Mathematische Logik - WS13/14

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These notes include part of the material discussed in the Tutorium and in the Exercises that correspond to the Vorlesung "Mathematische Logik" of Prof. Dr. Hans-Dieter Donder. Of course, possible mistakes in these notes are not related to Prof. Donder at all. Many extra, or optional exercises can be found here.

Please feel free to send me your comments, or your suggestions regarding these notes.

1. On Inductive Definitions

The most characteristic example of an inductive (or recursive) definition is that of a *natural number*. It can be given using the following two rules

$$\frac{}{0 \in \mathbb{N}} \qquad \frac{n \in \mathbb{N}}{S(n) \in \mathbb{N}},$$

where $S(n)$ denotes the successor of n. Note though, that this definition alone does not determine a unique set; for example the rationals $\mathbb{Q}$ or the reals $\mathbb{R}$ satisfy the same rules. We determine $\mathbb{N}$ by postulating that $\mathbb{N}$ is the least set satisfying the above rules. We do so by stating the following induction axiom:

$$A(0) \to \forall_n (A(n) \to A(S(n))) \to \forall_n (A(n)),$$

where A is a formula representing a predicate on natural numbers. If A is any formula, then the above principle is the well-known full induction principle on natural numbers (if we restrict the range of A's we get other weaker induction principles). Its interpretation is the following: Suppose that A satisfies the two rules, $A(0)$ and $\forall_n (A(n) \to A(S(n)))$, i.e., as it is usually said, A is a "competitor" predicate to $\mathbb{N}$, then $\mathbb{N} \subseteq A$, i.e, $\forall_n (A(n))$, or a bit more precisely, $\forall_x (\mathbb{N}(x) \to A(x))$. A consequence of this inductive characterization of $\mathbb{N}$ is that if we want to define a function $f : \mathbb{N} \to X$, where X is a set, it suffices to define it on 0, and provide a rule G which gives the value of f on $S(n)$ through the value of f on n. I.e., we have the following proposition:

Proposition 1. *If X is a set, $x_0 \in X$ and $G : X \to X$ is a function, then there exists a unique function $f : \mathbb{N} \to X$ such that $f(0) = x_0$ and $f(S(n)) = G(f(n))$.*

Proof. It is direct to see that the above defined f is a function with domain $\mathbb{N}$ and range in X. To show its uniqueness we suppose that there exists $g : \mathbb{N} \to X$ satisfying the above two conditions, and we apply the induction principle on $A(n) := (f(n) = g(n))$ to show that $\forall_n (A(n))$. □

So, we cannot see an inductive definition without its corresponding induction principle. If we consider now the inductive definition of a *formula* in Classical Propositional Calculus

$$\frac{p \in P}{p \in \text{Form}} \qquad \frac{\phi \in \text{Form}}{\neg\phi \in \text{Form}} \qquad \frac{\phi, \psi \in \text{Form}}{\phi \vee \psi \in \text{Form}},$$

the corresponding full induction principle is

$$A(p) \to \forall_\phi (A(\phi) \to A(\neg\phi)) \to \forall_{\phi,\psi} (A(\phi) \to A(\psi) \to A(\phi \vee \psi)) \to \forall_\phi (A(\phi)),$$

where A is any formula of our meta-language, for example this could be the language of set theory. Some times in the bibliography one can find one more clause in the above inductive definition of the form "there are no other formulas except the ones determined by the previous rules", and then the induction principle is proved as a theorem. It is clear though, that the added rule is a disguised form of the induction principle. Since this added rule is not also a formal statement, we propose in these notes to understand an inductive definition as a pair

$$(\text{Inductive Rules}, \ \text{Induction Principle}).$$

Again, a consequence of this inductive characterization of Form is that if we want to define a function $F : \text{Form} \to X$, where X is a set, it suffices to define it on the prime formulas P, and then provide firstly a rule $G_\neg$ which gives the value of F on $\neg\phi$ through the value of F on ϕ, and secondly a rule $G_\vee$ which gives the value of F on $\phi \vee \psi$ through the value of F on ϕ and ψ. I.e., we have the following proposition:

Proposition 2. *If X is a set, $f : P \to X$, $G_\neg : X \to X$ and $G_\vee : X \times X \to X$ are given functions, then there exists a unique function $F : \text{Form} \to X$ satisfying*

$$F(p) = f(p), \ \textit{for each } p,$$

$$F(\neg\phi) = G_\neg(F(\phi)),$$

$$F(\phi \vee \psi) = G_\vee(F(\phi) \vee F(\psi)).$$

Proof. **Exercise 1**. □

Consider the function $V : \text{Form} \to \mathcal{P}(P)$, where $\mathcal{P}(P)$ denotes the power set of P, and $V(\phi)$ is the set of propositional variables occurring in ϕ defined by

$$V(p) := \{p\},$$

$$V(\neg\phi) := V(\phi),$$

$$V(\phi \vee \psi) := V(\phi) \cup V(\psi).$$

The unique existence of V is guaranteed by Proposition 2.

