_4"] \arrow[ddl, bend left=60, "f_B^{\mathbb{2}}"] \\
& X & \\
& \mathbb{2} &
\end{tikzcd}$$

The outer commutativities ($\#_1$) and ($\#_2$) imply that the functions

$$(e_{AB})_{|\delta_0^1(f_A^{\mathbb{2}})} : \delta_0^1(f_A^{\mathbb{2}}) \to \delta_0^1(f_B^{\mathbb{2}}) \text{ and } (e_{AB})_{|\delta_0^0(f_A^{\mathbb{2}})} : \delta_0^0(f_A^{\mathbb{2}}) \to \delta_0^0(f_B^{\mathbb{2}}),$$
$$(e_{BA})_{|\delta_0^1(f_B^{\mathbb{2}})} : \delta_0^1(f_B^{\mathbb{2}}) \to \delta_0^1(f_A^{\mathbb{2}}) \text{ and } (e_{BA})_{|\delta_0^0(f_B^{\mathbb{2}})} : \delta_0^0(f_B^{\mathbb{2}}) \to \delta_0^0(f_A^{\mathbb{2}})$$

are well-defined. The commutativities ($\#_3$) and ($\#_4$) of the above inner diagrams (A, B, X) imply the commutativity of the following diagrams

$$\begin{tikzcd}
\delta_0^1(f_A^{\mathbb{2}}) \arrow[rr, bend left, hook, "(e_{AB})_{|\delta_0^1(f_A^{\mathbb{2}})}"] \arrow[dr, hook, "(i_A^X)_{|\delta_0^1(f_A^{\mathbb{2}})}"'] & & \delta_0^1(f_B^{\mathbb{2}}) \arrow[ll, bend left, hook, "(e_{BA})_{|\delta_0^1(f_B^{\mathbb{2}})}"] \arrow[dl, hook, "(i_B^X)_{|\delta_0^1(f_B^{\mathbb{2}})}"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\delta_0^0(f_A^{\mathbb{2}}) \arrow[rr, bend left, hook, "(e_{AB})_{|\delta_0^0(f_A^{\mathbb{2}})}"] \arrow[dr, hook, "(i_A^X)_{|\delta_0^0(f_A^{\mathbb{2}})}"'] & & \delta_0^0(f_B^{\mathbb{2}}) \arrow[ll, bend left, hook, "(e_{BA})_{|\delta_0^0(f_B^{\mathbb{2}})}"] \arrow[dl, hook, "(i_B^X)_{|\delta_0^0(f_B^{\mathbb{2}})}"] \\
& X &
\end{tikzcd}$$

which proves that $\delta^X(f_A) =_{\mathcal{P}^{\mathbb{II}}(X)} \delta^X(f_B)$ and therefore δ^X is a proper class-function.
(iii) Let $\mathbf{A} \in \mathscr{P}^{\mathbb{II}}(X)$, then

$$\begin{aligned}
\delta^X\big(\chi^X(\mathbf{A})\big) &:= \delta^X\big(A^1 \cup A^0, i_{A^1\cup A^0}^X, \chi_{A^1\cup A^0}^{\mathbb{2}}\big) \\
&:= \Big(\delta_0^1(\chi_{A^1\cup A^0}^{\mathbb{2}}), (i_{A^1\cup A^0}^X)_{|\delta_0^1(\chi_{A^1\cup A^0}^{\mathbb{2}})}, \delta_0^0(\chi_{A^1\cup A^0}^{\mathbb{2}}), (i_{A^1\cup A^0}^X)_{|\delta_0^0(\chi_{A^1\cup A^0}^{\mathbb{2}})}\Big) \\
&=_{\mathscr{P}^{\mathbb{II}}(X)} (A^1, i_{A^1}^X, A^0, i_{A^0}^X) \\
&:= \mathbf{A}
\end{aligned}$$

Let $f_A \in \mathcal{F}^{se}(X, \mathbb{2})$, then

$$\begin{aligned}
\chi^X\big(\delta^X(f_A)\big) &:= \chi^X\big(\delta^X(A, i_A^X, f_A^{\mathbb{2}})\big) \\
&:= \chi^X\Big(\delta_0^1(f_A^{\mathbb{2}}), (i_A^X)_{|\delta_0^1(f_A^{\mathbb{2}})}, \delta_0^0(f_A^{\mathbb{2}}), (i_A^X)_{|\delta_0^0(f_A^{\mathbb{2}})}\Big) \\
&:= \chi^X\big([f_A^{\mathbb{2}} =_{\mathbb{2}} 1], i_{[f_A^{\mathbb{2}}=_{\mathbb{2}}1]}^X, [f_A^{\mathbb{2}} =_{\mathbb{2}} 0], i_{[f_A^{\mathbb{2}}=_{\mathbb{2}}0]}^X)\big) \\
&:= \big([f_A^{\mathbb{2}} =_{\mathbb{2}} 1] \cup [f_A^{\mathbb{2}} =_{\mathbb{2}} 0], i_{[f_A^{\mathbb{2}}=_{\mathbb{2}}1]\cup[f_A^{\mathbb{2}}=_{\mathbb{2}}0]}^X, \chi_{[f_A^{\mathbb{2}}=_{\mathbb{2}}1]\cup[f_A^{\mathbb{2}}=_{\mathbb{2}}0]}^{\mathbb{2}}\big) \\
&=_{\mathcal{F}(X,\mathbb{2})} (A, i_A^X, f_A^{\mathbb{2}}) =: f_A
\end{aligned}$$

Hence δ^X and χ^X are inverse to each other. $\square$

Remark 2.5.2. We notice that the use of strong extensionality is crucial to the proof that δ^X is well-defined. The statement would not hold for $\mathscr{F}(X, \mathbb{2})$.

3 Categorical aspects

After having discussed Bishop set theory, we will in this chapter talk about the categorical aspects of Bishop set theory. Following [6], we consider the category of complemented subsets and Chu-representation of different categories.

3.1 Basic definitions

We will first introduce some basic definitions of category theory following [1].

Definition 3.1.1. A *category* consists of *objects* $A, B, C, ...$ and *arrows* $f, g, h, ...$, for $f : A \to B$ we call A the *domain* of f and B the *codomain* of f. For given arrows $f : A \to B$ and $g : B \to C$ there is a given arrow $g \circ f : A \to C$ called the *composite* of f and g. For each object A there is a given *identity arrow* $1_A : A \to A$.
The following is required to hold:
(i) Associativity:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for all $f : A \to B$, $g : B \to C$, $h : C \to D$.
(ii) Unit:

$$f \circ 1_A = f = 1_B \circ f$$

for all $f : A \to B$.
If $\mathcal{C}$ is a category, we denote by C_0 the objects and by C_1 the arrows of $\mathcal{C}$.

Definition 3.1.2. A *functor*

$$F : \mathcal{C} \to \mathcal{D}$$

between two categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objets to objects and arrows to arrows, such that
(i) $F(f : A \to B) = F(f) : F(A) \to F(B)$,
(ii) $F(1_A) = 1_{F(A)}$,
(iii) $F(g \circ f) = F(g) \circ F(f)$.
A functor $F : \mathcal{C} \to \mathcal{C}$ is called an *endofunctor*.

Definition 3.1.3. Let the category $\mathcal{C}$ have binary product $\times$. An *exponential* of objects B and C consists of an object C^B and an arrow $\epsilon : C^B \times B \to C$ such that, for any object A and arrow $f : A \times B \to C$ there is a unique arrow $\tilde{f} : A \to C^B$ such that

$$\epsilon \circ (\tilde{f} \times 1_B) = f,$$

meaning that the following diagram commutes

$$\begin{tikzcd}
C^B \times B \arrow[r, "\epsilon"] & C \\
A \times B \arrow[u, "\tilde{f}\times 1_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Definition 3.1.4. A category is called *cartesian closed*, if it has finite products and exponentials.

Definition 3.1.5. In a category $\mathcal{C}$, an arrow $f : A \to B$ is called
(i) a *monomorphism*, if given any $g, h : C \to A$, $fg = fh$ implies $g = h$.
(ii) an *epimorphism*, if given any $i, j : B \to D$, $if = jf$ implies $i = j$.

Definition 3.1.6. A category is called *thin*, if for any objects A, B and morphisms f, g from A to B

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \Rightarrow f = g.$$

Definition 3.1.7. The *pullback* of $f : A \to C$ and $g : B \to C$ consists of an object P with morphisms $p : P \to A$ and $q : P \to B$

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

such that $f \circ p = g \circ q$ and universal with this property. That means for any given $s_1 : S \to A$ and $s_2 : S \to B$ with $f \circ s_1 = g \circ s_2$, there is a unique $u : S \to P$ with $s_1 = p \circ u$ and $s_2 = q \circ u$.

$$\begin{tikzcd}
S \arrow[dr, dashed, "u"] \arrow[drr, "s_1"] \arrow[ddr, "s_2"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

The *pushout* of $f : C \to A$ and $g : C \to B$ consists of an object P with morphisms $p : A \to P$ and $q : B \to P$

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "p"] \\
B \arrow[r, "q"'] & P
\end{tikzcd}$$

such that $p \circ f = q \circ g$ and universal with this property.

3.2 The category of complemented subsets

After having discussed complemented subsets in the last chapter, we will now present the category of complemented subsets.

Definition 3.2.1. If X is a set , the poset category $\boldsymbol{\mathcal{P}}^{\mathbb{II}}(X)$ has as objects the complemented subsets of X and a morphism $f : \mathbf{A} \to \mathbf{B}$ is a pair $f = (f_1, f_0) : \mathbf{A} \subseteq \mathbf{B}$, i.e. $f_1 : A^1 \subseteq B^1$ and $f_0 : B^0 \subseteq A^0$. The unit morphism $1_{\mathbf{A}}$ of $\mathbf{A}$ is the pair $(\mathrm{id}_{A^1}, \mathrm{id}_{A^0})$ and if $g = (g_1, g_0) : \mathbf{B} \subseteq \mathbf{C}$, then $g \circ f := (g_1 \circ f_1, f_0 \circ g_0)$.

Proposition 3.2.2. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \boldsymbol{\mathcal{P}}^{\mathbb{II}}(X)$.
(i) If $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{C} \to \mathbf{B}$, then $\mathbf{A} \times_{\mathbf{B}} \mathbf{C} := (A^1 \times_{B^1} C^1, A^0 +_{B^0} C^0)$ is a pullback, where

$$A^1 \times_{B^1} C^1 := \big\{(a_1, c_1) \in A^1 \times C^1 \mid f_1(a_1) =_{B^1} g_1(c_1)\big\},$$
$$i_{A^1 \times_{B^1} C^1} : A^1 \times_{B^1} C^1 \to X, \quad i_{A^1 \times_{B^1} C^1}(a_1, c_1) := i^X_{A^1}(a_1); \quad (a_1, c_1) \in A^1 \times_{B^1} C^1,$$
$$A^0 +_{B^0} C^0 := A^0 \cup C^0.$$

(ii) If $f : \mathbf{B} \to \mathbf{A}$ and $g : \mathbf{B} \to \mathbf{C}$, then $\mathbf{A} +_{\mathbf{B}} \mathbf{C} := (A^1 +_{B^1} C^1, A^0 \times_{B^0} C^0)$ is a pushout, where

$$A^1 +_{B^1} C^1 := A^1 \cup C^1,$$
$$A^0 \times_{B^0} C^0 := \big\{(a_0, c_0) \in A^0 \times C^0 \mid f_0(a_0) =_{B^0} g_0(c_0)\big\},$$
$$i_{A^0 \times_{B^0} C^0} : A^0 \times_{B^0} C^0 \to X, \quad i_{A^0 \times_{B^0} C^0}(a_0, c_0) := i^X_{A^0}(a_0); \quad (a_0, c_0) \in A^0 \times_{B^0} C^0.$$

Proof. (i) Let $f = (f_1, f_0) : \mathbf{A} \subseteq \mathbf{B}$ and $g = (g_1, g_0) : \mathbf{C} \subseteq \mathbf{B}$, i.e. $f_1 : A^1 \subseteq B^1, f_0 : B^0 \subseteq A^0$ and $g_1 : C^1 \subseteq B^1, g_0 : B^0 \subseteq C^0$. We define the morphism $p = (p_1, p_0)$, where

$$p_1 : A^1 \times_{B^1} C^1 \to A^1, \quad p_1(a_1, c_1) := a_1,$$
$$p_0 : A^0 \to A^0 +_{B^0} C^0, \quad p_0(a_0) := (a_0, c_0) \in A^0 \cup C^0$$

and the morphism $q = (q_1, q_0)$, where

$$q_1 : A^1 \times_{B^1} C^1 \to C^1, \quad q_1(a_1, c_1) := c_1,$$
$$q_0 : C^0 \to A^0 +_{B^0} C^0, \quad q_0(c_0) := (a_0, c_0) \in A^0 \cup C^0.$$

Let $(a_1, c_1) \in A^1 \times_{B^1} C^1$, then

$$f_1\big(p_1(a_1, c_1)\big) = f_1(a_1) =_{B^1} g_1(c_1) = g_1\big(q_1(a_1, c_1)\big),$$

so $f_1 p_1 = g_1 q_1$.
Now let $b_0 \in B^0$, then there is a $a_0 \in A^0$ and a $c_0 \in C^0$, such that $f_0(b_0) = a_0$ and $c_0 = g_0(b_0)$. We then get

$$p_0\big(f_0(b_0)\big) = p_0(a_0) = (a_0, c_0) = q_0(c_0) = q_0\big(g_0(b_0)\big),$$

so $p_0 f_0 = q_0 g_0$. Therefore $fp = gq$, which proves that $\mathbf{A} \times_{\mathbf{B}} \mathbf{C}$ is a pullback.
(ii) We precede similarly as in (i):
Let $f = (f_1, f_0) : \mathbf{B} \subseteq \mathbf{A}$ and $g = (g_1, g_0) : \mathbf{B} \subseteq \mathbf{C}$, i.e. $f_1 : B^1 \subseteq A^1, f_0 : A^0 \subseteq B^0$ and $g_1 : B^1 \subseteq C^1, g_0 : C^0 \subseteq B^0$. We define the morphism $p = (p_1, p_0)$, where

$$p_1 : A^1 \to A^1 +_{B^1} C^1, \quad p_1(a_1) := (a_1, c_1) \in A^1 \cup C^1,$$
$$p_0 : A^0 \times_{B^0} C^0 \to A^0, \quad p_0(a_0, c_0) := a_0$$

and the morphism $q = (q_1, q_0)$, where

$$q_1 : C^1 \to A^1 +_{B^1} C^1, \quad q_1(c_1) := (a_1, c_1) \in A^1 \cup C^1,$$
$$q_0 : A^0 \times_{B^0} C^0 \to C^0, \quad q_0(a_0, c_0) := c_0$$

Let $b_1 \in B^1$, then there is a $a_1 \in A^1$ and a $c_1 \in C^1$, such that $f_1(b_1) = a_1$ and $g_1(b_1) = c_1$. We then get

$$p_1\big(f_1(b_1)\big) = p_1(a_1) = (a_1, c_1) = q_1(c_1) = q_1\big(g_1(b_1)\big),$$

so $p_1 f_1 = q_1 g_1$. Now let $b_1 \in B^1$, then $f(b) \in A^1$ and $g(b) \in C^1$, since $f : B^1 \subseteq A^1$ and $f : B^1 \subseteq C^1$

$$f_0\big(p_0(a_0, c_0)\big) = f_0(a_0) =_{B^0} g_0(c_0) = g_0\big(q_0(a_0, c_0)\big),$$

so $f_0 p_0 = g_0 q_0$.

Hence $pf = pg$, which proves that $\mathbf{A} +_{\mathbf{B}} \mathbf{C}$ is a pushout. □

Proposition 3.2.3. Let the sets $(X, =_X, \neq_X^f)$ and $(Y, =_Y, \neq_Y)$, where $f : X \to Y$ and the inequality $\neq_X^f$ on X induced by f is defined by

$$x \neq_X^f x' :\Leftrightarrow f(x) \neq_Y f(x').$$

Let $\mathbf{K} := (K^1, K^0)$, $\mathbf{L} := (L^1, L^0)$ in $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$ and $\mathbf{A} := (A^1, A^0)$, $\mathbf{B} := (B^1, B^0)$ in $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(Y)$.
(i) $f^{-1}(\mathbf{A}) := \big(f^{-1}(A^1), f^{-1}(A^0)\big) \in \boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$
(ii) $f^{-1}(\mathbf{A} \cup \mathbf{B}) =_{\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)} f^{-1}(\mathbf{A}) \cup f^{-1}(\mathbf{B})$
(iii) $f^{-1}(\mathbf{A} \cap \mathbf{B}) =_{\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)} f^{-1}(\mathbf{A}) \cap f^{-1}(\mathbf{B})$
(iv) $f^{-1}(-\mathbf{A}) =_{\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)} -f^{-1}(\mathbf{A})$
(v) $f^{-1}(\mathbf{A} - \mathbf{B}) =_{\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)} f^{-1}(\mathbf{A}) - f^{-1}(\mathbf{B})$
(vi) $\mathbf{A} \subseteq \mathbf{B} \Rightarrow f^{-1}(\mathbf{A}) \subseteq f^{-1}(\mathbf{B})$
(vii) $f(\mathbf{K}) := \big(f(K^1), f(K^0)\big) \in \boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(Y)$
(viii) $\mathbf{K} \subseteq \mathbf{L} \Rightarrow f(\mathbf{K}) \subseteq f(\mathbf{L})$
(ix) $f(\mathbf{K}) \cup f(\mathbf{L}) \subseteq_1 f(\mathbf{K} \cup \mathbf{L})$
(x) $f(\mathbf{K} \cap \mathbf{L}) \subseteq_0 f(\mathbf{K}) \cap f(\mathbf{L})$

Proof. We will only show (i) and (vii)-(viii):
(i) By definition we have

$$f^{-1}(A^1) := \{(x, a_1) \in X \times A^1 \mid f(x) =_Y i^X_{A^1}(a_1)\}, \quad i_{f^{-1}(A^1)}(x, a_1) := x,$$
$$f^{-1}(A^0) := \{(x, a_0) \in X \times A^0 \mid f(x) =_Y i^X_{A^0}(a_0)\}, \quad i_{f^{-1}(A^0)}(x, a_0) := x.$$

Let $(x, a_1) \in f^{-1}(A^1)$ and $(z, a_0) \in f^{-1}(A^0)$. Then we have, by extensionality of $\neq_Y$ that

$$i_{f^{-1}(A^1)}(x, a_1) \neq_X^f i_{f^{-1}(A^0)}(z, a_0) :\Leftrightarrow x \neq_X^f z :\Leftrightarrow f(x) \neq_Y f(z) \Leftrightarrow i^X_{A^1}(a_1) \neq_Y i^X_{A^0}(a_0),$$

which holds by the hypothesis $\mathbf{K} \in \boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$.
(vii) By definition we have

$$f(K^1) := \{f(k_1) \mid k_1 \in K^1\}, \quad f_{K^1} = f \circ i^X_{K^1},$$
$$f(K^0) := \{f(k_0) \mid k_0 \in K^0\}, \quad f_{K^0} = f \circ i^X_{K^0}.$$

Let $k_1 \in K^1$ and $k_0 \in K^0$. Then we have that

$$k_1 \neq_{f(\mathbf{K})} k_0 :\Leftrightarrow f(i^X_{K^1}(k_1)) \neq_Y f(i^X_{K^0}(k_0)) :\Leftrightarrow i^X_{K^1}(k_1) \neq_X^f i^X_{K^0}(k_0) :\Leftrightarrow i^X_{K^1}(k_1) \neq_X i^X_{K^0}(k_0),$$

which holds by the hypothesis $\mathbf{A} \in \boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(Y)$.
(viii) Let $\mathbf{K} \subseteq \mathbf{L}$, i.e. $K^1 \subseteq L^1$ and $L^0 \subseteq K^0$.

$$f(K^1) := \{f(k) \mid k \in K^1\}, \quad f(L^0) := \{f(l) \mid l \in L^0\}$$

Let $k \in K^1$ and $l \in L^0$, then $k \in L^1$ and $l \in K^0$. Therefore $f(k) \in f(L^1)$ and $f(l) \in f(K^0)$ for every $k \in K^1$ and $l \in L^0$. So we have $f(K^1) \subseteq f(L^1)$ and $f(L^0) \subseteq f(K^0)$, which proves the statement. □

3.3 The Chu category

The Chu category can be defined on a category with products.

Definition 3.3.1. Let $\mathcal{C}$ be a cartesian closed category. The *Chu category* $\mathbf{Chu}(\mathcal{C}, \gamma)$ over $\mathcal{C}$ and $\gamma \in C_0$ has objects *Chu spaces*, i.e. triplets (a, f, x) with $a, x \in C_0$ and $f : a \times x \to \gamma \in C_1$. A morphism $\phi : (a, f, x) \to (b, g, y)$ in $\mathbf{Chu}(\mathcal{C}, \gamma)$, or a *Chu transform*, is a pair $\phi = (\phi^+, \phi^-)$, where $\phi^+ : a \to b$ and $\phi^- : y \to x$ are in C_1 such that the following diagram commutes

$$\begin{tikzcd}
a \times y \arrow[r, "1_a \times \phi^-"] \arrow[d, "\phi^+ \times 1_y"'] & a \times x \arrow[d, "f"] \\
b \times y \arrow[r, "g"'] & \gamma
\end{tikzcd}$$

If $\theta = (\theta^+, \theta^-) : (b, g, y) \to (c, h, z)$, then $\theta \circ \phi = (\theta^+ \circ \phi^+, \phi^- \circ \theta^-)$.

$$\begin{tikzcd}
a \times y \arrow[r, "1_a \times \phi^-"] \arrow[d, "\phi^+ \times 1_y"'] & a \times x \arrow[d, "f"] & \\
b \times y \arrow[r, "g"'] & \gamma & a \times z \arrow[ul, "1_a \times (\phi^- \circ \theta^-)"'] \arrow[dl, "(\theta^+ \circ \phi^+) \times 1_z"] \\
b \times z \arrow[u, "1_b \times \theta^-"] \arrow[r, "\theta^+ \times 1_z"'] & c \times z \arrow[u, "h"'] &
\end{tikzcd}$$

Moreover, $1_{(a,f,x)} = (1_a, 1_x)$.

$$\begin{tikzcd}
a \times x \arrow[r, "1_a \times 1_x"] \arrow[d, "1_a \times 1_x"'] & a \times x \arrow[d, "f"] \\
a \times x \arrow[r, "f"'] & \gamma
\end{tikzcd}$$

Remark 3.3.2. To show that composition is well-defined in $\mathbf{Chu}(\mathcal{C}, \gamma)$, we show commutativity of the triangle in the above definition:

$$\begin{aligned}
\big[f \circ [1_a \times (\phi^- \circ \theta^-)]\big](a, z) &= \big[f \circ (1_a \times \phi^-) \circ (1_a \times \theta^-)\big](a, z) \\
&= \big[g \circ (\phi^+ \times 1_y) \circ (1_a \times \theta^-)\big](a, z) \\
&= \big[g \circ (\phi^+ \times +_z)\big](a, y) \\
&= g(g, y)
\end{aligned}$$

$$
\begin{aligned}
&= h(c, z) \\
&= \big[h \circ (\theta^+ \times 1_z)\big](b, z) \\
&= \big[h \circ (\theta^+ \times 1_z) \circ (\phi^+ \times 1_z)\big](a, z) \\
&= \big[h \circ [(\theta^+ \circ (\phi^+) \times 1_z)]\big](a, z).
\end{aligned}
$$

Definition 3.3.3. A functor $F : \mathcal{C} \to \mathcal{D}$, with $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$, is called
i) *injective (sujective) on objects*, if F_0 is an injection (surjection).
ii) *injective (sujective) on arrows*, if F_1 is an injection (surjection).
iii) *faithful*, if for every $a, b \in C_0$

$$F_{(a,b)} : C_1(a, b) \to D_1(F_0(a), F_0(b)),$$

where $F_{(a,b)}(f) = F_1(f)$, is an injection.
iv) *full*, if for every $a, b \in C_0$, $F_{(a,b)}$ is a surjection.
v) an *embedding*, if F is injective on objects and faithful.
vi) a *representation*, if F is a full embedding.
vii) a *strict representation*, if F is injective on objects, arrows and is full.
viii) a *Chu representation*, if $\mathcal{D}$ is a Chu category and F is a representation.

3.3.1 A Chu-representation of the category of subsets

Definition 3.3.4. The *category of sets* **Set** has as objects sets $(A, =_A)$. If $(A, =_A), (B, =_B)$ are objects of **Set**, the morphisms are functions $f : A \to B$.

Definition 3.3.5. The *category of subsets* $\mathcal{P}(X)$ has as objects the subsets (A, i_A^X) of the set X. If (A, i_A^X) and (B, i_B^X) are objects of $\mathcal{P}(X)$, a morphism $f : (A, i_A^X) \to (B, i_B^X)$ is a function, such that the following diagram commutes

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, hook, "i_A^X"'] & & B \arrow[dl, hook', "i_B^X"] \\
& X &
\end{tikzcd}
$$

If $g : (B, i_B^X) \to (C, i_C^X)$, we define $g \circ f : (A, i_A^X) \to (C, i_C^X)$. Moreover $1_{(A, i_A^X)} = \mathrm{id}_A$.

Proposition 3.3.6. $\big($Chu-representation of $\mathcal{P}(X)\big)$
If $(X, =_X)$ is a set, then the functor

$$
\begin{gathered}
E^X : \mathcal{P}(X) \to \mathbf{Chu}(\mathbf{Set}, X) \\
E_0^X(A, i_A^X) = (A, I_A^X, \mathbb{1}), \\
I_A^X : A \times \mathbb{1} \to X,\ I_A^X(a, 0) = i_A^X(a) \qquad a \in A, \\
E_1^X(f : (A, i_A^X) \to (B, i_B^X)) = (f, \mathrm{id}_{\mathbb{1}}) : (A, I_A^X, \mathbb{1}) \to (B, I_B^X, \mathbb{1}),
\end{gathered}
$$

is a full embedding of $\mathcal{P}(X)$ into $\mathbf{Chu}(\mathbf{Set}, X)$.

Proof. First we will show that E^X is a well-defined functor. Clearly E^X is a functor. Let $(A, i_A^X) \in \mathcal{P}(X)$, then

$$E_0^X(A, i_A^X) = (A, I_A^X, \mathbb{1}).$$

$A, \mathbb{1}$ are objects in **Set**. Let $(a, 0), (b, 0) \in A \times \mathbb{1}$ with $(a, 0) = (b, 0)$, i.e $a = b$, then

$$I_A^X(a, 0) = i_A^X(a) = i_A^X(b) = I_A^X(b, 0).$$

Hence I_A^X is a function in **Set**. Let $(B, i_B^X) \in \mathcal{P}(X)$ and $f : (A, i_A^X) \to (B, i_B^X)$, meaning that the following triangle commutes

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "i_A^X"'] & & B \arrow[dl, hook', "i_B^X"] \\ & X & \end{tikzcd}$$

This implies the commutativity of the following rectangle

$$\begin{tikzcd} A \times \mathbb{1} \arrow[r, "\mathrm{id}_A \times \mathrm{id}_\mathbb{1}"] \arrow[d, "f \times \mathrm{id}_\mathbb{1}"'] & A \times \mathbb{1} \arrow[d, "I_A^X"] \\ B \times \mathbb{1} \arrow[r, "I_B^X"'] & X \end{tikzcd}$$

since $I_A^X(\mathrm{id}_A \times \mathrm{id}_\mathbb{1}(a, 0)) = I_A^X(a, 0) = i_A^X(a) = i_B^X(f(a)) = I_B^X(f(a), 0) = I_B^X(f \times \mathrm{id}_\mathbb{1}(a, 0))$, for $(a, 0) \in A \times \mathbb{1}$. Therefore $E_1^X(f) : (A, I_A^X, \mathbb{1}) \to (B, I_B^X, \mathbb{1})$ is in $\mathbf{Chu}(\mathbf{Set}, X)$ and E^X is well-defined. Now let $(g_1, g_2) : (A, I_A^X, \mathbb{1}) = (B, I_B^X, \mathbb{1})$, then the following triangle commutes

$$\begin{tikzcd} A \times \mathbb{1} \arrow[rr, bend left, "g_1"] \arrow[dr, hook, "I_A^X"'] & & B \times \mathbb{1} \arrow[ll, bend left, "g_2"] \arrow[dl, hook', "I_B^X"] \\ & X & \end{tikzcd}$$

which implies that $(A, i_A^X) = (B, i_B^X)$, therefore E^X is injective on objects. Now let $(f, \mathrm{id}_\mathbb{1}), (g, \mathrm{id}_\mathbb{1}) : (A, I_A^X, \mathbb{1}) \to (B, I_B^X, \mathbb{1})$, with $(f, \mathrm{id}_\mathbb{1}) = (g, \mathrm{id}_\mathbb{1})$, i.e.

$$(f(x), 0) = (f, \mathrm{id}_\mathbb{1})(x, 0) = (g, \mathrm{id}_\mathbb{1})(x, 0) = (g(x), 0)$$

for all $(x, 0) \in A \times \mathbb{1}$. Therefore $f = g$ and E^X is injective on arrows. Hence E^X is an embedding. We still need to show that E^X is full. By the commutativity of the above rectangle, we get the commutativity of the first triangle. Hence, if $(f, \mathrm{id}_\mathbb{1}) : (A, I_A^X, \mathbb{1}) \to (B, I_B^X, \mathbb{1})$ is in $\mathbf{Chu}(\mathbf{Set}, X)$, then $f : A \to B$ is in $\mathcal{P}(X)$, so E^X is a full embedding. $\square$

Definition 3.3.7. If $\mathcal{C}$ is a category and $\gamma \in C_0$, the *category* $\mathbf{Sub}(\mathcal{C}, \gamma)$ *of subobjects of* γ has as objects the monomorphisms of $\mathcal{C}$ with codomain γ and a morphism $f : i \to j$, where $i : a \hookrightarrow \gamma$ and $j : b \hookrightarrow \gamma$ is a morphism $f : a \to b$ such that the following diagram commutes

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & b \arrow[dl, hook', "j"] \\ & \gamma & \end{tikzcd}$$

Remark 3.3.8. Let $g, h : k \to i$, where $k : c \hookrightarrow \gamma$ and $i : a \hookrightarrow \gamma$. Suppose $fg = fh$, then the following diagram commutes

$$\begin{tikzcd}
c \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[dr, hook, "k"'] & a \arrow[r, "f"] \arrow[d, hook, "i"] & b \arrow[dl, "j"] \\
& \gamma &
\end{tikzcd}$$

and it immediatly follows that $g = h$, i.e. f is a monomorphism. Moreover the diagram shows that $\mathbf{Sub}(\mathcal{C}, \gamma)$ is thin.

Proposition 3.3.9. $\big($Chu-representation of $\mathbf{Sub}(\mathcal{C}, \gamma)\big)$
If $\mathcal{C}$ is cartesian closed, the functor

$$\begin{gathered}
E^{\mathbf{Sub}(\mathcal{C},\gamma)} : \mathbf{Sub}(\mathcal{C}, \gamma) \to \mathbf{Chu}(\mathcal{C}, \gamma), \\
E_0^{\mathbf{Sub}(\mathcal{C},\gamma)}(i : a \hookrightarrow \gamma) = (a, i \circ \mathtt{pr}_a, 1), \\
a \times 1 \xhookrightarrow{\mathtt{pr}_a} a \xhookrightarrow{i} \gamma, \\
E_1^{\mathbf{Sub}(\mathcal{C},\gamma)}(f : i \to j) = (f, 1_1) : (a, i \circ \mathtt{pr}_a, 1) \to (b, j \circ \mathtt{pr}_b, 1)
\end{gathered}$$

is a full embedding of $\mathbf{Sub}(\mathcal{C}, \gamma)$ into $\mathbf{Chu}(\mathcal{C}, \gamma)$.

Proof. First we show that $E^{\mathbf{Sub}(\mathcal{C},\gamma)}$ is a well-defined functor. Clearly it is a functor, so we only show that it is well-defined. Let $i : a \to \gamma$ be an object of $\mathbf{Sub}(\mathcal{C}, \gamma)$. Then

$$E_0^{\mathbf{Sub}(\mathcal{C},\gamma)}(i : a \hookrightarrow \gamma) = (a, i \circ \mathtt{pr}_a, 1).$$

Since the morphism $\mathtt{pr}_a$ is an isomorphism, it is a monomorphism and therefore $i \circ \mathtt{pr}_a$ is a monomorphism. To show that $E_1^{\mathbf{Sub}(\mathcal{C},\gamma)}(f) : (a, i \circ \mathtt{pr}_a, 1) \to (b, j \circ \mathtt{pr}_b, 1)$ is a morphism in $\mathbf{Chu}(\mathcal{C}, \gamma)$, we need to show that the following rectangle commutes

$$\begin{tikzcd}
a \times 1 \arrow[r, "1_a \times 1_1"] \arrow[d, "f \times 1_1"'] & a \times 1 \arrow[d, "i \circ \mathtt{pr}_a"] \\
b \times 1 \arrow[r, "j \circ \mathtt{pr}_b"'] & \gamma
\end{tikzcd}$$

It holds that

$$i \circ \mathtt{pr}_a = (j \circ f) \circ \mathtt{pr}_a = j \circ (f \circ \mathtt{pr}_a) = j \circ [\mathtt{pr}_b \circ (f \times 1_1)] = (j \circ \mathtt{pr}_b) \circ (f \times 1_1),$$

as the equality $f \circ \mathtt{pr}_a = \mathtt{pr}_b \circ (f \times 1_1)$ follows from the definition of $f \times 1_1$

$$\begin{tikzcd}
& & a \times 1 \arrow[dl, "\mathtt{pr}_1"'] \arrow[dr, "\mathtt{pr}_a"] \arrow[dd, dashed, "f \times 1_1"] & & \\
& a \arrow[dl, "f"'] & & 1 \arrow[dr, "1_1"] & \\
b & & b \times 1 \arrow[ll, "\mathtt{pr}_b"'] \arrow[rr, "\mathtt{pr}_1"] & & 1
\end{tikzcd}$$

Therefore, we have $(i \circ \mathtt{pr}_a) \circ (1_a \times 1_1) = (j \circ \mathtt{pr}_b) \circ (f \times 1_1)$, which proves that the above rectangle commutes. Hence $E^{\mathbf{Sub}(\mathcal{C},\gamma)}$ is well-defined. Let $(a, i \circ \mathtt{pr}_a, 1) = (b, j \circ \mathtt{pr}_b, 1)$, then $a = b$ and $i \circ \mathtt{pr}_a = j \circ \mathtt{pr}_a$. As $\mathtt{pr}_a$ is a monomorphism, we get $i = j$, which shows that $E^{\mathbf{Sub}(\mathcal{C},\gamma)}$ is injective on objects and it is trivially injective on arrows. To show that the functor is full, let $(\phi^+, \phi^-) : (a, i \circ \mathtt{pr}_a, 1) \to (b, j \circ \mathtt{pr}_b, 1)$ in $\mathbf{Chu}(\mathcal{C}, \gamma)$. Clearly $\phi^- = 1_1$. By the previous equalities we get $i \circ \mathtt{pr}_a = (j \circ \phi^+) \circ \mathtt{pr}_a$ and since $\mathtt{pr}_a$ is a monomorphism, it follows that $j \circ \phi^+ = i$. Therefore $\phi^+ : i \to j$ is in $\mathbf{Sub}(\mathcal{C}, \gamma)$. Hence $E^{\mathbf{Sub}(\mathcal{C},\gamma)}$ is a full embedding. $\square$

3.3.2 A Chu-representation of the category of complemented subsets

Proposition 3.3.10. (Chu-representation of $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$)
If $(X, =_X, \neq_X)$ is a set with an inequality, then the functor

$$E^{\mathbf{X}} : \boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X) \to \mathbf{Chu}(\mathbf{Set}, X \times X),$$
$$E_0^{\mathbf{X}}(A^1, i^X_{A^1}, A^0, i^X_{A^0}) = (A^1, i^X_{A^1} \times i^X_{A^0}, A^0),$$
$$E_1^{\mathbf{X}}((f^1, f^0) : \mathbf{A} \to \mathbf{B}) = (f^1, f^0) : (A^1, i^X_{A^1} \times i^X_{A^0}, A^0) \to (B^1, i^X_{B^1} \times i^X_{B^0}, B^0)$$

is a full embedding of $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$ into $\mathbf{Chu}(\mathbf{Set}, X \times X)$.

Proof. Clearly $E^{\mathbf{X}}$ is a functor, but we still need to show that it is well-defined. Let $(A^1, i^X_{A^1}, A^0, i^X_{A^0})$ be in $\mathcal{P}^{\rrbracket\llbracket}(X)$, then

$$E_0^{\mathbf{X}}(A^1, i^X_{A^1}, A^0, i^X_{A^0}) = (A^1, i^X_{A^1} \times i^X_{A^0}, A^0).$$

Let

$$i^X_{A^1} \times i^X_{A^0} : A^1 \times A^0 \to X \times X$$
$$[i^X_{A^1} \times i^X_{A^0}](a^1, a^0) = (i^X_{A^1}(a^1), i^X_{A^0}(a^0)),$$

for every $(a^1, a^0) \in A^1 \times A^0$. A^1 and A^0 are objects in $\mathbf{Set}$ and $i^X_{A^1} \times i^X_{A^0}$ is a function in $\mathbf{Set}$, hence $(A^1, i^X_{A^1} \times i^X_{A^0}, A^0)$ is in $\mathbf{Chu}(\mathbf{Set}, X \times X)$. If $(f^1, f^0) : \mathbf{A} \to \mathbf{B}$, then the commutativity of the following two triangles

$$\begin{tikzcd} A^1 \arrow[rr, hook, bend left, "f^1"] \arrow[dr, hook, "i^X_{A^1}"'] & & B^1 \arrow[dl, hook', "i^X_{B^1}"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} B^0 \arrow[rr, hook, bend left, "f^0"] \arrow[dr, hook, "i^X_{B^0}"'] & & A^0 \arrow[dl, hook', "i^X_{A^0}"] \\ & X & \end{tikzcd}$$

implies that

$$\begin{aligned} \big[\big(i^X_{A^1} \times i^X_{A^0}\big) \circ \big(\mathtt{id}_{A^1} \times f^0\big)\big](a^1, b^0) &= \big[i^X_{A^1} \times i^X_{A^0}\big](a^1, f^0(b^0)) \\ &= \big(i^X_{A^1}(a^1), i^X_{A^0}(f^0(b^0))\big) \\ &= \big(i^X_{B^1}(f^1(a^1)), i^X_{B^0}(b^0)\big) \\ &= \big[i^X_{B^1} \times i^X_{B^0}\big]\big(f^1(a^1), b^0\big) \\ &= \big[\big(i^X_{B^1} \times i^X_{B^0}\big) \circ \big(f^1 \times \mathtt{id}_{B^0}\big)\big](a^1, b^0). \end{aligned}$$

Therefore the following rectangle commutes

$$\begin{tikzcd}
A^1 \times B^0 \arrow[r, "\mathtt{id}_{A^1} \times f^0"] \arrow[d, "f^1 \times \mathtt{id}_{B^0}"'] & A^1 \times A^0 \arrow[d, "i^X_{A^1} \times i^X_{A^0}"] \\
B^1 \times B^0 \arrow[r, "i^X_{B^1} \times i^X_{B^0}"'] & X \times X
\end{tikzcd}$$

and $(f^1, f^0) : (A^1, i^X_{A^1} \times i^X_{A^0}, A^0) \to (B^1, i^X_{B^1} \times i^X_{B^0}, B^0)$ is a morphism in $\mathbf{Chu}(\mathbf{Set}, X \times X)$. Hence $E^{\mathbf{X}}$ is a well-defined functor. Clearly, $E^{\mathbf{X}}$ is injective on objects and arrows, hence it is an embedding. We still need to show that it is full. Let $(f^1, f^0) : (A^1, i^X_{A^1} \times i^X_{A^0}, A^0) \to (B^1, i^X_{B^1} \times i^X_{B^0}, B^0)$ in $\mathbf{Chu}(\mathbf{Set}, X \times X)$. The above equalities also show that the commutativity of the above rectangle implies the commutativity of the above triangles. Hence, $(f^1, f^0) : \mathbf{A} \to \mathbf{B}$. Therefore $E^{\mathbf{X}}$ is a full embedding of $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$ into $\mathbf{Chu}(\mathbf{Set}, X \times X)$. $\square$

Consequently, we can identify $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$ with the full subcategory of $\mathbf{Chu}(\mathbf{Set}, X \times X)$ with objects triplets $(A^1, i^X_{A^1} \times i^X_{A^0}, A^0)$, where $i^X_{A^1} : A^1 \hookrightarrow X$ and $i^X_{A^0} : A^0 \hookrightarrow X$, such that

$$\forall_{a^1 \in A^1} \forall_{a^0 \in A^0} \big(i^X_{A^1}(a^1) \neq_X i^X_{A^0}(a^0) \big).$$

We notice that the $\mathbf{Chu}(\mathbf{Set}, X \times X)$ "captures" the behavior of the morphisms in $\boldsymbol{\mathcal{P}}^{\rrbracket\llbracket}(X)$, but not the positive disjointness of A^1, A^0, as there are objects (A, f, B) of $\mathbf{Chu}(\mathbf{Set}, X \times X)$, with $A \between B$. As an example we consider the triplet $(X, \mathtt{id}_{X \times X}, X)$. Obviously $X \between X$, but $(X, \mathtt{id}_{X \times X}, X)$ is an object in $\mathbf{Chu}(\mathbf{Set}, X \times X)$.

Can we characterize categorically the full subcategory of $\mathbf{Chu}(\mathbf{Set}, X \times X)$, that corresponds to Bishop's complemented subsets, in order to grasp apartness of subsets categorically?