Exercise 2: Which are the functions $f, G_\neg, G_\vee$ that correspond to the function V?

The set $2 = \{0, 1\}$ (in the lecture course it is also written as $\{w, f\}$, or you can see it elsewhere as $\{\mathtt{tt}, \mathtt{ff}\}$) is the simplest *boolean algebra* i.e., a complemented distributive lattice, or a commutative ring with 1 in which every element is idempotent ($p^2 = p$). These algebraic structures are very important in mathematical logic and topology. So, we define on 2 the following operations:

$$\neg 0 = 1, \quad \neg 1 = 0, \quad 0 \vee b = b, \quad 1 \vee b = 1,$$

for each $b \in 2$. Also the ring operations are defined from $\neg, \vee$ as follows:

$$b \cdot c := b \wedge q,$$

$$b + c := (b \wedge \neg c) \vee (\neg b \wedge c),$$

where $b \wedge c := \neg b \vee c$.

We have now all the tools to understand how a *truth valuation* $W : \mathrm{Form} \to 2$ works. If $W : P \to 2$, $G_{\neg} = \neg$, and $G_{\vee} = \vee$ are given, then by Proposition 2 there exists a unique function $W^* : \mathrm{Form} \to 2$ satisfying:

$$W^*(p) = W(p), \text{ for each } p,$$

$$W^*(\neg\phi) = \neg W^*(\phi),$$
$$W^*(\phi \vee \psi) = W^*(\phi) \vee W^*(\psi),$$

where the $\vee$ on the left side of the last equality is the logical connective, and the $\vee$ on the right side is the boolean operation.

Exercise 3: Give an example of a function $F : \mathrm{Form} \to 2$ which is not an extension of a truth valuation.

We are in position now to fully grasp the formulation of the following proposition given in the lecture course.

Proposition 3. *If $W_1, W_2 : P \to 2$ are truth valuations, then*

$$\forall_{\phi}(W_{1\restriction V(\phi)} = W_{2\restriction V(\phi)} \to W_1^*(\phi) = W_2^*(\phi)).$$

Proof. **Exercise 4**.
Hint: Apply the induction principle corresponding to the inductive definition of formulas on

$$A(\phi) := W_{1\restriction V(\phi)} = W_{2\restriction V(\phi)} \to W_1^*(\phi) = W_2^*(\phi),$$

using the definition of $V(\phi)$. $\square$

Exercise 5: (i) Given the field structure $(\mathbb{R}, +, \cdot, 0, 1)$ of the real numbers, define inductively the set of rationals $\mathbb{Q}$.
(ii) Which is the corresponding induction principle?

2. On Classical Propositional Calculus

If $\phi \in \mathrm{Form}$, then it is called a *tautology*, if

$$\forall_{W \in 2^P}(W^*(\phi) = 1),$$

where X^Y denotes the set of all functions $f : Y \to X$. A formula ϕ is called a *contradiction*, if

$$\forall_{W \in 2^P}(W^*(\phi) = 0).$$

Exercise 6: (i) Give an example of a tautology, and an example of a contradiction.
(ii) Show that ϕ is a tautology iff $\neg\phi$ is a contradiction.

(iii) Explain why Proposition 3 guarantees that there is a (semantic) algorithm deciding if a formula ϕ is a tautology or not. Although the notion of a Yes/No-algorithm is not yet formally defined, what we mean by it is a recipe which can be executed in a finite amount of time and provides effectively a Yes or No-answer to a given question.
(iv) Describe explicitly the above algorithm for a specific formula ϕ of your choice.
(v) After you learn the definition of a tautology in Classical Predicate Calculus, try to guess if there is a similar (semantic) algorithm deciding if a formula in Predicate Calculus is a tautology or not.

Proposition 4. *If* $n \in \mathbb{N}$, $M_n = 2^{\{p_0,\ldots,p_n\}}$ *and* $F : M_n \to 2$, *then*

$$\exists_\phi (V(\phi) \subseteq \{p_0, \ldots, p_n\} \ \wedge \ \forall_W (W^*(\phi) = F(W_{\restriction \{p_0,\ldots,p_n\}})).$$

The fact that the above proposition expresses the *completeness of the connectives* $\{\neg, \vee\}$, that is the sufficiency of $\{\neg, \vee\}$ in writing equivalent forms to all formulas, is related to the general definition of a connective. Maybe more will be added on that later.

Exercise 7. Show that the set of connectives $\{\wedge, \to\}$ is not complete.

Hint: Since $\{\neg, \vee\}$ is a complete set of connectives, the set $\{\neg, \wedge\}$ is also complete (why?). Then try to show that $\neg$ is not expressible within $\{\wedge, \to\}$. If $\{\wedge, \to\}$ was complete, there would exist some formula ϕ including $p, \wedge, \to$ and equivalent to $\neg p$ i.e., $\forall_W (W^*(\phi) = W^*(\neg p))$, for some fixed $p \in P$. Show that this cannot happen.