3.4 The generalized Chu construction over a cartesian closed category $\mathcal{C}$ and an endofunctor on $\mathcal{C}$

We need to generalise the Chu construction in order to embed **Pred** and $\mathbf{Pred}^{\neq}$ into a Chu category.

Definition 3.4.1. (The Chu construction over a ccc and an endofunctor)
Let $\mathcal{C}$ be a cartesian closed category, and let $\Gamma : \mathcal{C} \to \mathcal{C}$ be an endofunctor on $\mathcal{C}$. The *(generalised) Chu category* $\mathbf{Chu}(\mathcal{C}, \Gamma)$ over $\mathcal{C}$ and Γ has objects quadruples $(x; a, f, b)$ with $x, a, b \in C_0$ and $f : a \times b \to \Gamma_0(x) \in C_1$. A morphism $\phi : (x; a, f, b) \to (y; c, g, d)$ in $\mathbf{Chu}(\mathcal{C}, \Gamma)$, or a *(generalised) Chu transform*, is a triplet $\phi = (\phi^0, \phi^+, \phi^-)$, where $\phi^0 : x \to y$, $\phi^+ : a \to c$ and $\phi^- : d \to b$ are in C_1 such that the following diagram commutes

$$\begin{tikzcd}
a \times d \arrow[r, "1_a \times \phi^-"] \arrow[dd, "\phi^+ \times 1_d"'] & a \times b \arrow[d, "f"] \\
 & \Gamma_0(x) \arrow[d, "\Gamma_1(\phi^0)"] \\
c \times d \arrow[r, "g"'] & \Gamma_0(y)
\end{tikzcd}$$

If $\theta = (\theta^0, \theta^+, \theta^-) : (y; c, g, d) \to (z; i, h, j)$, let $\theta \circ \phi : (\theta^0 \circ \phi^0, \theta^+ \circ \phi^+, \theta^- \circ \phi^-)$.

$$
\begin{tikzcd}
a \times d \arrow[r, "1_a \times \phi^-"] \arrow[dd, "\phi^+ \times 1_d"'] & a \times b \arrow[d, "f"] & \\
& \Gamma_0(x) \arrow[d, "\Gamma_1(\phi^0)"'] \arrow[dd, bend left, "\Gamma_1(\theta^0 \circ \phi^0)"] & \\
c \times d \arrow[r, "g"'] & \Gamma_0(y) \arrow[d, "\Gamma_1(\theta^0)"'] & a \times j \arrow[uul, "1_a \times (\phi^- \circ \theta^-)"'] \arrow[ddl, "(\theta^+ \circ \phi^+) \times 1_j"] \\
& \Gamma_0(z) & \\
c \times j \arrow[uu, "1_c \times \theta^-"] \arrow[r, "\theta^+ \times 1_j"'] & i \times j \arrow[u, "h"] &
\end{tikzcd}
$$

Moreover, $1_{(x;a,f,b)} = (1_x, 1_a, 1_b)$.

$$
\begin{tikzcd}
a \times b \arrow[r, "1_a \times 1_b"] \arrow[dd, "1_a \times 1_b"'] & a \times b \arrow[d, "f"] \\
& \Gamma_0(x) \arrow[d, "\Gamma_1(1_x) = 1_{\Gamma_0(x)}"] \\
a \times b \arrow[r, "f"'] & \Gamma_0(x)
\end{tikzcd}
$$

Remark 3.4.2. To show that composition in $\mathbf{Chu}(\mathcal{C}, \Gamma)$ is well-defined, we show that the triangle in the above definition commutes:

$$
\begin{aligned}
\big[\Gamma_1(\theta^0 \circ \phi^0) \circ f \circ [1_a \times (\phi^- \circ \theta^-)]\big](a, j) &= \big[\Gamma_1(\theta^0) \circ \Gamma_1(\phi^0) \circ f \circ [1_a \times (\phi^- \circ \theta^-)]\big](a, j) \\
&= \big[\Gamma_1(\theta^0) \circ [\Gamma_1(\phi^0) \circ f \circ (1_a \times \phi^-)] \circ (1_a \times \theta^-)\big](a, j) \\
&= \big[\Gamma_1(\theta^0) \circ g \circ (\phi^+ \times 1_d) \circ (1_a \times \theta^-)\big](a, j) \\
&= \big[\Gamma_1(\theta^0) \circ g \circ (\phi^+ \times 1_d)\big](a, d) \\
&= (\Gamma_1(\theta^0) \circ g)(c, d) \\
&= \big[\Gamma_1(\theta^0) \circ g \circ (1_c \times \theta^-)\big](c, j) \\
&= \big[\Gamma_1(\theta^0) \circ g \circ (1_c \times \theta^-) \circ (\phi^+ \times 1_j)\big](a, j) \\
&= \big[[\Gamma_1(\theta^0) \circ g \circ (1_c \times \theta^-)] \circ (\phi^+ \times 1_j)\big](a, j) \\
&= \big[[h \circ (\theta^+ \times 1_j)] \circ (\phi^+ \times 1_j)\big](a, j) \\
&= \big[h \circ [(\theta^+ \circ \phi^+) \times 1_j)]\big](a, j).
\end{aligned}
$$

Proposition 3.4.3. Let $\mathcal{C}$ be a ccc, $\gamma \in C_0$ and let $\Gamma^\gamma : \mathcal{C} \to \mathcal{C}$ be the constant endofunctor with value γ, i.e. $\Gamma_0^\gamma(a) = \gamma$, for every $a \in C_0$, and $\Gamma_1^\gamma(f) = 1_\gamma$, for every $f \in C_1$. Then the functor

$$
\begin{gathered}
E^\gamma : \mathbf{Chu}(\mathcal{C}, \gamma) \to \mathbf{Chu}(\mathcal{C}, \Gamma^\gamma), \\
E_0^\gamma(a, f, b) = (\gamma; a, f, b), \\
E_1^\gamma((\phi^+, \phi^-) : (a, f, b) \to (c, g, d)) = (1_\gamma, \phi^+, \phi^-) : (\gamma; a, f, b) \to (\gamma; c, g, d)
\end{gathered}
$$

is an embedding of $\mathbf{Chu}(\mathcal{C}, \gamma)$ into $\mathbf{Chu}(\mathcal{C}, \Gamma^\gamma)$.

Proof. Let $(\phi^+, \phi^-) : (a, f, b) \to (c, g, d)$, then the following upper inner diagram commutes

$$\begin{tikzcd}
a \times d \arrow[r, "1_a \times \phi^-"] \arrow[dd, "\phi^+ \times 1_d"'] & a \times b \arrow[d, "f"] \\
 & \Gamma_0(x) \arrow[d, "1_\gamma"] \\
c \times d \arrow[ur, "g"] \arrow[r, "g"'] & \Gamma_0(y)
\end{tikzcd}$$

This implies that

$$\begin{aligned}
\big[1_\gamma \circ f \circ (1_a \times \phi^-)\big](a \times d) &= 1_\gamma\big(\Gamma_0(x)\big) \\
&= \Gamma_0(y) \\
&= \big[g \circ (\phi^+ \times +_d)\big](a \times d),
\end{aligned}$$

which shows that the above outer diagram commutes and $(1_\gamma, \phi^+, \phi^-) : (\gamma; a, f, b) \to (\gamma; c, g, d)$. Therefore E^γ is a well-defined functor. Clearly E^γ is injective on objects and arrows, hence it is an embedding. $\square$

3.4.1 A generalized Chu-representation of the category of predicates

We present the definition of the category of predicates within BST.

Definition 3.4.4. The objects of the *category of predicates* **Pred** are triplets (X, i_A^X, A), where X is a set and (A, i_A^X) is a subset of X. If (X, i_A^X, A) and (Y, i_B^Y, B) are objects of **Pred**, a morphism $u : (X, i_A^X, A) \to (Y, i_B^Y, B)$ in **Pred** is a pair of functions $u = (u^0, u^+)$, where $u^0 : X \to Y$ and $u^+ : A \to B$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "u^+"] \arrow[d, hook, "i_A^X"'] & B \arrow[d, hook, "i_B^Y"] \\
X \arrow[r, "u^0"'] & Y
\end{tikzcd}$$

If $v = (v^0, v^+) : (Y, i_B^Y, B) \to (Z, i_C^Z, C)$, we define $v \circ u : (X, i_A^X, A) \to (Z, i_C^Z, C)$ by $v \circ u = (v^0 \circ u^0, v^+ \circ u^+)$. Moreover, $1_{(X, i_A^X, A)} = (\mathrm{id}_X, \mathrm{id}_A)$.

Proposition 3.4.5. (Generalized Chu-representations of **Set** and **Pred**)
(i) The functor $E^{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Chu}(\mathbf{Chu}, \mathrm{Id})$, defined by

$$\begin{gathered}
E_0^{\mathbf{Set}}(X) = (X; X, I_X^X, \mathbb{1}), \\
I_X^X : X \times \mathbb{1} \to \mathrm{Id}_0(X) = X, \quad I_X^X(x, 0) = x; \quad x \in X, \\
E_1^{\mathbf{Set}}\big(f : X \to Y = (f, f) : (X, i_X^X, X) \to (Y, i_Y^Y, Y)\big) = (f, f, \mathrm{id}_{\mathbb{1}}) : (X; X, I_X^X, \mathbb{1}) \to (Y; Y, I_Y^Y, \mathbb{1}),
\end{gathered}$$

is a full embedding of **Set** into **Chu**(**Set**,Id).
(ii) The functor $E^{\mathbf{Pred}} : \mathbf{Pred} \to \mathbf{Chu}(\mathbf{Chu}, \mathrm{Id})$, defined by

$$E_0^{\mathbf{Pred}}(X, i_A^X, A) = (X; A, I_A^X, \mathbb{1}),$$
$$I_A^X : A \times \mathbb{1} \to \mathrm{Id}_0(X) = X, \quad I_A^X(a, 0) = i_A^X(a); \quad a \in A,$$
$$E_1^{\mathbf{Pred}}\big(u = (u^0, u^+) : (X, i_A^X, A) \to (Y, i_B^Y, B)\big) = (u^0, u^+, \mathrm{id}_{\mathbb{1}}) : (X; A, I_A^X, \mathbb{1}) \to (Y; B, I_B^Y, \mathbb{1}),$$

is a full embedding of **Pred** into **Chu**(**Set**,Id).
(iii) If $F : \mathbf{Set} \to \mathbf{Pred}$ is the full embedding of **Set** into **Pred**, defined by

$$F_0(X) = (X, \mathrm{id}_X, X),$$
$$F_1(f : X \to Y) = (f, f),$$

then the following diagram commutes

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, hook, "E^{\mathbf{Set}}"] \arrow[d, hook, "F"'] & \mathbf{Chu}(\mathbf{Set}, \mathrm{Id}) \arrow[d, hook, "\mathrm{Id}"] \\
\mathbf{Pred} \arrow[r, hook, "E^{\mathbf{Pred}}"'] & \mathbf{Chu}(\mathbf{Set}, \mathrm{Id})
\end{tikzcd}$$

Proof.
(i) First we show that $E^{\mathbf{Set}}$ is a well-defined functor. Let X be in **Set**, then

$$E_0^{\mathbf{Set}}(X) = (X; X, I_X^X, \mathbb{1}).$$

X and $\mathbb{1}$ are objects in **Set** and I_X^X is a function in **Set**. If $f : X \to Y$, then the commutativity of the following rectangle

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_X^X"'] & Y \arrow[d, hook, "i_Y^Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

implies the commutativity of the following diagram

$$\begin{tikzcd}
X \times \mathbb{1} \arrow[r, "\mathrm{id}_X \times \mathrm{id}_{\mathbb{1}}"] \arrow[dd, "f \times \mathrm{id}_{\mathbb{1}}"'] & X \times \mathbb{1} \arrow[d, "I_X^X"] \\
& X \arrow[d, "f"] \\
Y \times \mathbb{1} \arrow[r, "I_Y^Y"'] & Y
\end{tikzcd}$$

since

$$\big[f \circ I_X^X \circ (\mathrm{id}_X \times \mathrm{id}_{\mathbb{1}})\big](x, 0) = \big(f \circ I_X^X\big)(x, 0) = f(x) = I_Y^Y\big(f(x), 0\big) = \big[I_Y^Y \circ (f \times \mathrm{id}_{\mathbb{1}})\big](x, 0).$$

Therefore $(u^0, u^+, \mathrm{id}_{\mathbb{1}}) : (X; X, I_X^X, \mathbb{1}) \to (Y; Y, I_Y^Y, \mathbb{1})$ is in $\mathbf{Chu}(\mathbf{Chu}, \mathrm{Id})$. Clearly $E^{\mathbf{Set}}$ is a functor, hence it is a well-defined functor. $E^{\mathbf{Set}}$ is injective on objects and arrows, so it is an embedding. We still have to show that it is full. If $(f, f, \mathrm{id}_{\mathbb{1}}) : (X; X, I_X^X, \mathbb{1}) \to (Y; Y, I_Y^Y, \mathbb{1})$, then the commutativity of the above rectangle implies the commutativity of the first rectangle, hence $(f, f) = (X, i_X^X, X) \to (Y, i_Y^Y, Y)$.
(ii) Clearly $E^{\mathbf{Pred}}$ is a functor. Let (X, i_A^X, A) be in **Pred**, then

$$E_0^{\mathbf{Pred}}(X, i_A^X, A) = (X; A, I_A^X, \mathbb{1}).$$

X, A and $\mathbb{1}$ are objects in **Set**. I_A^X is a function in **Set**. If $u = (u^0, u^+) : (X, i_A^X, A) \to (Y, i_B^Y, B)$, then the following rectangle commutes

$$
\begin{tikzcd}
A \arrow[r, "u^+"] \arrow[d, hook, "i_A^X"'] & B \arrow[d, hook, "i_B^Y"] \\
X \arrow[r, "u^0"'] & Y
\end{tikzcd}
$$

which implies the commutativity of the following diagram

$$
\begin{tikzcd}
A \times \mathbb{1} \arrow[r, "\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}}"] \arrow[dd, hook, "u^+ \times \mathrm{id}_{\mathbb{1}}"'] & A \times \mathbb{1} \arrow[d, "I_A^X"] \\
 & X \arrow[d, "\mathrm{Id}_1(u^0) = u^0"] \\
B \times \mathbb{1} \arrow[r, "I_B^Y"'] & Y
\end{tikzcd}
$$

since

$$
\begin{aligned}
\big[u^0 \circ I_A^X \circ (\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}})\big](a, 0) &= u^0\big(I_A^X(a, 0)\big) \\
&= u^0\big(i_A^X(a)\big) \\
&= i_B^Y\big(u^+(a)\big) \\
&= I_B^Y\big(u^+(a), 0\big) \\
&= \big[I_B^Y \circ (u^+ \circ \mathrm{id}_{\mathbb{1}})\big](a, 0).
\end{aligned}
$$

Hence $(u^0, u^+, \mathrm{id}_{\mathbb{1}}) : (X; A, I_A^X, \mathbb{1}) \to (Y; B, I_B^Y, \mathbb{1})$ is in $\mathbf{Chu}(\mathbf{Chu}, \mathrm{Id})$ and $E^{\mathbf{Pred}}$ is a well-defined functor. Clearly it is injective on objects and arrows, hence it is an embedding. $E^{\mathbf{Pred}}$ is also full. If $(u^0, u^+, \mathrm{id}_{\mathbb{1}}) : (X; A, I_A^X, \mathbb{1}) \to (Y; B, I_B^Y, \mathbb{1})$, then the commutativity of the last diagram implies the commutativity of the first rectangle, therefore $u = (u^0, u^+) : (X, i_A^X, A) \to (Y, i_B^Y, B)$. Hence $E^{\mathbf{Pred}}$ is a full embedding.

(iii) Let X be an object in **Set** and $f : X \to Y$ in **Set**. By (i) and (ii) follows

$$
\begin{aligned}
\big(\mathrm{Id}_0 \circ E_0^{\mathbf{Set}}\big)(X) &= \mathrm{Id}_0(X; X, I_X^X, \mathbb{1}) = (X; X, I_X^X, \mathbb{1}) \\
&= E_1^{\mathbf{Pred}}(X, i_X^X, X) = E_0^{\mathbf{Pred}}(X, \mathrm{id}_X, X) \\
&= \big(E_0^{\mathbf{Pred}} \circ F_0\big)(X)
\end{aligned}
$$

and

$$
\begin{aligned}
\big(\mathrm{Id}_1 \circ E_1^{\mathbf{Set}}\big)(f : X \to Y) &= \mathrm{Id}_1\big((f, f, \mathrm{id}_{\mathbb{1}}) : (X; X, I_X^X, \mathbb{1}) \to (Y; Y, I_Y^Y, \mathbb{1})\big) \\
&= \big((f, f, \mathrm{id}_{\mathbb{1}}) : (X; X, I_X^X, \mathbb{1}) \to (Y; Y, I_Y^Y, \mathbb{1})\big) \\
&= E_1^{\mathbf{Pred}}\big((f, f) : (X, i_X^X, X) \to (Y, i_Y^Y, Y)\big) \\
&= \big(E_1^{\mathbf{Pred}} \circ F_1\big)(f : X \to Y).
\end{aligned}
$$

Hence the diagram commutes. $\square$

Definition 3.4.6. If $\mathcal{C}$ is a category, the category $\mathbf{Pred}(\mathcal{C})$ of $\mathcal{C}$ has objects pairs $(x, i : a \hookrightarrow x)$, where $x \in C_0$ and $i \in C_1(a, x)$ is a monomorphism, and morphisms $(f^0, f^+) : (x, i : a \hookrightarrow x) \to (y, j : b \hookrightarrow y)$ with $j \circ f^+ = f^0 \circ i$.

$$\begin{tikzcd} a \arrow[r, "f^+"] \arrow[d, hook, "i"'] & b \arrow[d, hook, "j"] \\ x \arrow[r, "f^0"'] & y \end{tikzcd}$$

If $(g^0, g^+) : (y, j : b \hookrightarrow y) \to (z, k : c \hookrightarrow z)$, then $(g^0, g^+) \circ (f^0, f^+) = (g^0 \circ f^0, g^+ \circ f^+)$. Moreover, $1_{(x,i:a \hookrightarrow x)} = (1_x, 1_a)$.

Proposition 3.4.7. (Generalized Chu-representation of $\mathbf{Pred}(\mathcal{C})$)
Let $\mathcal{C}$ be a cartesian closed category. The functor

$$E^{\mathbf{Pred}(\mathcal{C})} : \mathbf{Pred}(\mathcal{C}) \to \mathbf{Chu}(\mathcal{C}, \mathrm{Id}^{\mathcal{C}}),$$
$$E_0^{\mathbf{Pred}(\mathcal{C})}(x, i : a \hookrightarrow x) = (x; a, i \circ \mathtt{pr}_a, 1),$$

$$a \times 1 \xhookrightarrow{\mathtt{pr}_a} a \xhookrightarrow{i} x$$

$$E_1^{\mathbf{Pred}(\mathcal{C})}\big((f^0, f^+){:}(x, i : a \hookrightarrow x) \to (y, j : b \hookrightarrow y)\big) = (f^0, f^+, 1_1) : (x; a, i{\circ}\mathtt{pr}_a, 1) \to (y; b, j{\circ}\mathtt{pr}_b, 1),$$

is a full embedding of $\mathbf{Pred}(\mathcal{C})$ into $\mathbf{Chu}(\mathcal{C}, \mathrm{Id}^{\mathcal{C}})$.

Proof. First we show that $E^{\mathbf{Pred}(\mathcal{C})}$ is well-defined. Let $(x, i : a \hookrightarrow x)$ be an object in $\mathbf{Pred}(\mathcal{C})$, then

$$E_0^{\mathbf{Pred}(\mathcal{C})}(x, i : a \hookrightarrow x) = (x; a, i \circ \mathtt{pr}_a, 1).$$

The morphism $\mathtt{pr}_a$ is an isomorphism, hence a monomorphism. Therefore $i \circ \mathtt{pr}_a$ is a monomorphism. If $(f^0, f^+){:}(x, i : a \hookrightarrow x) \to (y, j : b \hookrightarrow y)$, the following diagram commutes

$$\begin{tikzcd} a \arrow[r, "f^+"] \arrow[d, hook, "i"'] & b \arrow[d, hook, "j"] \\ x \arrow[r, "f^0"'] & y \end{tikzcd}$$

To show that $E_1^{\mathbf{Pred}(\mathcal{C})}(f^0, f^+) : (x; a, i \circ \mathtt{pr}_a, 1) \to (y; b, j \circ \mathtt{pr}_b, 1)$ is in $\mathbf{Chu}(\mathcal{C}, \mathrm{Id}^{\mathcal{C}})$, we need to show that the following diagram commutes

$$\begin{tikzcd} a \times 1 \arrow[r, "1_{a \times 1}"] \arrow[dd, hook, "f^+ \times 1_1"'] & a \times 1 \arrow[d, hook, "i \circ \mathtt{pr}_a"] \\ & x \arrow[d, "f^0"] \\ b \times 1 \arrow[r, "j \circ \mathtt{pr}_b"'] & y \end{tikzcd}$$

By the commutativity of the first rectangle it holds that

$$\big[f^0 \circ (i \circ \mathtt{pr}_a)\big](a, 1) = \big[(f^0 \circ i) \circ \mathtt{pr}_a\big](a, 1)$$

$$
\begin{aligned}
&= \big[(j \circ f^+) \circ \mathtt{pr}_a\big](a, 1) \\
&= j\big(f^+(a)\big) = j(b) \\
&= j\big(\mathtt{pr}_b(b, 1)\big) \\
&= j \circ \big[\mathtt{pr}_b \circ (f^+ \times 1_1)\big](a, 1) \\
&= \big[(j \circ \mathtt{pr}_b) \circ (f^+ \times 1_1)\big](a, 1).
\end{aligned}
$$

Hence, the last diagram commutes. Clearly $E^{\mathbf{Pred}(\mathcal{C})}$ is a functor, hence it is a well-defined functor. Let $(x; a, i \circ \mathtt{pr}_a, 1) = (y; b, j \circ \mathtt{pr}_b, 1)$, then $x = y$, $a = b$ and $i \circ \mathtt{pr}_a = j \circ \mathtt{pr}_a$. Since $\mathtt{pr}_a$ is a monomorphism, we have $i = j$, therefore $E^{\mathbf{Pred}(\mathcal{C})}$ is injective on objects. It is trivially injective on arrows, hence it is an embedding. We still need to show that $E^{\mathbf{Pred}(\mathcal{C})}$ is full. Let $(\phi^0, \phi^+, \phi-) : (x; a, i \circ \mathtt{pr}_a, 1) \to (y; b, j \circ \mathtt{pr}_b, 1)$. Clearly $\phi^- = 1_1$. By the previous equalities and since $\mathtt{pr}_a$ is a monomorphism, we get

$$(\phi^0 \circ i) \circ \mathtt{pr}_a = (j \circ \phi^1) \circ \mathtt{pr}_a \Rightarrow \phi^0 \circ i = j \circ \phi^+,$$

i.e. $(\phi^0, \phi^+) : (x, i : a \hookrightarrow x) \to (y, j : b \hookrightarrow y)$. Finally we get that $E^{\mathbf{Pred}(\mathcal{C})}$ is a full embedding. $\square$

3.4.2 A generalized Chu-representation of the category of complemented predicates

In this section, we present the complemented predicates on sets that are equipped with a fixed inequality in the category $\mathbf{Pred}^{\neq}$. By $\mathbf{Pred}^{\neq}_{se}$ we denote the subcategory of $\mathbf{Pred}^{\neq}$, where we consider strongly extensional functions in the definition of the morphisms. The motivation for the next definition is to get a full embedding of $\mathbf{Pred}^{\neq}(\mathbf{Set})$ into the Chu category over $\mathbf{Set}$ and the endofunctor $\mathrm{Id}^2 : \mathbf{Set} \to \mathbf{Set}$, defined by

$$
\begin{gathered}
\mathrm{Id}^2_0(X) = X \times X, \\
\mathrm{Id}^2_0(f : X \to Y) : X \times X \to Y \times Y, \\
\big[\mathrm{Id}^2_1(f)\big](x, x') = \big(f(x), f(x')\big).
\end{gathered}
$$

This result is analogous to the full embedding of $\mathbf{Pred}$ into $\mathbf{Chu}(\mathbf{Set}, \mathrm{Id})$.

Definition 3.4.8. The *category* $\mathbf{Pred}^{\neq}(\mathbf{Set})$ *of complemented predicates* has objects pairs $(X, \mathbf{A})$, where X is in $\mathbf{Set}^{\neq}$, the category of sets equipped with a fixed inequality and strongly extensional functions between them, and $\mathbf{A} := (A^1, A^0)$ is a complemented subset of X. If $(X, \mathbf{A})$ and $(Y, \mathbf{B})$ are objects of $\mathbf{Pred}^{\neq}$, a morphism $u : (X, \mathbf{A}) \to (Y, \mathbf{B})$ is a triplet $u = (u^0, u^+, u^-)$, where $u^0 : X \to Y$, $u^+ : A^1 \to B^1$, and $u^- : B^0 \to A^0$ such that the following rectangles commute

$$
\begin{tikzcd}
A^1 \arrow[r, "u^+"] \arrow[d, hook, "i^X_{A^1}"'] & B^1 \arrow[d, hook, "i^Y_{B^1}"] \\
X \arrow[r, "u^0"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B^0 \arrow[r, "u^-"] \arrow[d, hook, "i^Y_{B^0}"'] & A^0 \arrow[d, hook, "i^X_{A^0}"] \\
Y & X \arrow[l, "u^0"]
\end{tikzcd}
$$

If $v = (v^0, v^+, v^-) : (Y, \mathbf{B}) \to (Z, \mathbf{C})$, we define the composite morphism $v \circ u : (X, \mathbf{A}) \to (Z, \mathbf{C})$ by $v \circ u = (v^0 \circ u^0, v^+ \circ u^+, v^- \circ u^-)$. Moreover, $1_{(X,\mathbf{A})} = (\mathrm{id}_X, \mathrm{id}_{A^1}, \mathrm{id}_{A^0})$.

Proposition 3.4.9. (Generalized Chu-representation of $\mathbf{Pred}^{\neq}(\mathbf{Set})$)
The functor

$$\begin{gathered} E^{\mathbf{Pred}^{\neq}(\mathbf{Set})} : \mathbf{Pred}^{\neq} \to \mathbf{Chu}(\mathbf{Set}, \mathrm{Id}^2), \\ E_0^{\mathbf{Pred}^{\neq}(\mathbf{Set})}(X, \mathbf{A}) = (X; A^1, i^X_{A^1} \times i^X_{A^0}, A^0), \\ i^X_{A^1} \times i^X_{A^0} : A^1 \times A^0 \to \mathrm{Id}^2_0(X) = X \times X, \\ E_1^{\mathbf{Pred}^{\neq}(\mathbf{Set})}\big((u^0, u^+, u^-) : (X, \mathbf{A}) \to (Y, \mathbf{B})\big) = (u^0, u^+u^-) : (X; A^1, i^X_{A^1} \times i^X_{A^0}, A^0) \to \\ (Y; B^1, i^Y_{B^1} \times i^Y_{B^0}, B^0), \end{gathered}$$

is a full embedding of $\mathbf{Pred}^{\neq}(\mathbf{Set})$ into $\mathbf{Chu}(\mathbf{Set}, \mathrm{Id}^2)$.

Proof. Let $(X, \mathbf{A})$ be in $\mathbf{Pred}^{\neq}$, then $(X; A^1, i^X_{A^1} \times i^X_{A^0}, A^0)$ is an object in $\mathbf{Chu}(\mathbf{Set}, \mathrm{Id}^2)$. If $(u^0, u^+, u^-) : (X, \mathbf{A}) \to (Y, \mathbf{B})$, then the following two rectangles commute

$$\begin{tikzcd} A^1 \arrow[r, "u^+"] \arrow[d, hook, "i^X_{A^1}"'] & B^1 \arrow[d, hook, "i^Y_{B^1}"] \\ X \arrow[r, "u^0"'] & Y \end{tikzcd} \qquad \begin{tikzcd} B^0 \arrow[r, "u^-"] \arrow[d, hook, "i^Y_{B^0}"'] & A^0 \arrow[d, hook, "i^X_{A^0}"] \\ Y & X \arrow[l, "u^0"] \end{tikzcd}$$

which implies the commutativity of the following diagram

$$\begin{tikzcd} A^1 \times B^0 \arrow[r, "\mathrm{id}_{A^1} \times u^-"] \arrow[dd, hook, "u^+ \times \mathrm{id}_{B^0}"'] & A^1 \times A^0 \arrow[d, "i^X_{A^1} \times i^X_{A^0}"] \\ & X \times X \arrow[d, "\mathrm{Id}^2_1(u^0)"] \\ B^1 \times B^0 \arrow[r, "i^Y_{B^1} \times i^Y_{B^0}"'] & Y \times Y \end{tikzcd}$$

since

$$\begin{aligned} \big[\mathrm{Id}^2_1(u^0) \circ (i^X_{A^1} \times i^X_{A^0}) \circ (\mathrm{id}_{A^1} \times u^-)\big](a^1, b^0) &= \big[\mathrm{Id}^2_1(u^0) \circ (i^X_{A^1} \times i^X_{A^0})\big](a^1, u^-(b^0)) \\ &= \mathrm{Id}^2_1(u^0)\big(i^X_{A^1}(a^1), i^X_{A^0}(u^-(b^0))\big) \\ &= \big(u^0(i^X_{A^1}(a^1)), u^0(i^X_{A^0}(u^-(b^0)))\big) \\ &= \big(i^Y_{B^1}(u^+(a^1)), i^Y_{B^0}(b^0)\big) \\ &= \big[i^Y_{B^1} \times i^Y_{B^0}\big](u^+(a^1), b^0) \\ &= \big[(i^Y_{B^1} \times i^Y_{B^0}) \circ (u^+ \times \mathrm{id}_{B^0})\big](a^1, b^0). \end{aligned}$$

Hence $(u^0, u^+u^-) : (X; A^1, i^X_{A^1} \times i^X_{A^0}, A^0) \to (Y; B^1, i^Y_{B^1} \times i^Y_{B^0}, B^0)$ is in $\mathbf{Chu}(\mathbf{Set}, \mathrm{Id}^2)$. Clearly $E^{\mathbf{Pred}^{\neq}}$ is injective on objects and arrows, therefore it is an embedding. We still need to show that $E^{\mathbf{Pred}^{\neq}}$ is full. If $(u^0, u^+u^-) : (X; A^1, i^X_{A^1} \times i^X_{A^0}, A^0) \to (Y; B^1, i^Y_{B^1} \times i^Y_{B^0}, B^0)$, then the commutativity of the last diagram implies the commutativity of the first two rectangles, so $(u^0, u^+, u^-) : (X, \mathbf{A}) \to (Y, \mathbf{B})$. Therefore $E^{\mathbf{Pred}^{\neq}}$ is a full embedding. $\square$

4 Apartness relations in categories

Our aim is to translate the notions of Bishop's set theory into category theory. We want to introduce subsets and complemented subsets into the language of categories. In the terms of Bishop, an equality and inequality were defined through some formula. In set theory, a relation between sets is defined as a subset of their cartesian product, and this one can translate into the theory of categories. We are based on Petrakis' work *Equality and apartness relations in categories*, [8].

Definition 4.0.1. A *subobject* of an object X in a category $\mathcal{C}$ is a monomorphisms

$$m : M \hookrightarrow X$$

with codomain X, where isomorphic subobjects are identified.

Definition 4.0.2. Let $\mathcal{C}$ be a category.
(i) An object 0 of $\mathcal{C}$ is called *initial*, if for any object $X \in C_0$ there is a unique morphism $i : 0 \to X$.
(ii) An object 1 of $\mathcal{C}$ is called *terminal*, if for any object $X \in C_0$ there is a unique morphism $t : X \to 1$.

4.1 Binary relations

Definition 4.1.1. A *binary relation* between two objects x_1, x_2 of a category $\mathcal{C}$ is a triplet $\mathbf{r} := (r \in C_0, r_1 : r \to x_1, r_2 : r \to x_2)$, where r_1, r_2 are *jointly monic* arrows, i.e. for every $g, h : a \to r$ such that $r_1 \circ g = r_1 \circ h$ and $r_2 \circ g = r_2 \circ h$ we have that $g = h$,

$$\begin{tikzcd} & & x_1 \\ a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & r \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & \\ & & x_2 \end{tikzcd}
\qquad
\begin{tikzcd} & & x_1 \\ & & \vdots \\ a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & r \arrow[uur, "r_1"] \arrow[r, "r_i"'] \arrow[ddr, "r_2"'] & x_i \\ & & \vdots \\ & & x_2 \end{tikzcd}$$

and we may also write $(r_1, r_2) : r \hookrightarrow (x_1, x_2)$. If $n \in \mathbb{N}^+$, an *n-ary relation* is a structure $\mathbf{r} := (r \in C_0, r_1 : r \to x_1, ..., r_n : r \to x_n)$, where $r_1, ..., r_n$ are jointly monic arrows, i.e. for every $g, h : a \to r$ such that $r_1 \circ g = r_1 \circ h, ..., r_n \circ g = r_n \circ h$, we have that $g = h$. In the general case we may also write $(r_1, ..., r_n) : r \hookrightarrow (x_1, ..., x_n)$.

Definition 4.1.2. If $\mathcal{C}$ is a category and $x_1, x_2 \in C_0$, the category $\mathbf{Rel}_{\mathcal{C}}(x_1, x_2)$ of binary relations between x_1, x_2 , or simply $\mathbf{Rel}(x_1, x_2)$, has objects binary relations $\mathbf{r} := (r, r_1, r_2)$ and an arrow $f : \mathbf{r} \to \mathbf{s}$, where $\mathbf{s} := (s, s_1, s_2)$, is an arrow $f : r \to s$ such that the following inner diagrams commute

$$\begin{tikzcd}
& x_1 & \\
r \arrow[ur, "r_1"] \arrow[rr, "f"] \arrow[dr, "r_2"'] & & s \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\
& x_2 &
\end{tikzcd}$$

The *composition* of arrows in $\mathbf{Rel}(x_1, x_2)$ is the composition of arrows in $\mathcal{C}$, and $1_{(r,r_1,r_2)} = 1_r$. If $n \in \mathbb{N}$ and $x_1, ..., x_n \in C_0$, the category $\mathbf{Rel}(x_1, ..., x_n)$ of n-ary relations between $x_1, ..., x_n$ is defined similarly.

Definition 4.1.3. Let $\mathcal{C}$ be a category with a terminal object 1. An *element* of an object x in $\mathcal{C}$ is an arrow $e : 1 \to x$.

Definition 4.1.4. If 1 is a terminal object of a category $\mathcal{C}$, a *member* of a binary relation $\mathbf{r} \in \mathbf{Rel}(x_1, x_2)$ is a triplet $\boldsymbol{i} := (1, i_1, i_2)$, where $i_1 : 1 \hookrightarrow x_1$ and $i_2 : 1 \hookrightarrow x_2$, hence an object of $\mathbf{Rel}(x_1, x_2)$, such that there is an arrow $m : \boldsymbol{i} \to \mathbf{r}$ in $\mathbf{Rel}(x_1, x_2)$, i.e. the following inner diagrams commute

$$\begin{tikzcd}
& x_1 & \\
1 \arrow[ur, hook, "i_1"] \arrow[rr, hook, "m"] \arrow[dr, hook, "i_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
& x_2 &
\end{tikzcd}$$

and we write

$$\boldsymbol{i} \ \mathtt{Memb} \ \mathbf{r} \qquad \& \qquad m : \boldsymbol{i} \ \mathtt{Memb} \ \mathbf{r},$$

to denote that m realises the membership of $\boldsymbol{i}$ to $\mathbf{r}$.

A relation between two sets X_1, X_2 is a subset $R \subseteq X_1 \times X_2$. Using Bishop's notion of a subset, this is a set R and an embedding $i : R \hookrightarrow X_1 \times X_2$. Then one can show that there is a relation $(r_1, r_2) : R \hookrightarrow (X_1 \times X_2)$.

Proposition 4.1.5. Let $R \subseteq X_1 \times X_2$.
(i) Let $i : R \hookrightarrow X_1 \times X_2$ be an embedding. If $r_1 = \mathtt{pr}_1 \circ i$ and $r_2 = \mathtt{pr}_2 \circ i$, then $(r_1, r_2) : R \hookrightarrow (X_1, X_2)$ is a relation.
(ii) Conversely, if $(r_1, r_2) : R \hookrightarrow (X_1, X_2)$, then $i = r_1 \times r_2 : R \hookrightarrow X_1 \times X_2$ is an embedding, where $(r_1 \times r_2)(x) = \big(r_1(x), r_2(x)\big)$ for every $x \in R$.

Proof.

$$\begin{tikzcd}
& & & X_1 \\
A \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & R \arrow[r, hook, "i"] \arrow[urr, bend left, "r_1"] \arrow[drr, bend right, "r_2"'] & X_1 \times X_2 \arrow[ur, "\mathtt{pr}_1"] \arrow[dr, "\mathtt{pr}_2"'] & \\
& & & X_2
\end{tikzcd}$$

(i) Let $g, h : A \to R$, with $r_1 \circ g = r_1 \circ h$ and $r_2 \circ g = r_2 \circ h$. It holds that $r_1 \circ g = \mathtt{pr}_1 \circ i \circ g = \mathtt{pr}_1 \circ i \circ h = r_1 \circ h$ and $r_2 \circ g = \mathtt{pr}_2 \circ i \circ g = \mathtt{pr}_2 \circ i \circ h = r_2 \circ h$. Since $\mathtt{pr}_1$ and $\mathtt{pr}_2$ are isomorphisms, hence monomorphisms, it follows

$$i \circ g = i \circ h.$$

Since i is an embedding we get $g = h$. Hence r_1 and r_2 are jointly monic. Therefore $(r_1, r_2) : R \hookrightarrow (X_1, X_2)$ is a binary relation.
(ii) Now, let $(r_1, r_2) : R \hookrightarrow (X_1, X_2)$ and let $i \circ g = i \circ h$, for $g, h : A \to R$. For $x \in A$ the following holds

$$i\big(g(x)\big) = \big(r_1(g(x)), r_2(g(x))\big) = \big(r_1(h(x)), r_2(h(x))\big) = i\big(h(x)\big),$$

which implies that $r_1(g(x)) = r_1(h(x))$ and $r_2(g(x)) = r_2(h(x))$. Since r_1 and r_2 are jointly monic, it holds $g = h$. Hence $i = r_1 \times r_2 : R \hookrightarrow X_1 \times X_2$ is an embedding. $\square$

Remark 4.1.6. Let $\mathcal{C}$ be a category with $1,\ x, x_1, x_2, y_1, y_2 \in C_0$ and let $\mathbf{r} := (r, r_1, r_2), \mathbf{s} := (s, s_1, s_2)$ and $\boldsymbol{i} := (1, i_1, i_2)$ in $\mathbf{Rel}(x_1, x_2)$.
(i) If $\boldsymbol{i}\ \mathtt{Memb}\ \mathbf{r}$ and $f : \mathbf{r} \to \mathbf{s}$ in $\mathbf{Rel}(x_1, x_2)$, then $\boldsymbol{i}\ \mathtt{Memb}\ \mathbf{s}$.
(ii) If $e_1 : a \hookrightarrow x_1$ is a subobject of x_1 and $e_2 : a \hookrightarrow x_2$ is a subobject of x_2, then $(a, e_1, e_2) \in \mathbf{Rel}(x_1, x_2)$.
(iii) If $e : a \hookrightarrow x$ is a subobject of x, then $(a, e, e) \in \mathbf{Rel}(x, x)$.
(iv) If $f_1 : x_1 \hookrightarrow y_1$ and $f_2 : x_2 \hookrightarrow y_2$, then $(f_1, f_2)^* : \mathbf{Rel}(x_1, x_2) \to \mathbf{Rel}(y_1, y_2)$ is a functor, where $(f_1, f_2)^*_0(\mathbf{r}) := (f_1, f_2) \circ \mathbf{r} := (r, f_1 \circ r_1, f_2 \circ r_2)$, and $(f_1, f_2)^*_1(h : \mathbf{r} \to \mathbf{s}) := h$.
(v) If $e : a \hookrightarrow x$ is a subobject of x, let $e^* : \mathbf{Rel}(a, a) \to \mathbf{Rel}(x, x)$ be the functor $(e, e)^*$, i.e. $e^*_0(\mathbf{r}) := (r, e \circ r_1, e \circ r_2) =: e \circ \mathbf{r}$. Then, $\boldsymbol{i}\ \mathtt{Memb}\ \mathbf{r} \Rightarrow e \circ \boldsymbol{i}\ \mathtt{Memb}\ e \circ \mathbf{r}$.
(vi) The category $\mathbf{Rel}(x_1, x_2)$ is thin.
(vii) If $e : 1 \hookrightarrow r$ is an element of r, then $\mathbf{e} := (1, r_1 \circ e, r_2 \circ e)\ \mathtt{Memb}\ \mathbf{r}$ and $e : \mathbf{e}\ \mathtt{Memb}\ \mathbf{r}$.

Proof.
(i) Let $\boldsymbol{i}\ \mathtt{Memb}\ \mathbf{r}$ and $f : \mathbf{r} \to \mathbf{s}$ in $\mathbf{Rel}(x_1, x_2)$, i.e. the following left and right inner diagrams commute

$$\begin{tikzcd}
& x_1 & \\
1 \arrow[ur, hook, "i_1"] \arrow[r, hook, "m"] \arrow[dr, hook', "i_2"'] & r \arrow[u, "r_1"'] \arrow[d, "r_2"] \arrow[r, "f"] & s \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\
& x_2 &
\end{tikzcd}$$

since $i_1 = m \circ r_1 = m \circ f \circ s_1$ and $i_2 = m \circ r_2 = m \circ f \circ s_2$. Hence $\boldsymbol{i}\ \mathtt{Memb}\ \mathbf{s}$.
(ii) Let $e_1 : a \hookrightarrow x_1$ be a subobject of x_1 and $e_2 : a \hookrightarrow x_2$ be a subobject of x_2, i.e. e_1 and e_2 are monomorphisms. Let $g, h : b \to a$

$$\begin{tikzcd}
& & x_1 \\
b \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & a \arrow[ur, hook, "e_1"] \arrow[dr, hook', "e_2"'] & \\
& & x_2
\end{tikzcd}$$

with $e_1 \circ g = e_1 \circ h$ and $e_2 \circ g = e_2 \circ h$. Since e_1 and e_2 are monomorphisms, $g = h$, hence e_1 and e_2 are jointly monic and $(a, e_1, e_2) \in \mathbf{Rel}(x_1, x_2)$.

(iii) Let $e : a \hookrightarrow x$ be a subobject of x, i.e. e is a monomorphism. Let $g, h : b \to a$

$$\begin{tikzcd}
 & & x \\
b \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & a \arrow[ur, hook, "e"] \arrow[dr, hook, "e"'] & \\
 & & x
\end{tikzcd}$$

with $e \circ g = e \circ h$. Since e is a monomorphism, $g = h$, hence e and e are jointly monic and $(a, e, e) \in \mathbf{Rel}(x, x)$.

(iv) Let $f_1 : x_1 \hookrightarrow y_1$ and $f_2 : x_2 \hookrightarrow y_2$ be monic, then we need to show that $(r, f_1 \circ r_1, f_2 \circ r_2) \in \mathbf{Rel}(y_1, y_2)$. Obviously $r \in C_0$. Let $g, h : a \to r$ with $f_1 \circ r_1 \circ g = f_1 \circ r_1 \circ h$ and $f_2 \circ r_2 \circ g = f_2 \circ r_2 \circ h$. Since f_1, f_2 are monic we have

$$r_1 \circ g = r_1 \circ h \text{ and } r_2 \circ g = r_2 \circ h$$

and since r_1, r_2 are jointly monic by definition, we get

$$g = h.$$

Hence $f_1 \circ r_1$ and $f_2 \circ r_2$ are jointly monic and $(r, f_1 \circ r_1, f_2 \circ r_2)$ is in $\mathbf{Rel}(y_1, y_2)$. Let $h : \mathbf{r} \to \mathbf{s}$ be an arrow in $\mathbf{Rel}(x_1, x_2)$. $(f_1, f_2)^*_1(h) := h$. Then the commutativity of the following right inner diagrams follows from the commutativity of the left inner diagrams

$$\begin{tikzcd}
 & x_1 & \\
r \arrow[ur, "r_1"] \arrow[rr, "h"] \arrow[dr, "r_2"'] & & s \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\
 & x_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y_1 & \\
r \arrow[ur, "f_1 \circ r_1"] \arrow[rr, "h"] \arrow[dr, "f_2 \circ r_2"'] & & s \arrow[ul, "f_1 \circ s_1"'] \arrow[dl, "f_2 \circ s_2"] \\
 & y_2 &
\end{tikzcd}$$

since $f_1 \circ r_1 = f_1 \circ (s_1 \circ h) = (f_1 \circ s_1) \circ h$ and $f_2 \circ r_2 = f_2 \circ (s_2 \circ h) = (f_2 \circ s_2) \circ h$. Hence $h : \mathbf{r} \to \mathbf{s}$ is an arrow in $\mathbf{Rel}(y_1, y_2)$. Therefore $(f_1, f_2)^*$ is a functor from $\mathbf{Rel}(x_1, x_2)$ to $\mathbf{Rel}(y_1, y_2)$.

(v) Let $e : a \hookrightarrow x$ be a subobject of x and $e^* : \mathbf{Rel}(a, a) \to \mathbf{Rel}(x, x)$ be the functor $(e, e)^*$. Let $\boldsymbol{i}$,$\mathbf{r} \in \mathbf{Rel}(a, a)$. If $m : \boldsymbol{i}$ `Memb` $\mathbf{r}$, then the following inner diagrams commute

$$\begin{tikzcd}
 & a & \\
1 \arrow[ur, hook, "i_1"] \arrow[rr, hook, "m"] \arrow[dr, hook, "i_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & a &
\end{tikzcd}$$

By definiton $e \circ \boldsymbol{i} := (1, e \circ i_1, e \circ 1_2)$ and $e \circ \mathbf{r} := (r, e \circ r_1, e \circ r_2)$. Then the following inner diagrams commute since, by the commutativity of the first diagram, $e \circ i_1 = e \circ (r_1 \circ m) = e \circ r_1 \circ m$ and

$e \circ i_2 = e \circ (r_2 \circ m) = e \circ r_2 \circ m.$

$$\begin{tikzcd}
 & & x & & \\
 & a \arrow[ur, hook, "e"] & & a \arrow[ul, hook', "e"'] & \\
1 \arrow[ur, hook, "i_1"] \arrow[rrrr, hook, "m"] \arrow[dr, hook', "i_2"'] & & & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & a \arrow[dr, hook', "e"'] & & a \arrow[dl, hook, "e"] & \\
 & & x & &
\end{tikzcd}$$

Hence $m : e \circ \boldsymbol{i}$ `Memb` $e \circ \mathbf{r}$.

(vi) Let $f, g : \mathbf{r} \to \mathbf{s}$ be in $\mathbf{Rel}(x_1, x_2)$, then the following inner diagrams commute

$$\begin{tikzcd}
 & x_1 & \\
r \arrow[ur, "r_1"] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "r_2"'] & & s \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\
 & x_2 &
\end{tikzcd}$$

and it holds

$$r_1 = s_1 \circ f = s_1 \circ g \text{ and } r_2 = s_2 \circ f = s_2 \circ g.$$

Since s_1, s_2 are jointly monic we get $f = g$. Hence $\mathbf{Rel}(x_1, x_2)$ is thin.

(vii) Let $e : 1 \hookrightarrow r$ be an element of r, then $\mathbf{e} := (1, r_1 \circ e, r_2 \circ e)$ is in $\mathbf{Rel}(x_1, x_2)$, since $1 \in C_0$ and $r_1 \circ e, r_2 \circ e$ are jointly monic, because e is monic and r_1, r_2 are jointly monic. The following inner diagrams obviously commute

$$\begin{tikzcd}
 & x_1 & \\
1 \arrow[ur, hook, "r_1 \circ e"] \arrow[rr, hook, "e"] \arrow[dr, hook', "r_2 \circ e"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x_2 &
\end{tikzcd}$$

Hence $\mathbf{e}$ `Memb` $\mathbf{r}$ and $e : \mathbf{e}$ `Memb` $\mathbf{r}$. $\square$

Because of the thinness of $\mathbf{Rel}(x_1, x_2)$, we write $\mathbf{r} \leqslant \mathbf{s}$, if there is (unique) arrow $h : \mathbf{r} \to \mathbf{s}$. In this case, we also write $h : \mathbf{r} \leqslant \mathbf{s}$.

Remark 4.1.7. Let $\boldsymbol{i}, \mathbf{r} \in \mathbf{Rel}(x_1, x_2)$, then

$$\boldsymbol{i} \ \texttt{Memb} \ \mathbf{r} \quad \Leftrightarrow \quad \boldsymbol{i} \leqslant \mathbf{r}.$$

Proof. "⇒" Let $\boldsymbol{i}$ `Memb` $\mathbf{r}$, then the following inner diagrams commute

$$\begin{tikzcd}
 & x_1 & \\
1 \arrow[ur, "i_1"] \arrow[rr, "f"] \arrow[dr, "i_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x_2 &
\end{tikzcd}$$

Let $g : 1 \to r$ be another arrow from 1 to r, then $i_1 = r_1 \circ f = r_1 \circ g$ and $i_2 = r_2 \circ f = r_2 \circ g$. Since r_1, r_2 are jointly monic we get $f = g$, hence $f : 1 \to r$ is unique, so $\boldsymbol{i} \leqslant \mathbf{r}$.
"$\Leftarrow$" Now let $\boldsymbol{i} \leqslant \mathbf{r}$, then there exist a unique arrow $f : 1 \to r$ such that the above inner diagrams commute. Hence $\boldsymbol{i}$ `Memb` $\mathbf{r}$. $\square$

In a category with products the product $x_1 \times x_2$ is the largest binary relation between x_1, x_2.

Proposition 4.1.8. Let $\mathcal{C}$ be a category with products, x_1, x_2 objects of $\mathcal{C}$ and $\mathbf{r} \in \mathbf{Rel}(x_1, x_2)$.
(i) $\mathbf{x_1} \times \mathbf{x_2} := (x_1 \times x_2, \mathtt{pr}_1, \mathtt{pr}_2) \in \mathbf{Rel}(x_1, x_2)$, and $\langle r_1, r_2 \rangle : \mathbf{r} \leqslant \mathbf{x_1} \times \mathbf{x_2}$.
(ii) r_1, r_2 are jointly monic if and only if $\langle r_1, r_2 \rangle : r \to x_1 \times x_2$ is monic.
(iii) If 0 is an initial object of $\mathcal{C}$ and $\mathbf{0} := (0, 0_{x_1}, 0_{x_2}) \in \mathbf{Rel}(x_1, x_2)$, then $0_r : \mathbf{0} \leqslant \mathbf{r}$.

Proof. (i) Let $\mathtt{pr}_1 : x_1 \times x_2 \to x_1$ and $\mathtt{pr}_2 : x_1 \times x_2 \to x_2$. We need to show that $\mathtt{pr}_1$ and $\mathtt{pr}_2$ are jointly monic. Let $g, h : a \to x_1 \times x_2$ with $\mathtt{pr}_1 \circ g = \mathtt{pr}_1 \circ h$ and $\mathtt{pr}_2 \circ g = \mathtt{pr}_2 \circ h$.

$$\begin{tikzcd}
 & & x_1 & \\
a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & x_1 \times x_2 \arrow[ur, "\mathtt{pr}_1"] \arrow[dr, "\mathtt{pr}_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \arrow[ll, "\langle r_1, r_2 \rangle"'] \\
 & & x_2 &
\end{tikzcd}$$

Since $\mathtt{pr}_1$ and $\mathtt{pr}_2$ are isomorphisms, hence monomorphisms, it holds that $g = h$, so $\mathtt{pr}_1, \mathtt{pr}_2$ are jointly monic. Hence $\mathbf{x_1} \times \mathbf{x_2}$ is a binary relation. Let $\langle r_1, r_2 \rangle : r \to x_1 \times x_2$, then the above inner diagrams commute and $\langle r_1, r_2 \rangle$ is unique since $\mathtt{pr}_1, \mathtt{pr}_2$ are jointly monic. Hence $\langle r_1, r_2 \rangle : \mathbf{r} \leqslant \mathbf{x_1} \times \mathbf{x_2}$.
(ii) First, let r_1 and r_2 be jointly monic and $g, h : a \to r$ such that $\langle r_1, r_2 \rangle \circ g = \langle r_1, r_2 \rangle \circ h$.

$$\begin{tikzcd}
 & & x_1 \\
a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & r \arrow[ur, "r_1"] \arrow[dr, "r_2"'] \arrow[r, "\langle r_1, r_2 \rangle"] & x_1 \times x_2 \\
 & & x_2
\end{tikzcd}$$

Since $\langle r_1, r_2 \rangle = r_1 \times r_2$ we have for $x \in a$

$$\langle r_1, r_2 \rangle \big(g(x)\big) = \big(r_1(g(x)), r_2(g(x))\big) = \big(r_1(h(x)), r_2(h(x))\big) = \langle r_1, r_2 \rangle \big(h(x)\big),$$

which implies $r_1(g(x)) = r_1(h(x))$ and $r_2(g(x)) = r_2(h(x))$. Since r_1, r_2 are jointly monic we get $g = h$, hence $\langle r_1, r_2 \rangle : r \to x_1 \times x_2$ is monic. Now let $\langle r_1, r_2 \rangle$ be monic, and $r_1 \circ g = r_1 \circ h$ and $r_2 \circ g = r_2 \circ h$, then

$$\langle r_1, r_2 \rangle \circ g = (r_1 \times r_2) \circ g = (r_1 \circ g) \times (r_2 \circ g) = (r_1 \circ h) \times (r_2 \circ h) = (r_1 \times r_2) \circ h = \langle r_1, r_2 \rangle \circ h.$$

Since $\langle r_1, r_2 \rangle$ is monic, we get $g = h$. Hence r_1, r_2 are jointly monic.
(iii) Let 0 be an initial object of $\mathcal{C}$ and $\mathbf{0} \in \mathbf{Rel}(x_1, x_2)$. By definition of an initial object there is

a unique arrow $0_r : 0 \to r$. Then the following inner diagrams commute

$$\begin{tikzcd}
& x_1 & \\
0 \arrow[ur, "0_{x_1}"] \arrow[rr, "0_r"] \arrow[dr, "0_{x_2}"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
& x_2 &
\end{tikzcd}$$

Hence $0_r : \mathbf{0} \leqslant \mathbf{r}$. $\square$

Now let $\mathcal{C}$ be a category with pullbacks and pushouts. Let x be an object in $\mathcal{C}$, then we want to define the intersection and union of two binary relations $\mathbf{r}$ and $\mathbf{s}$. First consider the pullback of the following diagram

$$\begin{tikzcd}
r \times_x s \arrow[r, "q"] \arrow[d, "p"'] & s \arrow[r, "s_1"] \arrow[d, "s_2"] & x \\
r \arrow[r, "r_1"] \arrow[d, "r_2"] & x & \\
x & &
\end{tikzcd}$$

We can define the intersection of $\mathbf{r}$ and $\mathbf{s}$ as their pullback.

Definition 4.1.9. Let $\mathbf{r}, \mathbf{s} \in \mathbf{Rel}(x, x)$. The *intersection* of $\mathbf{r} := (r, r_1, r_2)$ and $\mathbf{s} := (s, s_1, s_2)$ is defined as

$$\mathbf{r} \cap \mathbf{s} := \big\{(a, b) \in r \times_x s \mid (r_1 \circ p)(a) = (s_2 \circ q)(b)\big\}$$

where $p : r \times_x s \to r$ and $q : r \times_x s \to s$.

Let $i_r : \mathbf{r} \cap \mathbf{s} \to r$ and $i_s : \mathbf{r} \cap \mathbf{s} \to s$ be the obvious inclusions of $\mathbf{r} \cap \mathbf{s}$ into $\mathbf{r}$ and $\mathbf{s}$. Taking the pushout of i_r along i_s gives us the disjoint union of r and s,

$$\begin{tikzcd}
\mathbf{r} \cap \mathbf{s} \arrow[r, "i_s"] \arrow[d, "i_r"'] & s \arrow[r, "s_1"] \arrow[d, "s_2"] & x \\
r \arrow[r, "r_1"] \arrow[d, "r_2"] & r +_x s & \\
x & &
\end{tikzcd}$$

where we identify $i_r(x)$ with $i_s(x)$ for every $x \in \mathbf{r} \cap \mathbf{s}$. Since the pullback of r and s is the intersection, the pushout is in fact the union of r and s. Hence we get the following definition:

Definition 4.1.10. Let $\mathbf{r}, \mathbf{s} \in \mathbf{Rel}(x, x)$. The *union* of $\mathbf{r}$ and $\mathbf{s}$ is defined as

$$\mathbf{r} \cup \mathbf{s} := \big\{x \in r +_x s \mid i_r(x) = i_s(x) \ \forall x \in \mathbf{r} \cap \mathbf{s}\big\}$$

where $i_r : \mathbf{r} \cap \mathbf{s} \to r$ and $i_s : \mathbf{r} \cap \mathbf{s} \to s$ are the inclusions.

In the following we assume that the category $\mathcal{C}$ has finite products and an epi-mono-factorization. In such a category every arrow $f : a \to b$ can be factorized by an epimorphism $e : a \to c$ and a monomorphism $m : c \to b$ such that $f = m \circ e$, meaning that the following diagram commutes

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[rr, bend right, "f"'] & c \arrow[r, "m"] & b. \end{tikzcd}$$

c is called the *image* of f. The epi-mono factorization is unique up to isomorphisms in the sense that, if there exists another epimorphism $e' : a \to c'$ and monomorphism $m' : c' \to b$ with $f = m' \circ e'$, then there is an isomorphism $i : c \to c'$ such that $h \circ e = e'$ and $m' \circ h = m$.

We now want to present the composition of two relations $\mathbf{r}$ and $\mathbf{s}$ as a relation $\mathbf{s} \circ \mathbf{r}$. Let $\mathcal{C}$ be a category with pullbacks, $x, y, z \in \mathcal{C}$. Let $\mathbf{r} := (r, r_1, r_2)$ be a relation between x, y and $\mathbf{s} := (s, s_1, s_2)$ be a relation between y, z, then $r \subseteq x \times y$ and $s \subseteq y \times z$. The composition of r with s is expressed through the following subset

$$s \circ r = \big\{(a, c) \in x \times z \mid \exists b \in y \ ((a, b) \in r \wedge (b, c) \in s)\big\} \subseteq x \times z.$$

First consider the pullback of r_2 along s_1,

$$\begin{tikzcd} r \times_y s \arrow[r, "q"] \arrow[d, "p"'] & s \arrow[r, "s_2"] \arrow[d, "s_1"] & z \\ r \arrow[r, "r_2"] \arrow[d, "r_1"] & y & \\ x & & \end{tikzcd}$$

where $r \times_y s = \{(a, b) \in r \times s \mid r_2(a) = s_1(b)\}$. We factorize the morphism $(r_1 \circ p, s_2 \circ q)$ by

$$(r_1 \circ p, s_2 \circ q) = \big((s \circ r)_1, (s \circ r)_2\big) \circ e : r \times_y s \to s \circ r \to x \times z,$$

where $\big((s \circ r)_1, (s \circ r)_2\big)$ is a monomorphism and e is an epimorphism. We call

$$\mathbf{s} \circ \mathbf{r} := (s \circ r, (s \circ r)_1, (s \circ r)_2)$$

the *composition* of $\mathbf{r}$ and $\mathbf{s}$. Let $i := \big((s \circ r)_1, (s \circ r)_2\big)$. If $(s \circ r)_1 = \mathtt{pr}_1 \circ i$ and $(s \circ r)_2 = \mathtt{pr}_2 \circ i$, then $\mathbf{s} \circ \mathbf{r}$ indeed defines a relation between x and z.

$$\begin{tikzcd} & & r \times_y s \arrow[dl, "p"'] \arrow[d, two heads, "e"] \arrow[dr, "q"] & & \\ & r \arrow[dl, "r_1"'] & s \circ r \arrow[dll, "(s \circ r)_1"'] \arrow[d, hook, "i"] \arrow[drr, "(s \circ r)_2"] & s \arrow[dr, "s_2"] & \\ x & & x \times z \arrow[ll, "\mathtt{pr}_1"] \arrow[rr, "\mathtt{pr}_2"'] & & z \end{tikzcd}$$

Let $g, h : a \to s \circ r$ with $(s \circ r)_1 \circ g = (s \circ r)_1 \circ h$ and $(s \circ r)_2 \circ g = (s \circ r)_2 \circ h$. Then it holds $\mathtt{pr}_1 \circ i \circ g = (s \circ r)_1 \circ g = (s \circ r)_1 \circ h = \mathtt{pr}_1 \circ i \circ h$ and $\mathtt{pr}_2 \circ i \circ g = (s \circ r)_2 \circ g = (s \circ r)_2 \circ h = \mathtt{pr}_2 \circ i \circ h$. As $\mathtt{pr}_1, \mathtt{pr}_2$ and i are monomorphisms, it holds $g = h$. Hence $(s \circ r)_1, (s \circ r)_2$ are jointly monic and $\mathbf{s} \circ \mathbf{r}$ is a relation between x and z.

Definition 4.1.11. Let $f : X \to Y$ be a morphism. A morphism $g : Y \to X$ is called an *inverse* of f, if $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Let $f : x \to y$ and $f^{-1} : y \to x$ be the inverse. Let $\mathbf{r} := (r, r_1, r_2)$ be a relation on x, x, then $f(r)$ is a subset of $y \times y$. Using Bishop's notion of a subset $f(r)$ is a set and $i : f(r) \to y \times y$ is an embedding. If $f \circ r_1 = \mathtt{pr}_1 \circ i \circ F$ and $f \circ r_2 = \mathtt{pr}_2 \circ i \circ f$, then $f(\mathbf{r}) := (r, f \circ r_1, f \circ r_2)$ is a relation between y, y.

$$\begin{tikzcd}
 & & x \arrow[r, "f"] & y \\
a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & r \arrow[ur, "r_1"] \arrow[dr, "r_2"'] \arrow[r, "f"] & f(r) \arrow[r, hook, "i"] & y \times y \arrow[u, "\mathtt{pr}_1"'] \arrow[d, "\mathtt{pr}_2"] \\
 & & x \arrow[r, "f"'] & y
\end{tikzcd}$$

Let $g, h : a \to r$, then $f \circ r_1 \circ g = \mathtt{pr}_1 \circ i \circ f \circ g = \mathtt{pr}_1 \circ i \circ f \circ h = f \circ r_1 \circ h$, then it follows that $i \circ f \circ g = i \circ f \circ h \Rightarrow f \circ g = f \circ h$, since i is an embedding. Hence $g = h$. Similarly for $f \circ r_2 \circ g = f \circ r_2 \circ h$. Hence $f \circ r_1, f \circ r_2$ are jointly monic and $f(\mathbf{r})$ is a relation between y, y.

Let $\mathbf{s} := (s, s_1, s_2)$ be a relation on y, y. Then s is a subset of $y \times y$. We consider the pullback of the corner $< s_1, s_2 >: s \to y \times y$ and $f \times f : x \times x \to y \times y$:

$$\begin{tikzcd}
f^{-1}(s) \arrow[r, "<s'_1, s'_2>"] \arrow[d, "f"'] & x \times x \arrow[d, "f \times f"] \\
s \arrow[r, "<s_1, s_2>"'] & y \times y.
\end{tikzcd}$$

We want to show that $f^{-1}(s)$ is a relation on x, x. Again using Bishop's language, s is a set and $i : s \hookrightarrow y \times y$ is an embedding.

$$\begin{tikzcd}
 & & x \arrow[r, "f"] & y \\
a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & f^{-1}(s) \arrow[ur, "s'_1"] \arrow[dr, "s'_2"'] \arrow[r, "f"] & s \arrow[r, hook, "i"] & y \times y \arrow[u, "\mathtt{pr}_1"'] \arrow[d, "\mathtt{pr}_2"] \\
 & & x \arrow[r, "f"'] & y
\end{tikzcd}$$

If $f \circ s'_1 = \mathtt{pr}_1 \circ i \circ f$ and $f \circ s'_2 = \mathtt{pr}_2 \circ i \circ f$, then $f^{-1}(\mathbf{s}) := (f^{-1}(s), s'_1, s'_2)$ is a relation between x, x. Let $g, h : a \to f^{-1}(s)$, then $f \circ s'_1 \circ g = \mathtt{pr}_1 \circ i \circ f \circ g = \mathtt{pr}_1 \circ i \circ f \circ h = f \circ s'_1 \circ h$. It follows that $i \circ f \circ g = i \circ f \circ h \Rightarrow f \circ g = f \circ h$, since i is an embedding. Hence $g = h$. Similarly for $f \circ s'_2 \circ g = f \circ s'_2 \circ h$. Hence s'_1 and s'_2 are jointly monic and $f^{-1}(\mathbf{s})$ is a relation between x, x.

4.2 Equality relations

4.2.1 Local equality relations

After having introduced binary relations in the last section, we now define the notion of an equality relation. We describe it as a binary relation satisfying specific properties. The description of these properties requires quantification over elements of an object. In order for this elementwise description to be predicative we need to work in a locally small category, where for given objects x and y the collection of morphisms from x to y is actually a set and not a proper class. We therefore refer to *local* equality relations.

Definition 4.2.1. Let $\mathcal{C}$ be a category with 1, $x \in C_0$ and $\mathbf{r} := (r, r_1, r_2) \in \mathbf{Rel}(x, x)$. We call $\mathbf{r}$ (*locally*) *reflexive*, if for every element $e : 1 \hookrightarrow x$ of x we have that $(1, e, e)$ $\texttt{Memb}$ $\mathbf{r}$

$$\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e"] \arrow[rr, hook, "\texttt{refl}^r_e"] \arrow[dr, hook, "e"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}$$

and in this case we write $\texttt{refl}^r_e : (1, e, e)$ $\texttt{Memb}$ $\mathbf{r}$. We call $\mathbf{r}$ (*locally*) *symmetric*, if for every $(1, e_1, e_2) \in \mathbf{Rel}(x, x)$ we have that $(1, e_1, e_2)$ $\texttt{Memb}$ $\mathbf{r} \Rightarrow (1, e_2, e_1)$ $\texttt{Memb}$ $\mathbf{r}$

$$\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e_1"] \arrow[rr, hook, "m"] \arrow[dr, hook, "e_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e_2"] \arrow[rr, hook, "\texttt{sym}^r_m"] \arrow[dr, hook, "e_1"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}$$

If $\mathbf{r}$ is symmetric with $m : (1, e_1, e_2)$ $\texttt{Memb}$ $\mathbf{r}$, we write $\texttt{sym}^r_m : (1, e_2, e_1)$ $\texttt{Memb}$ $\mathbf{r}$. We call $\mathbf{r}$ (*locally*) *transitive*, if for every $(1, e_1, e_2), (1, e_2, e_3) \in \mathbf{Rel}(x, x)$ we have that

$$\big[(1, e_1, e_2) \ \texttt{Memb}\ \mathbf{r}\ \&\ (1, e_2, e_3)\ \texttt{Memb}\ \mathbf{r}\big] \Rightarrow (1, e_1, e_3)\ \texttt{Memb}\ \mathbf{r}$$

$$\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e_1"] \arrow[rr, hook, "m"] \arrow[dr, hook, "e_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e_2"] \arrow[rr, hook, "n"] \arrow[dr, hook, "e_3"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e_1"] \arrow[rr, hook, "\texttt{tr}^r_{m,n}"] \arrow[dr, hook, "e_3"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}$$

If $m : (1, e_1, e_2)$ $\texttt{Memb}$ $\mathbf{r}$ and $n : (1, e_2, e_3)$ $\texttt{Memb}$ $\mathbf{r}$, we write $\texttt{tr}^r_{m,n} : (1, e_1, e_3)$ $\texttt{Memb}$ $\mathbf{r}$.

Definition 4.2.2. If $\mathbf{eq}^x := (\mathrm{eq}^x, \mathrm{eq}^x_1, \mathrm{eq}^x_2) \in \mathbf{Rel}(x, x)$ is reflexive, symmetric and transitive, we call it a (*local*) *equality relation*, or simply an *equality*, on x. In this case, we call the pair $(x, \mathbf{eq}^x)$, or simply x, a *set-like object* in $\mathcal{C}$, or a *set* in $\mathcal{C}$.

Proposition 4.2.3. Let $\mathcal{C}$ be a category with 1, $x \in C_0$ and $\mathbf{eq}^x := (\mathrm{eq}^x, \mathrm{eq}^x_1, \mathrm{eq}^x_2)$ an equality relation between x, x.
(i) If $h : \mathbf{eq}^x \to (s, s_1, s_2)$ in $\mathbf{Rel}(x, x)$, then (s, s_1, s_2) is an equality on x, with $\texttt{refl}^s_e = h \circ \texttt{refl}^{\mathrm{eq}^x}_e$, $\texttt{sym}^s_h = h \circ \texttt{sym}^{\mathrm{eq}^x}_h$ and $\texttt{tr}^s_{m,n} = h \circ \texttt{tr}^{\mathrm{eq}^x}_{m,n}$.
(ii) If $e : a \hookrightarrow x$ is a subobject of x, and $\mathbf{eq}^a$ is an equality on a, then $e^*_0(\mathbf{eq}^a) := e \circ \mathbf{eq}^a$ is an equality on x.

Proof. (i) Let $h : \mathbf{eq}^x \to (s, s_1, s_2)$ be in $\mathbf{Rel}(x, x)$ and $e : 1 \hookrightarrow x$ be an element of x. Since $\mathbf{eq}^x$ is an equality we have that $(1, e, e)$ `Memb` $\mathbf{eq}^x$. Then the following inner diagrams commute

$$\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e"] \arrow[rr, hook, "\mathtt{refl}_e^{\mathrm{eq}^x}"] \arrow[dr, hook, "e"'] & & \mathrm{eq}^x \arrow[ul, "\mathrm{eq}_1^x"'] \arrow[dl, "\mathrm{eq}_2^x"] \\
& x &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x & \\
\mathrm{eq}^x \arrow[ur, "\mathrm{eq}_1^x"] \arrow[rr, "h"] \arrow[dr, "\mathrm{eq}_2^x"'] & & s \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\
& x &
\end{tikzcd}$$

which implies that $e = \mathtt{refl}_e^{\mathrm{eq}^x} \circ \mathrm{eq}_1^x = \mathtt{refl}_e^{\mathrm{eq}^x} \circ h \circ s_1$ and $e = \mathtt{refl}_e^{\mathrm{eq}^x} \circ \mathrm{eq}_2^x = \mathtt{refl}_e^{\mathrm{eq}^x} \circ h \circ s_2$. Hence $\mathtt{refl}_e^s : (1, e, e)$ `Memb` s and $m : \mathtt{refl}_e^s = h \circ \mathtt{refl}_e^{\mathrm{eq}^x}$. Now let $(1, e_1, e_2) \in \mathbf{Rel}(x, x)$ and $(1, e_1, e_2)$ `Memb` s. Then the following left inner diagrams commute

$$\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e_1"] \arrow[rr, hook, "m"] \arrow[dr, hook, "e_2"'] & & s \arrow[ul, "s_1"'] \arrow[dl, "s_2"] \\
& x &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e_2"] \arrow[rr, hook, "\mathtt{sym}_h^{\mathrm{eq}^x}"] \arrow[dr, hook, "e_1"'] & & \mathrm{eq}^x \arrow[ul, "\mathrm{eq}_1^x"'] \arrow[dl, "\mathrm{eq}_2^x"] \\
& x &
\end{tikzcd}$$

and since $\mathbf{eq}^x$ is symmetric, the above right inner diagrams commute. From $h : \mathbf{eq}^x \to (s, s_1, s_2)$ we get $e_2 = \mathrm{eq}_1^x \circ \mathtt{sym}_h^{\mathrm{eq}^x} = s_1 \circ h \circ \mathtt{sym}_h^{\mathrm{eq}^x}$ and $e_1 = \mathrm{eq}_2^x \circ \mathtt{sym}_h^{\mathrm{eq}^x} = s_2 \circ h \circ \mathtt{sym}_h^{\mathrm{eq}^x}$. Hence $\mathtt{sym}_h^s : (1, e_2, e_1)$ `Memb` s and $\mathtt{sym}_h^s = h \circ \mathtt{sym}_h^{eq^x}$. Let $(1, e_1, e_2), (1, e_2, e_3) \in \mathbf{Rel}(x, x)$ with

$$m : (1, e_1, e_2) \ \mathtt{Memb\ s} \ \& \ n : (1, e_2, e_3) \ \mathtt{Memb\ s}.$$

Since $\mathbf{eq}^x$ is reflexive the following inner diagrams commute

$$\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e_1"] \arrow[rr, hook, "\mathtt{tr}_{m,n}^{\mathrm{eq}^x}"] \arrow[dr, hook, "e_3"'] & & \mathrm{eq}^x \arrow[ul, "\mathrm{eq}_1^x"'] \arrow[dl, "\mathrm{eq}_2^x"] \\
& x &
\end{tikzcd}$$

and by $h : \mathbf{eq}^x \to (s, s_1, s_2)$ we get $e_1 = \mathrm{eq}_1^x \circ \mathtt{tr}_{m,n}^{\mathrm{eq}^x} = s_1 \circ h \circ \mathtt{tr}_{m,n}^{\mathrm{eq}^x}$ and $e_3 = \mathrm{eq}_2^x \circ \mathtt{tr}_{m,n}^{\mathrm{eq}^x} = s_2 \circ h \circ \mathtt{tr}_{m,n}^{\mathrm{eq}^x}$. Hence $\mathtt{tr}_{m,n}^s : (1, e_1, e_3)$ `Memb` s and $\mathtt{tr}_{m,n}^s = h \circ \mathtt{tr}_{m,n}^{eq^x}$. Therefore (s, s_1, s_2) is an equality on x.

(ii) Let $e : a \hookrightarrow x$ be a subobject of x and $\mathbf{eq}^a$ be an equality on a. $e \circ \mathbf{eq}^a = (\mathrm{eq}^a, e \circ \mathrm{eq}_1^a, e \circ \mathrm{eq}_2^a)$ is in $\mathbf{Rel}(x, x)$. Let $i : 1 \hookrightarrow a$ be an element of a, then $e \circ i : 1 \hookrightarrow x$ and $\mathtt{refl}_i^{\mathbf{eq}^a} : (1, i, i)$ `Memb` $\mathbf{eq}^a$. The following inner diagrams commute

$$\begin{tikzcd}
& & x & & \\
& a \arrow[ur, hook, "e"] & & a \arrow[ul, hook', "e"'] & \\
1 \arrow[ur, hook, "i"] \arrow[rrrr, hook, "\mathtt{refl}_i^{\mathrm{eq}^a}"] \arrow[dr, hook, "i"'] & & & & \mathrm{eq}^a \arrow[ul, "\mathrm{eq}_1^a"'] \arrow[dl, "\mathrm{eq}_2^a"] \\
& a \arrow[dr, hook, "e"'] & & a \arrow[dl, hook', "e"] & \\
& & x & &
\end{tikzcd}$$

since $e\circ i = e\circ \mathrm{eq}_1^a \circ \mathtt{refl}_i^{\mathrm{eq}^a}$ and $e\circ i = e\circ \mathrm{eq}_2^a \circ \mathtt{refl}_i^{\mathrm{eq}^a}$. Hence $(1, e\circ i, e\circ i)$ $\mathtt{Memb}$ $e\circ \mathbf{eq}^a$, so $e\circ \mathbf{eq}^a$ is reflexive. Let $(1, e_1, e_2) \in \mathbf{Rel}(a, a)$ with $(1, e_1, e_2)$ $\mathtt{Memb}$ $\mathbf{eq}^a$, then $(1, e\circ e_1, e\circ e_2) \in \mathbf{Rel}(x, x)$ and $m : (1, e\circ e_1, e\circ e_2)$ $\mathtt{Memb}$ $e\circ \mathbf{eq}^a$. The commutativity of the following inner diagrams follows from the symmetry of $\mathbf{eq}^a$,

$$\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e\circ e_2"] \arrow[rr, hook, "\mathtt{sym}_m^{\mathrm{eq}^a}"] \arrow[dr, hook, "e\circ e_1"'] & & \mathrm{eq}^a \arrow[ul, "e\circ \mathrm{eq}_1^a"'] \arrow[dl, "e\circ \mathrm{eq}_2^a"] \\
& x &
\end{tikzcd}$$

$e\circ e_2 = e\circ \mathrm{eq}_1^a \circ \mathtt{sym}_m^{\mathrm{eq}^a}$ and $e\circ e_1 = e\circ \mathrm{eq}_2^a \circ \mathtt{sym}_m^{\mathrm{eq}^a}$. Hence $(1, e\circ e_2, e\circ e_1)$ $\mathtt{Memb}$ $e\circ \mathbf{eq}^a$, so $e\circ \mathbf{eq}^a$ is symmetric. Moreover, let $(1, e_2, e_3) \in \mathbf{Rel}(a, a)$ with $(1, e_2, e_3)$ $\mathtt{Memb}$ $\mathbf{eq}^a$, then $(1, e\circ e_2, e\circ e_3) \in \mathbf{Rel}(x, x)$ and $n : (1, e\circ e_2, e\circ e_3)$ $\mathtt{Memb}$ $\mathbf{eq}^a$. The commutativity of the following inner diagrams follows from the transitivity of $\mathbf{eq}^a$,

$$\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e\circ e_1"] \arrow[rr, hook, "\mathtt{tr}_{m,n}^{\mathrm{eq}^a}"] \arrow[dr, hook, "e\circ e_3"'] & & \mathrm{eq}^a \arrow[ul, "e\circ \mathrm{eq}_1^a"'] \arrow[dl, "e\circ \mathrm{eq}_2^a"] \\
& x &
\end{tikzcd}$$

$e\circ e_1 = e\circ \mathrm{eq}_1^a \circ \mathtt{tr}_{m,n}^{\mathrm{eq}^a}$ and $e\circ e_3 = e\circ \mathrm{eq}_2^a \circ \mathtt{tr}_{m,n}^{\mathrm{eq}^a}$. Hence $(1, e\circ e_1, e\circ e_3)$ $\mathtt{Memb}$ $e\circ \mathbf{eq}^a$, so $e\circ \mathbf{eq}^a$ is transitive. Therefore $e\circ \mathbf{eq}^a$ is an equality on x. $\square$

Remark 4.2.4. Case (ii) above is the categrocial formulation of the equivalence $a =_A a' \Leftrightarrow i_A^X(a) =_X i_A^X(a')$ in the case of a subset (A, i_A^X) of a Bishop set X.

Every arrow $f : x \to y$ in a category $\mathcal{C}$ with 1 is an *operation-arrow*, or an *assignement-routine-arrow*, as it sends elements of x to elements of y. Namely, if $f : x \to y$ is an arrow in $\mathcal{C}$ and $e : 1 \hookrightarrow x$ is an element of x, the *application* $f(e)$ of f on e is an element $f\circ e : 1 \hookrightarrow y$ of y

$$\begin{tikzcd}
1 \arrow[r, hook, "e"] \arrow[rr, hook, bend right, "f(e)"'] & x \arrow[r, "f"] & y
\end{tikzcd}$$

Moreover, if $\boldsymbol{i} := (1, i_1, i_2) \in \mathbf{Rel}(x, x)$, then $f(\boldsymbol{i}) := \big(1, f(i_1), f(i_2)\big) \in \mathbf{Rel}(y, y)$, as the arrows $f(i_1), f(i_2)$ are both monic

$$\begin{tikzcd}
& x \arrow[r, "f"] & y \\
1 \arrow[ur, hook, "i_1"] \arrow[urr, hook, bend right, "f(i_1)"] \arrow[dr, hook, "i_2"'] \arrow[drr, hook, bend left, "f(i_2)"'] & & \\
& x \arrow[r, "f"'] & y
\end{tikzcd}$$

Next we define when $f : x \to y$ behaves like a function, with respect to given equalities x and y.

Definition 4.2.5. Let $\mathcal{C}$ be a category with 1, and let $(x, \mathbf{eq}^x)$ and $(y, \mathbf{eq}^y)$ be sets in $\mathcal{C}$. An arrow

$f : x \to y$ in $\mathcal{C}$ is *function-like* (*with respect to* $\mathbf{eq}^x$ *and* $\mathbf{eq}^y$), or simply a *function*, if for every $\boldsymbol{i}:= (1, i_1, i_2) \in \mathbf{Rel}(x,x)$ we have that

$$\boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x \Rightarrow f(\boldsymbol{i}) \ \texttt{Memb} \ \mathbf{eq}^y$$

$$\begin{tikzcd}
& x \arrow[rr, "f"] & & y & \\
\mathrm{eq}^x \arrow[ur, "\mathrm{eq}_1^x"] \arrow[dr, "\mathrm{eq}_2^x"'] & & 1 \arrow[ll, hook', "m"'] \arrow[ul, "i_1"'] \arrow[dl, "i_2"] \arrow[ur, "f(i_1)"] \arrow[dr, "f(i_2)"'] \arrow[rr, hook, "f(m)"] & & \mathrm{eq}^y \arrow[ul, "\mathrm{eq}_1^y"'] \arrow[dl, "\mathrm{eq}_2^y"] \\
& x \arrow[rr, "f"'] & & y &
\end{tikzcd}$$

In this case, if $m : \boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x$, we write $f(m) : f(\boldsymbol{i}) \ \texttt{Memb} \ \mathbf{eq}^y$. If for every $\boldsymbol{i} \in \mathbf{Rel}(x,x)$ the converse inclusion $f(\boldsymbol{i}) \ \texttt{Memb} \ \mathbf{eq}^y \Rightarrow \boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x$ is satisfied, then we call f an *embedding-arrow*, or simply an *embedding* (*with respect to* $\mathbf{eq}^y$ *and* $\mathbf{eq}^x$).

Remark 4.2.6. Let $\mathcal{C}$ be a category with 1, and let $(x, \mathbf{eq}^x), (y, \mathbf{eq}^y), (z, \mathbf{eq}^z)$ be sets in $\mathcal{C}$.
(i) 1_x is function-like, where if $m : \boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x$, then $1_x(m) = m : 1_x(\boldsymbol{i}) \ \texttt{Memb} \ \mathbf{eq}^x$.
(ii) If $f : x \to y$ and $g : y \to z$ are function-like, then $g \circ f$ is function-like, where if $m : \boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x$ and $f(m) : f(\boldsymbol{i}) \ \texttt{Memb} \ \mathbf{eq}^y$, then $(g \circ f)(\boldsymbol{i}) = g(f(\boldsymbol{i})) \ \texttt{Memb} \ \mathbf{eq}^z$.
(iii) If $f : x \to y$ is monic, i.e. x is a subobject of y, and if $\mathbf{eq}^y = f \circ \mathbf{eq}^x$, then f is an embedding.

Proof. (i) Let $1_x : x \to x$ and $\boldsymbol{i} \in \mathbf{Rel}(x,x)$. Let $m : \boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x$, then $1_x(m) = m : 1_x(\boldsymbol{i}) \ \texttt{Memb}$ $\mathbf{eq}^x$. Hence, by definition, 1_x is function-like.
(ii) Let $f : x \to y$ and $g : y \to z$ be function-like. Let $m : \boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x$, then, since f is function-like, $f(m) : f(\boldsymbol{i}) \ \texttt{Memb} \ \mathbf{eq}^y$. Since g is function-like, we have $(g \circ f)(\boldsymbol{i}) = g(f(\boldsymbol{i})) \ \texttt{Memb} \ \mathbf{eq}^z$. Hence $g \circ f$ is function-like.
(iii) Let $l : f(\boldsymbol{i}) \ \texttt{Memb} \ f \circ \mathbf{eq}^x$,

$$\begin{tikzcd}
& x \arrow[rr, "f"] & & y & x \arrow[l, "f"'] & \\
\mathrm{eq}^x \arrow[ur, "\mathrm{eq}_1^x"] \arrow[dr, "\mathrm{eq}_2^x"'] & & 1 \arrow[ll, hook', "l"'] \arrow[ul, "i_1"'] \arrow[dl, "i_2"] \arrow[ur, "f(i_1)"] \arrow[dr, "f(i_2)"'] \arrow[rrr, hook, "l"] & & & \mathrm{eq}^x \arrow[ul, "\mathrm{eq}_1^x"'] \arrow[dl, "\mathrm{eq}_2^x"] \\
& x \arrow[rr, "f"'] & & y & x \arrow[l, "f"] &
\end{tikzcd}$$

then we have $f \circ (\mathrm{eq}_1^x \circ l) = f(i_1) = f \circ i_1$ and $f \circ (\mathrm{eq}_2^x \circ l) = f(i_2) = f \circ i_2$. Since f is monic, $\mathrm{eq}_1^x \circ l = i_1$ and $\mathrm{eq}_2^x \circ l = i_2$. Hence $l : \boldsymbol{i} \ \texttt{Memb} \ \mathbf{eq}^x$, so f is an embedding. $\square$

4.2.2 Global equality relations

In the previous section we restricted our definition of an equality relation to categories that admit a terminal object 1. We now want to consider more general categories that do not necessarily have a terminal object. Hence we will give a definition of a global equality relation without mentioning elements of an object.

First we define some notions that we will need in our definition. Let $\mathcal{C}$ be a category with products and $X \in C_0$. We call $\boldsymbol{\delta}_X : (X, 1_X, 1_X) \in \mathbf{Rel}(X,X)$ the *identity relation*, where 1_X :

$X \to X$ is the identity morphism. This relation is the subset $\Delta := \{(x,x) \mid x \in X\} \subseteq X \times X$ called the *diagonal* of X.

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[d] \arrow[dr, "1_X"] & \\ X & X \times X & X \end{tikzcd}$$

A relation $\mathbf{r} := (R, r_1, r_2)$ between X, X is a subset $\{(x,y) \mid x, y \in X\} \subseteq X \times X$. We call $\mathbf{r}^{\mathrm{op}} := (R, r_2, r_1)$ the *opposite relation* of $\mathbf{r}$, given by the subset $\{(y,x) \mid (x,y) \in \mathbf{r}\} \subseteq X \times X$.

Definition 4.2.7. Let $\mathcal{C}$ be a category with pullbacks and limits and $\mathbf{r} := (r, r_1, r_2) \in \mathbf{Rel}(x,x)$. We call $\mathbf{r}$ (*globally*) *reflexive*, if $\boldsymbol{\delta}_x \leqslant \mathbf{r}$

$$\begin{tikzcd} & x & \\ x \arrow[ur, "1_x"] \arrow[dr, "1_x"'] \arrow[rr, "\mathtt{refl}"] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\ & x & \end{tikzcd}$$

and in this case we write $\mathtt{refl} : \boldsymbol{\delta}_x \leqslant \mathbf{r}$. We call $\mathbf{r}$ (*globally*) *symmetric*, if $\mathbf{r} \leqslant \mathbf{r}^{\mathrm{op}}$

$$\begin{tikzcd} & x & \\ r \arrow[ur, "r_2"] \arrow[dr, "r_1"'] \arrow[rr, "\mathtt{sym}"] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\ & x & \end{tikzcd}$$

and we write $\mathtt{sym} : \mathbf{r} \leqslant \mathbf{r}^{\mathrm{op}}$. $\mathbf{r}$ is called (*globally*) *transitive*, if $\mathbf{r} \circ \mathbf{r} \leqslant \mathbf{r}$

$$\begin{tikzcd} & x & \\ r \circ r \arrow[ur, "r \circ r_1"] \arrow[dr, "r \circ r_2"'] \arrow[rr, "\mathtt{tr}"] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\ & x & \end{tikzcd}$$

and in this case we write $\mathtt{tr} : \mathbf{r} \circ \mathbf{r} \leqslant \mathbf{r}$.

Definition 4.2.8. If $\mathbf{eq}^x := (\mathrm{eq}^x, \mathrm{eq}^x_1, \mathrm{eq}^x_2) \in \mathbf{Rel}(x,x)$ is reflexive, symmetric and transitive, we call it a (*global*) *equality relation*, or simply an *equality* on x. We call the pair $(x, \mathbf{eq}^x)$ a *Bishop set object*.

Proposition 4.2.9. Let $\mathbf{s}, \mathbf{r} \in \mathbf{Rel}(x,x)$. If $h : \mathbf{s} \leqslant \mathbf{r}$, then for every $i = (1, i_1, i_2) \in \mathbf{Rel}(x,x)$

$$i \ \mathtt{Memb}\ \mathbf{s} \Rightarrow i \ \mathtt{Memb}\ \mathbf{r}$$

Proof. Let $h : \mathbf{s} \leqslant \mathbf{r}$ and $m : i$ `Memb` $\mathbf{s}$, then the following right and left inner diagrams commute

$$\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "i_1"] \arrow[r, hook, "m"] \arrow[dr, hook, "i_2"'] & s \arrow[u, "s_1"] \arrow[d, "s_2"] \arrow[r, "h"] & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}$$

and it holds $i_1 = m \circ s_1 = m \circ (h \circ r_1) = (m \circ h) \circ r_1$ and $i_2 = m \circ s_2 = m \circ (h \circ r_2) = (m \circ h) \circ r_2$. Hence the above outer diagrams commute and i `Memb` $\mathbf{r}$. □

We will use this proposition in the proof of the following theorem.

Theorem 4.2.10. Let $\mathcal{C}$ be a category with pullbacks, limits and a terminal object 1 and let x be an object in $\mathcal{C}$. If $\mathbf{r}$ is a global equality relation on x, then $\mathbf{r}$ is a local equality relation on x.

Proof. We show the properties of a local equality relation. First we show that $\mathbf{r}$ is locally reflexive. Since $\mathbf{r}$ is globally reflexive, it holds that $\delta_x \leqslant \mathbf{r}$. Now let $e : 1 \hookrightarrow x$ be an element of x, then

$$\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e"] \arrow[r, hook, "e"] \arrow[dr, hook, "e"'] & x \arrow[u, "1_x"] \arrow[d, "1_x"] \arrow[r, "\texttt{refl}"] & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}$$

$e = e \circ 1_x = e \circ (\texttt{refl} \circ r_1) = (e \circ \texttt{refl}) \circ r_1$ and $e = e \circ 1_x = e \circ (\texttt{refl} \circ r_2) = (e \circ \texttt{refl}) \circ r_2$. Therefore the above outer diagrams commute and $(1, e, e)$ `Memb` $\mathbf{r}$, hence $\mathbf{r}$ is locally reflexive. Next we show local symmetry. Since $\mathbf{r}$ is globally symmetric, $\mathbf{r}^{\mathrm{op}} \leqslant \mathbf{r}$. Let $(1, e_1, e_2) \in \mathbf{Rel}(x, x)$ with $h : (1, e_1, e_2)$ `Memb` $\mathbf{r}$,

$$\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e_1"] \arrow[r, hook, "h"] \arrow[dr, hook, "e_2"'] & r \arrow[u, "r_2"] \arrow[d, "r_1"] \arrow[r, "\texttt{sym}"] & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}$$

then it holds $e_1 = h \circ \texttt{sym} \circ r_1 = h \circ r_2$ and $e_2 = h \circ \texttt{sym} \circ r_2 = h \circ r_1$. Hence $(1, e_2, e_1)$ `Memb` $\mathbf{r}$, so $\mathbf{r}$ is locally symmetric. Now let $(1, e_1, e_2), (1, e_2, e_3) \in \mathbf{Rel}(x, x)$ with

$$m : (1, e_1, e_2) \ \texttt{Memb}\ \mathbf{r} \ \& \ n : (1, e_2, e_3) \ \texttt{Memb}\ \mathbf{r}.$$

The following inner diagrams commute

$$\begin{tikzcd}
 & x & \\
1 \arrow[ur, "e_1"] \arrow[rr, "m"] \arrow[dr, "e_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x & \\
1 \arrow[ur, "e_2"] \arrow[rr, "n"] \arrow[dr, "e_3"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
 & x &
\end{tikzcd}$$

To prove local transitivity we first show $(1, e_1, e_3)$ `Memb` $\mathbf{r} \circ \mathbf{r}$. Since $h : \mathbf{r} \circ \mathbf{r} \leqslant \mathbf{r}$ the following left inner diagrams commute.

$$\begin{tikzcd}
& x & \\
r \circ r \arrow[ur, "(r\circ r)_1"] \arrow[rr] \arrow[dr, "(r\circ r)_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
& x &
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "q"'] & r \arrow[r, "r_2"] \arrow[d, "r_1"] & {} \\
r \arrow[r, "r_2"'] \arrow[d, "r_1"'] & x & \\
{} & &
\end{tikzcd}$$

We take the pullback P of the two morphisms $r_1 : r \to x$ and $r_2 : r \to x$, as shown in the right diagram above. By the universal property of the pullback, since $r_1 \circ n = e_2 = m \circ r_2$, there is a unique arrow $u : 1 \to P$ such that $q \circ u = n$ and $p \circ u = m$. Moreover, there is an arrow from the pullback P to the product $x \times x$, which can be factorized through an epimorphism $e : P \to r \circ r$ and a monomorphism $m : r \circ r \to x \times x$, shown in the following diagram:

$$\begin{tikzcd}
& & 1 \arrow[d, hook, "u"] \arrow[dl, hook, bend right, "m"'] \arrow[dr, hook, bend left, "n"] \arrow[ddll, hook, bend right=50, "e_1"'] \arrow[ddrr, hook, bend left=50, "e_3"] & & \\
& & P \arrow[dl, "p"'] \arrow[d, "e"] \arrow[dr, "q"] & & \\
& r \arrow[dl, "r_2"'] & r \circ r \arrow[l, "h"'] \arrow[r, "h"] \arrow[d, "m"] \arrow[dll, bend left, "(r\circ r)_1"] \arrow[drr, bend right, "(r\circ r)_2"'] & r \arrow[dr, "r_1"] & \\
x & & x \times x \arrow[ll, "\mathtt{pr}_1"] \arrow[rr, "\mathtt{pr}_2"'] & & x
\end{tikzcd}$$

Hence we have an arrow $e \circ u : 1 \to r \circ r$ such that $e_1 = (r \circ r)_1 \circ e \circ u$ and $e_3 = (r \circ r)_2 \circ e \circ u$, meaning that the following left inner diagrams commute,

$$\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e_1"] \arrow[r, hook, "e\circ u"] \arrow[dr, hook, "e_3"'] & r \circ r \arrow[u, "(r\circ r)_1"] \arrow[r, "h"] \arrow[d, "(r\circ r)_2"'] & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
& x &
\end{tikzcd}$$

so it holds that $(1, e_1, e_3)$ `Memb` $\mathbf{r} \circ \mathbf{r}$. From Prop. 4.2.9 it follows that $(1, e_1, e_3)$ `Memb` $\mathbf{r}$. Hence $\mathbf{r}$ is locally transitive and therefore $\mathbf{r}$ is a local equality relation. $\square$

Remark 4.2.11. The converse does not generally hold. Suppose for every $i := (1, i_1, i_2) \in \mathbf{Rel}(x, x)$ it holds that i `Memb` s & i `Memb` $r \Rightarrow s \leqslant r$. Let $\mathbf{r}$ be a local equality relation on x. Then $\mathbf{r}$ is locally reflexive, i.e. for every $e : 1 \hookrightarrow x$, it holds that `refl` $: (1, e, e)$ `Memb` $\mathbf{r}$. Let $1_x : x \to x$

be the identity on x, then the following inner diagrams commute

$$
\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e"] \arrow[rr, hook, "e"] \arrow[dr, hook, "e"'] & & x \arrow[ul, "1_x"'] \arrow[dl, "1_x"] \\
 & x &
\end{tikzcd}
$$

By our assumption $\boldsymbol{\delta}_x \leqslant \mathbf{r}$. As $\mathbf{r}$ is locally symmetric, it holds for every $(1, e_1, e_2) \in \mathbf{Rel}(x, x)$ that $m : (1, e_1, e_2)$ `Memb` $\mathbf{r} \Rightarrow$ `sym` $: (1, e_2, e_1)$ `Memb` $\mathbf{r}$. Hence the following inner diagrams commute

$$
\begin{tikzcd}
 & x & \\
1 \arrow[ur, hook, "e_1"] \arrow[rr, hook, "\mathtt{sym}"] \arrow[dr, hook, "e_2"'] & & x \arrow[ul, "r_2"'] \arrow[dl, "r_1"] \\
 & x &
\end{tikzcd}
$$

and it holds $(1, e_1, e_2)$ `Memb` $\mathbf{r}^{\mathrm{op}}$. Again by our assumption we have $\mathbf{r}^{\mathrm{op}} \leqslant \mathbf{r}$. By now we showed global reflexiveness and global symmetry. However our assumption is not sufficient to prove global transitivity. It remains an open problem to find a good condition on $\mathcal{C}$, to prove local equality $\Rightarrow$ global equality.

The next proposition was originally stated on p. 103 of [10]. Here we give a full proof of the proposition with respect to our definition of a global equality.

Proposition 4.2.12. In any category with finite limits, the *kernel pair* of a morphism $f : x \to y$ is the pullback of f along itself:

$$
\begin{tikzcd}
r \arrow[r, "r_1"] \arrow[d, "r_2"'] & x \arrow[d, "f"] \\
x \arrow[r, "f"'] & y
\end{tikzcd}
$$

These maps r_1 and r_2 define a monomorphism $(r_1, r_2) : r \hookrightarrow x \times x$, so the object r is always a subobject of the product $x \times x$. Indeed, subobjects defined by kernel pairs are always global equivalence relations.

Proof. Let r be a subobject of $x \times x$ defined by the kernel pair of $f : x \to y$ with $(r_1, r_2) : r \hookrightarrow x \times x$. First, let $1_x : x \to x$ be the identity morphism, then it holds that $f \circ 1_x = f \circ 1_x$. By the universal property of the pullback there is a unique morphism $\mathtt{refl} : x \to r$

$$
\begin{tikzcd}
x \arrow[dr, "\mathtt{refl}"] \arrow[drr, bend left, "1_x"] \arrow[ddr, bend right, "1_x"'] & & \\
 & r \arrow[r, "r_1"] \arrow[d, "r_2"'] & x \arrow[d, "f"] \\
 & x \arrow[r, "f"'] & y
\end{tikzcd}
$$

such that $1_x = r_1 \circ \mathtt{refl}$ and $1_x = r_2 \circ \mathtt{refl}$. Hence the following inner diagrams commute

$$\begin{tikzcd}
& x & \\
x \arrow[ur, "1_x"] \arrow[rr, "\mathtt{refl}"] \arrow[dr, "1_x"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
& x &
\end{tikzcd}$$

and $\boldsymbol{\delta}_x = (x, 1_x, 1_x) \leqslant \mathbf{r}$. Next we want to prove symmetry. Since $f \circ r_1 = f \circ r_2$, again by the universal property of the pullback there is a unique morphism $\mathtt{sym} : r \to r$

$$\begin{tikzcd}
r \arrow[dr, "sym"] \arrow[drr, bend left, "r_2"] \arrow[ddr, bend right, "r_1"'] & & \\
& r \arrow[r, "r_1"] \arrow[d, "r_2"'] & x \arrow[d, "f"] \\
& x \arrow[r, "f"'] & y
\end{tikzcd}$$

such that $r_2 = r_1 \circ \mathtt{sym}$ and $r_1 = r_2 \circ \mathtt{sym}$. Hence the following inner diagrams commute

$$\begin{tikzcd}
& x & \\
r \arrow[ur, "r_2"] \arrow[rr, "\mathtt{sym}"] \arrow[dr, "r_1"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\
& x &
\end{tikzcd}$$

and it holds $\mathbf{r}^{\mathrm{op}} = (r, r_2, r_1) \leqslant \mathbf{r}$. Therefore $\mathbf{r}$ is globally symmetric. The pullback of r_1 and r_2 is the fiber product $r \times_x r$

$$\begin{tikzcd}
r \times_x r \arrow[r, "r_1'"] \arrow[d, "r_2'"'] & r \arrow[r, "r_2"] \arrow[d, "r_1"] & x \\
r \arrow[r, "r_2"'] \arrow[d, "r_1"'] & x & \\
x & &
\end{tikzcd}$$

such that $r_1 \circ r_1' = r_2 \circ r_2'$. Then $r_1 \circ r_1', r_2 \circ r_2' : r \times_x r \to x$ are two morphisms with $f \circ (r_1 \circ r_1') = f \circ (r_2 \circ r_2')$. Hence there is a unique morphism $u : r \times_x r \to r$

$$\begin{tikzcd}
r \times_x r \arrow[dr, "u"] \arrow[drr, bend left, "r_1 \circ r_1'"] \arrow[ddr, bend right, "r_2 \circ r_2'"'] & & \\
& r \arrow[r, "r_1"] \arrow[d, "r_2"'] & x \arrow[d, "f"] \\
& x \arrow[r, "f"'] & y
\end{tikzcd}$$

such that $r_1 \circ u = r_1 \circ r_1'$ and $r_2 \circ u = r_2 \circ r_2'$. Let $\mathbf{r} \circ \mathbf{r} = \big(r \circ r, (r \circ r)_1, (r \circ r)_2\big) \in \mathbf{Rel}(x, x)$. As $r \times_x r \subseteq x \times x$, there is a morphism from $r \times_x r$ to $x \times x$, which can be factorized by an epimorphism

$e : r \times_x r \to r \circ r$ and a monomorphism $m : r \circ r \to x \times x$.

$$\begin{tikzcd} & r \times_x r \arrow[d, "e"] \arrow[ddl, bend right, "r_1 \circ r_1'"'] \arrow[ddr, bend left, "r_2 \circ r_2'"] & \\ & r \circ r \arrow[d, hook, "m"] \arrow[dl, bend right, "(r\circ r)_1"] \arrow[dr, bend left, "(r\circ r)_2"] & \\ x & x \times x \arrow[l, "\mathtt{pr}_1"] \arrow[r, "\mathtt{pr}_2"'] & x \end{tikzcd}$$

As $r_1 \circ r_1' = r_2 \circ r_2'$, it holds that $\mathtt{pr}_1 \circ m \circ e = r_1 \circ r_1' = r_2 \circ r_2' = \mathtt{pr}_2 \circ m \circ e$. Since e is an epimorphism we have $(r \circ r)_1 = \mathtt{pr}_1 \circ m = \mathtt{pr}_2 \circ m = (r \circ r)_2$. Hence it holds $f \circ (r \circ r)_1 = f \circ (r \circ r)_2$, so there is a unique morphism $\mathtt{tr} : r \circ r \to r$

$$\begin{tikzcd} r \circ r \arrow[dr, "\mathtt{tr}"] \arrow[drr, bend left, "(r\circ r)_1"] \arrow[ddr, bend right, "(r\circ r)_2"'] & & \\ & r \arrow[r, "r_1"] \arrow[d, "r_2"'] & x \arrow[d, "f"] \\ & x \arrow[r, "f"'] & y \end{tikzcd}$$

such that $(r \circ r)_1 = r_1 \circ \mathtt{tr}$ and $(r \circ r)_2 = r_2 \circ \mathtt{tr}$. Therefore the following inner diagrams commute

$$\begin{tikzcd} & x & \\ r \circ r \arrow[ur, "(r\circ r)_1"] \arrow[rr, "\mathtt{tr}"] \arrow[dr, "(r\circ r)_2"'] & & r \arrow[ul, "r_1"'] \arrow[dl, "r_2"] \\ & x & \end{tikzcd}$$

and it holds $\mathbf{r} \circ \mathbf{r} \leqslant \mathbf{r}$, so $\mathbf{r}$ is globally transitive. We finally get that $\mathbf{r}$ is a global equality relation. □

4.3 Apartness relations

So far we have seen the definition of an equality in the categorical sense. We now also want to translate the notion of an inequality into categorical terms. For a given Bishop set object $(x, \mathbf{eq}^x)$ we will define an apartness relation $\mathbf{ineq}^x$.

4.3.1 Local apartness relations

As in the section about local equality relations, we give an elementwise description of an apartness relation, so we work in a locally small category in order to ensure that our definition is predicative.

Definition 4.3.1. Let $\mathcal{C}$ be a category with 1, $x \in C_0$ and $\mathbf{eq}^x := (\mathrm{eq}^x, \mathrm{eq}_1^x, \mathrm{eq}_2^x)$ be an equality relation on x. Let $\mathbf{ineq}^x := (\mathrm{ineq}^x, \mathrm{ineq}_1^x, \mathrm{ineq}_2^x) \in \mathbf{Rel}(x, x)$, then we call $\mathbf{ineq}^x$ an *apartness relation to* $(x, \mathbf{eq}^x)$, if it is *irreflexive*, i.e. for every $\boldsymbol{i} := (1, e_1, e_2) \in \mathbf{Rel}(x, x)$ we have that

$$\big(\boldsymbol{i} \ \mathtt{Memb} \ \mathbf{eq}^x \ \ \& \ \ \boldsymbol{i} \ \mathtt{Memb} \ \mathbf{ineq}^x\big) \Rightarrow \bot,$$

symmetric, i.e. for every $(1, e_1, e_2) \in \mathbf{Rel}(x, x)$ we have that

$$(1, e_1, e_2) \ \mathtt{Memb} \ \mathbf{ineq}^x \Rightarrow (1, e_2, e_1) \ \mathtt{Memb} \ \mathbf{ineq}^x,$$

$$
\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e_1"] \arrow[dr, hook, "e_2"'] \arrow[rr, hook, "m"] & & \mathrm{ineq}^x \arrow[ul, "\mathrm{ineq}^x_1"'] \arrow[dl, "\mathrm{ineq}^x_2"] \\
& x &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e_2"] \arrow[dr, hook, "e_1"'] \arrow[rr, hook, "\mathtt{sym}_m"] & & \mathrm{ineq}^x \arrow[ul, "\mathrm{ineq}^x_1"'] \arrow[dl, "\mathrm{ineq}^x_2"] \\
& x &
\end{tikzcd}
$$

and *cotransitive*, i.e. for every element $e : 1 \hookrightarrow x$ of x we have that

$$\boldsymbol{i} \;\texttt{Memb}\; \mathbf{ineq}^x \;\Rightarrow\; \big((1, e, e_1) \;\texttt{Memb}\; \mathbf{ineq}^x \;\vee\; (1, e, e_2) \;\texttt{Memb}\; \mathbf{ineq}^x\big).$$

$$
\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e_1"] \arrow[dr, hook, "e_2"'] \arrow[rr, hook, "m"] & & \mathrm{ineq}^x \arrow[ul, "\mathrm{ineq}^x_1"'] \arrow[dl, "\mathrm{ineq}^x_2"] \\
& x &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e"] \arrow[dr, hook, "e_1"'] \arrow[rr, hook, "\mathtt{cotr}_m"] & & \mathrm{ineq}^x \arrow[ul, "\mathrm{ineq}^x_1"'] \arrow[dl, "\mathrm{ineq}^x_2"] \\
& x &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x & \\
1 \arrow[ur, hook, "e"] \arrow[dr, hook, "e_2"'] \arrow[rr, hook, "\mathtt{cotr}_m"] & & \mathrm{ineq}^x \arrow[ul, "\mathrm{ineq}^x_1"'] \arrow[dl, "\mathrm{ineq}^x_2"] \\
& x &
\end{tikzcd}
$$

Definition 4.3.2. We call $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$ a *full Bishop object.*

We give an example of an apartness relation for a given equality relation.

Example 4.3.3. Let **Set** be the category of sets and functions. Let $X := \{1, 2, 3\}$ be an object of **Set** with the given equality relation

$$R = \{(1, 1), (2, 2), (3, 3)\} \subseteq X \times X.$$

R is obviously an equality relation as it is reflexive, symmetric and transitive. Then the apartness relation to (X, R) is the following set

$$S = \{(1, 2), (1, 3), (2, 1), (3, 1), (2, 3), (3, 2)\}.$$

It holds that $R \cap S = \emptyset$ and $R \cup S = X \times X$.

We now give a definition of when $f : x \to y$ behaves like a function, with respect to given inequalities on x and y.

Definition 4.3.4. Let $\mathcal{C}$ be a category with 1, and let $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$ and $(y, \mathbf{eq}^y, \mathbf{ineq}^y)$ be full Bishop objects in $\mathcal{C}$. An arrow $f : x \to y$ in $\mathcal{C}$ is *function-like* (*with respect to* $\mathbf{ineq}^x$ *and* $\mathbf{ineq}^y$), or simply a *function*, if for every $\boldsymbol{i} := (1, e_1, e_2) \in \mathbf{Rel}(x, x)$ we have that

$$\boldsymbol{i} \;\texttt{Memb}\; \mathbf{ineq}^x \Rightarrow f(\boldsymbol{i}) \;\texttt{Memb}\; \mathbf{ineq}^y$$

$$
\begin{tikzcd}
& x \arrow[rr, "f"] & & y & \\
\mathrm{ineq}^x \arrow[ur, "\mathrm{ineq}^x_1"] \arrow[dr, "\mathrm{ineq}^x_2"'] & & 1 \arrow[ul, "e_1"'] \arrow[dl, "e_2"] \arrow[ll, hook', "n"'] \arrow[ur, "f(e_1)"] \arrow[dr, "f(e_2)"'] \arrow[rr, hook, "f(n)"] & & \mathrm{ineq}^y \arrow[ul, "\mathrm{ineq}^y_1"'] \arrow[dl, "\mathrm{ineq}^y_2"] \\
& x \arrow[rr, "f"'] & & y &
\end{tikzcd}
$$

In this case, if $n : \boldsymbol{i} \;\texttt{Memb}\; \mathbf{ineq}^x$, we write $f(n) : f(\boldsymbol{i}) \;\texttt{Memb}\; \mathbf{ineq}^y$. An arrow $f : x \to y$ in $\mathcal{C}$ is called *strongly extensional*, if for every $\boldsymbol{i} := (1, e_1, e_2) \in \mathbf{Rel}(x, x)$ we have that

$$f(\boldsymbol{i}) \ \texttt{Memb}\ \mathbf{ineq}^y \Rightarrow \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x$$

Proposition 4.3.5. Let $\mathcal{C}$ be a category with 1, and let $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$, $(y, \mathbf{eq}^y, \mathbf{ineq}^y)$ and $(z, \mathbf{eq}^z, \mathbf{ineq}^z)$ be full Bishop objects in $\mathcal{C}$.
(i) 1_x is function-like, where if $n : \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x$, then $1_x(n) = n : 1_x(\boldsymbol{i})\ \texttt{Memb}\ \mathbf{ineq}^x$.
(ii) If $f : x \to y$ and $g : y \to z$ are function-like, then $g \circ f$ is function-like, where if $n : \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x$ and $f(n) : f(\boldsymbol{i})\ \texttt{Memb}\ \mathbf{ineq}^y$, then $(g \circ f)(\boldsymbol{i}) = g(f(\boldsymbol{i}))\ \texttt{Memb}\ \mathbf{ineq}^z$.
(iii) 1_x is strongly extensional.
(iv) If $f : x \to y$ and $g : y \to z$ are strongly extensional, then $g \circ f$ is strongly extensional.
(v) If $f : x \to y$ is monic and if $\mathbf{ineq}^y = f \circ \mathbf{ineq}^x$, then f is strongly extensional.

Proof. (i) Let $1_x : x \to x$ and $\boldsymbol{i} \in \mathbf{Rel}(x, x)$. Let $n : \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x$, then $1_x(n) = n : 1_x(\boldsymbol{i})\ \texttt{Memb}\ \mathbf{ineq}^x$. Hence, by definition, 1_x is function-like.
(ii) Let $f : x \to y$ and $g : y \to z$ be function-like. Let $n : \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x$, then, since f is function-like, $f(n) : f(\boldsymbol{i})\ \texttt{Memb}\ \mathbf{eq}^y$. Since g is function-like, we have $g(f(n)) : g(f(\boldsymbol{i}))\ \texttt{Memb}\ \mathbf{eq}^z$. Hence $g \circ f$ is function-like.
(iii) Let $\boldsymbol{i} := (1, e_1, e_2) \in \mathbf{Rel}(x, x)$ and let $1_x(n) : 1_x(\boldsymbol{i})\ \texttt{Memb}\ \mathbf{ineq}^x$. Since $1_x(n) = n$ and $1_x(\boldsymbol{i}) = \boldsymbol{i}$, we have $n : \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x$.
(iv) Let $f : x \to y$ and $g : y \to z$ be strongly extensional. Let $\boldsymbol{i} \in \mathbf{Rel}(x, x)$, then $f(\boldsymbol{i}) \in \mathbf{Rel}(y, y)$. Since g is strongly extensional, we have that

$$g(f(\boldsymbol{i}))\ \texttt{Memb}\ \mathbf{ineq}^z \Rightarrow f(\boldsymbol{i})\ \texttt{Memb}\ \mathbf{ineq}^y$$

and since f ist strongly extensional, we get

$$f(\boldsymbol{i})\ \texttt{Memb}\ \mathbf{ineq}^y \Rightarrow \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x.$$

Hence it holds

$$(g \circ f)(\boldsymbol{i}) = g(f(\boldsymbol{i}))\ \texttt{Memb}\ \mathbf{ineq}^z \Rightarrow \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x.$$

So $g \circ f$ is strongly extensional.
(v) Let $n : f(\boldsymbol{i})\ \texttt{Memb}\ f \circ \mathbf{ineq}^x$,

$$\begin{tikzcd}
 & x \arrow[rr, "f"] & & y & x \arrow[l, "f"'] & \\
\mathrm{ineq}^x \arrow[ur, "\mathrm{ineq}_1^x"] \arrow[dr, "\mathrm{ineq}_2^x"'] & & 1 \arrow[ll, hook', "n"'] \arrow[ul, "e_1"] \arrow[ur, "f(e_1)"'] \arrow[dl, "e_2"'] \arrow[dr, "f(e_2)"] \arrow[rrr, hook, "n"] & & & \mathrm{ineq}^x \arrow[ul, "\mathrm{ineq}_1^x"'] \arrow[dl, "\mathrm{ineq}_2^x"] \\
 & x \arrow[rr, "f"'] & & y & x \arrow[l, "f"] &
\end{tikzcd}$$

then we have $f \circ (\mathrm{ineq}_1^x \circ n) = f(e_1) = f \circ e_1$ and $f \circ (\mathrm{ineq}_2^x \circ n) = f(e_2) = f \circ e_2$. Since f is monic, $\mathrm{ineq}_1^x \circ n = e_1$ and $\mathrm{ineq}_2^x \circ n = e_2$. Hence $n : \boldsymbol{i}\ \texttt{Memb}\ \mathbf{ineq}^x$, so f is strongly extensional. $\square$

After having presented the notions of equality relations and apartness relations in categorical language, we can now define the notion of a complemented subobjects of $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$, which can be seen as the categorical version of Bishop's complemented subsets described in section 3.2.

Definition 4.3.6. Let $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$ be a full Bishop object in $\mathcal{C}$. We call a *complemented subobject of* x a pair of subobjects $(i : a \hookrightarrow x, j : b \hookrightarrow y)$ of x, such that for every element $x : 1 \hookrightarrow a$ of a and $y : 1 \hookrightarrow b$ of b, it holds

$$(1, i \circ x, j \circ y) \ \texttt{Memb} \ \mathbf{ineq}^x.$$

This means that the following inner diagrams commute

$$\begin{tikzcd}
 & & x & \\
 & a \arrow[ur, hook, "i"] & & \\
1 \arrow[ur, "x"] \arrow[rrr, "m"] \arrow[dr, "y"'] & & & \mathrm{ineq}^x \arrow[uul, "\mathrm{ineq}_1^x"'] \arrow[ddl, "\mathrm{ineq}_2^x"] \\
 & b \arrow[dr, hook, "j"'] & & \\
 & & x &
\end{tikzcd}$$

and we write $m : (1, i \circ x, j \circ y) \ \texttt{Memb} \ \mathbf{ineq}^x$. For simplicity we refer to the pair (a, b) instead of $(i : a \hookrightarrow x, j : b \hookrightarrow y)$.

Now we can define the category of complemented subobjects.

Definition 4.3.7. Let $\mathcal{C}$ be a category with 1 and let $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$ be a full Bishop object of $\mathcal{C}$. Then the *category* $\mathbf{Sub}_{(x,\mathbf{eq}^x,\mathbf{ineq}^x)}(\mathcal{C})$ *of complemented subobjects of* x has as objects the complemented subobjects of x. If (a^1, a^0) and (b^1, b^0) are objects of $\mathbf{Sub}_{(x,\mathbf{eq}^x,\mathbf{ineq}^x)}(\mathcal{C})$, a morphism $f : (a^1, a^0) \to (b^1, b^0)$ is a pair of morphisms (f_1, f_0), where $f_1 : a^1 \to b^1$ and $f_0 : b^0 \to a^0$, such that the following diagrams commute

$$\begin{tikzcd}
a^1 \arrow[rr, "f_1"] \arrow[dr, hook, bend right, "i_1"'] & & b^1 \arrow[dl, hook', bend left, "j_1"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
b^0 \arrow[rr, "f_0"] \arrow[dr, hook, bend right, "j_0"'] & & a^0 \arrow[dl, hook', bend left, "i_0"] \\
 & x &
\end{tikzcd}$$

If (c^1, c^0) is an object of $\mathbf{Sub}_{(x,\mathbf{eq}^x,\mathbf{ineq}^x)}(\mathcal{C})$ and $g = (g_1, g_0) : (b^1, b^0) \to (c^1, c^0)$, then $g \circ f := (g_1 \circ f_1, f_0 \circ g_0)$ is the composition of arrows in $\mathcal{C}$.

$$\begin{tikzcd}
a^1 \arrow[r, "f_1"] \arrow[dr, hook, bend right, "i_1"'] & b^1 \arrow[r, "g_1"] \arrow[d, hook, "j_1"] & c^1 \arrow[dl, hook', bend left, "k_1"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
c_0 \arrow[r, "g_0"] \arrow[dr, hook, bend right, "k_0"'] & b^0 \arrow[r, "f_0"] \arrow[d, hook, "j_0"] & a^0 \arrow[dl, hook', bend left, "i_0"] \\
 & x &
\end{tikzcd}$$

The unit morphism $1_{(a^1,a^0)} = (\mathtt{id}_{a^1}, \mathtt{id}_{a^0})$.

$$\begin{tikzcd}
a^1 \arrow[rr, "\mathtt{id}_{a^1}"] \arrow[dr, hook, bend right, "i_1"'] & & a^1 \arrow[dl, hook', bend left, "i_1"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
a^0 \arrow[rr, "\mathtt{id}_{a^0}"] \arrow[dr, hook, bend right, "i_0"'] & & a^0 \arrow[dl, hook', bend left, "i_0"] \\
 & x &
\end{tikzcd}$$

The following proposition shows that $\mathbf{Sub}_{(x,\mathbf{eq}^x,\mathbf{ineq}^x)}(\mathcal{C})$ can be fully embedded into the Chu category $\mathbf{Chu}(\mathcal{C}, x \times x)$.

Proposition 4.3.8. (Chu-representation of $\mathbf{Sub}_{(x,\mathbf{eq}^x,\mathbf{ineq}^x)}(\mathcal{C})$)
If $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$ is a full Bishop object, then the functor

$$E^x : \mathbf{Sub}_{(x,\mathbf{eq}^x,\mathbf{ineq}^x)}(\mathcal{C}) \to \mathbf{Chu}(\mathcal{C}, x \times x),$$
$$E_0^x(i_1 : a^1 \hookrightarrow x, i_0 : a^0 \hookrightarrow x) = (a^1, i_1 \times i_0, a^0),$$
$$E_1^x\big((f_1, f_0) : (i_1 : a^1 \hookrightarrow x, i_0 : a^0 \hookrightarrow x) \to (j_1 : b^1 \hookrightarrow x, j_0 : b^0 \hookrightarrow x)\big) = (f_1, f_0) : (a^1, i_1 \times i_0, a^0) \to (b^1, j_1 \times j_0, b^0)$$

is a full embedding of $\mathbf{Sub}_{(x,\mathbf{eq}^x,\mathbf{ineq}^x)}(\mathcal{C})$ into $\mathbf{Chu}(\mathcal{C}, x \times x)$.

Proof. The proof is similar to the proof of prop. 3.3.10. in section 3.3.2. $\square$

Remark 4.3.9. In prop. 3.3.10 we presented the full embedding of the category of complemented subsets $\boldsymbol{\mathcal{P}}^{]\!\![}(X)$ into the Chu category $\mathbf{Chu}(\mathbf{Set}, X \times X)$. In fact this is a specific example of the above proposition for the category of the complemented subobjects of X in $\mathbf{Set}$.

5 Conclusion

This thesis consisted of three parts. In the first part we were concerned with Bishop set theory, where we followed [9] to introduce the basic notions of BST and especially focused on complemented subsets. Moreover we presented partial functions between sets. Following [7] we gave a detailed proof of the existence of proper class-assignement routines between the class of complemented subsets $\mathcal{P}^{\mathbb{II}}(X)$ an $\mathscr{F}^{se}(X, \mathbb{2})$, the class of strongly extensional partial functions from X to $\mathbb{2}$.

In the second part we focused on the categorical aspects of complemented subsets by studying Chu representations of different categories. Following Petrakis' work [6] we presented the category of complemented subsets $\boldsymbol{\mathcal{P}}^{\mathbb{II}}(X)$ and showed it's embedding into the Chu category $\mathbf{Chu}(\mathbf{Set}, X \times X)$.

In the last part, we worked on translating some elements of Bishop set theory into category theory. Based on Petrakis' work [8] we defined the categorical notions of an equality relation and an apartness relation. We distinguished between local and global equality relations and proved that every global equality is indeed a local equality. We defined the notion of a full Bishop object $(x, \mathbf{eq}^x, \mathbf{ineq}^x)$, which gives us the categorical version of Bishop's complemented subset. Finally we defined the category of complemented subobjects of a category $\mathcal{C}$.

Of course, there are still a lot of open questions that arise. As mentioned in remark 4.2.11 it does not generally hold that a local equality relation is a global equality relation. It is still an open task to find a good condition on a category $\mathcal{C}$ such that this implication holds.

Another big task requiring some extra work is the global representation of an apartness relation, which we have not yet found categorically. In the global definition of an apartness relation, we need to define irreflexivity and cotransitivity in a global manner. Let $(x, \mathbf{eq}^x)$ be a Bishop set object, if $\mathbf{r} \in \mathbf{Rel}(x, x)$, then one could define $\mathbf{r}$ to be globally irreflexive if $\mathbf{eq}^x \cap \mathbf{r} = \mathbf{0}$, where $\mathbf{0} := (0, e_1, e_2)$ is the empty relation. However, if 0 is the initial object of the category $\mathcal{C}$, then e_1, e_2 are not jointly monic. To define global cotransitivity we would like to have a presentation of a categorical definition of cocomposition, the dual notion of composition. Let R and S be two subsets of $X \times X$, then the *cocomposition* of R with S is defined as follows

$$R * S := \big\{(x, z) \in X \times X \mid \forall y \in X((x, y) \in R \ \vee \ (y, z) \in S)\big\}.$$

If we had a categorical representation of cocomposition we could define global cotransitivity through $\mathbf{r} * \mathbf{r} \leqslant \mathbf{r}$, which would be dual to our definition of global transitivity. It would be really desirable to have a categorical representation of cocomposition of relations and to define the global version of an apartness relation.

An obvious question one may think of is whether a global apartness implies a local apartness and if so, if there exists a good condition on a category $\mathcal{C}$ to prove the converse. These open questions however are not straightforward and require some more work to be answered.

Finally, it would be really interesting to see more examples of equivalence and apartness relations and to elaborate some interesting applications of our work.

References

[1] S. Awodey: *Category theory*, Oxford University Press, 2010.

[2] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[3] E. Bishop, D. S. Bridges: *Constructive Analysis*, Grundlehren der Math. Wissenschaften 279, Springer-Verlag Berlin Heidelberg, 1985.

[4] A. Klein: Relations in Categories, Illinois J. Math. 14(4), 1970, 536-550.

[5] E. Palmgren: Constructivist and structuralist foundations: Bishop's and Lawvere's theories of sets, Annals of Pure and Applied Logic, 163, 2012, 1384-1399.

[6] I. Petrakis: Chu representations of categories related to constructive mathematics, 2021, https://arxiv.org/abs/2106.01878.

[7] I. Petrakis: From Daniell spaces to the integration spaces of Bishop and Cheng, preprint, 2021.

[8] I. Petrakis: Equality and apartness relations in categories, preprint, 2021.

[9] I. Petrakis: *Families of Sets in Bishop Set Theory*, Habilitationsschrift, LMU, Munich, 2020.

[10] E. Riehl: *Category Theory in Context*, Dover Publications Inc., 2016.

Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Masterarbeit eigenständig verfasst und keine weiteren als die angegebenen Quellen verwendet habe. Alle Passagen, die aus fremden Quellen wörtlich oder dem Sinn nach übernommenen wurden, wurden als solche gekennzeichnet. Die Arbeit wurde bisher weder vollständig noch in wesentlichen Teilen einem anderen Prüfungsamt vorgelegt und noch nicht veröffentlicht.

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