3. On the basic definitions of Classical Predicate Calculus

Exercise 8. Write down the Induction Principle that corresponds to the inductive definition of L-terms[1]

$$\frac{v \in \mathrm{Var}}{v \in \mathrm{Term}}, \quad \frac{c \in \mathrm{Konst}}{c \in \mathrm{Term}}, \quad \frac{t_1, \ldots, t_n \in \mathrm{Term}, \ f \in \mathrm{Funk}^n}{f(t_1, \ldots, t_n) \in \mathrm{Term}},$$

where Funk^n denotes the set of function symbols of L with arity n. Formulate the theorem of recursive definition on Term which corresponds to Propositions 1 or 2.

Exercise 9. Write down the Induction Principle that corresponds to the inductive definition of L-formulas

$$\frac{t_1, t_2 \in \mathrm{Term}}{t_1 = t_2 \in \mathrm{Form}}, \quad \frac{t_1, \ldots, t_n \in \mathrm{Term}, \ R \in \mathrm{Rel}^n}{R(t_1, \ldots, t_n) \in \mathrm{Form}},$$

[1] For simplicity we avoid the subscript L from the symbols $\mathrm{Var}, \mathrm{Konst}, \mathrm{Term}, \mathrm{Form}$.

$$\frac{\phi \in \text{Form}}{\neg\phi \in \text{Form}}, \quad \frac{\phi, \psi \in \text{Form}}{\phi \vee \psi \in \text{Form}}, \quad \frac{\phi \in \text{Form},\ x \in \text{Var}}{\exists_x \phi \in \text{Form}},$$

where Rel^n denotes the set of relation symbols of L with arity n. Formulate the theorem of recursive definition on Term which corresponds to Propositions 1 or 2. For that it will be helpful to write the last inductive rule as follows:

$$\frac{\phi \in \text{Form},\ x_i \in \text{Var}}{\exists_{x_i} \phi \in \text{Form}},$$

where $\text{Var} = \{x_n \mid n \in \mathbb{N}\}$.

4. On L-structures and L-interpretations

If $L = (\text{Rel}, \text{Funk}, \text{Konst})$ is a 1st-order language, then an *L-structure* is a structure $\mathfrak{A} = (A, \text{Rel}_{\mathfrak{A}}, \text{Funk}_{\mathfrak{A}}, c_{\mathfrak{A}})$, where $\text{Rel}_{\mathfrak{A}}$ is a set of relations on A, $\text{Funk}_{\mathfrak{A}}$ is a set of functions on products of A and with values in A, and $c_{\mathfrak{A}}$ is a fixed subset of A.

An *L-semi interpretation* is a pair $(\mathfrak{A}, \mathfrak{e})$, where $\mathfrak{A}$ is an L-structure, and $\mathfrak{e}$ is a triplet of functions, $\mathfrak{e} = (e_1, e_2, e_3)$, such that $e_1 : \text{Rel} \to \text{Rel}_{\mathfrak{A}}$, $e_2 : \text{Funk} \to \text{Funk}_{\mathfrak{A}}$ and $e_3 : \text{Konst} \to \text{Konst}_{\mathfrak{A}}$, where

$$R \overset{e_1}{\mapsto} R^{\mathfrak{A}},$$

$$f \overset{e_2}{\mapsto} f^{\mathfrak{A}},$$

$$c \overset{e_3}{\mapsto} c^{\mathfrak{A}},$$

such that the arity of each relation symbol R or each function symbol f of L is preserved.

Exercise 10. Give two different L-semi interpretations for some 1st-order language L.

An *L-assignment* η in an L-structure $\mathfrak{A}$ is a function

$$\eta : \text{Var} \to A,$$

where A is the *carrier set* of $\mathfrak{A}$.

An *L-interpretation* is a triplet $\mathcal{I} = (\mathfrak{A}, \mathfrak{e}, \eta)$, where $(\mathfrak{A}, \mathfrak{e})$ is an L-semi interpretation, and η is an L-assignment in $\mathfrak{A}$. An L-interpretation contains all the information necessary to extend the interpretation of Term in $\mathfrak{A}$ and define the notion of truth of a formula in $\mathfrak{A}$.

Since η is given in an L-interpretation beforehand, by the theorem of recursive definition on Term there exists a unique function

$$H : \text{Term} \to A$$

$$t \overset{H}{\mapsto} t^{\mathcal{I}},$$

such that

$$v^{\mathcal{I}} := \eta(v),$$
$$c^{\mathcal{I}} := e_3(c) = c^{\mathfrak{A}},$$
$$(f(t_1, \ldots, t_n))^{\mathcal{I}} := (e_2(f))(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}}) = f^{\mathfrak{A}}(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}}).$$

Note that in the lecture course notes it is used instead the notation $t^{\mathfrak{A}}$, where the use of a given assignment η is not explicitly mentioned.

From an assignment η we define the assignment $\eta(x \mapsto a)$, for each given $x \in \mathrm{Var}$ and $a \in A$, as follows:

$$\eta(x \mapsto a)(v) := \begin{cases} \eta(v) & \text{, if } v \neq x \\ a & \text{, if } v = x, \end{cases}$$

i.e., the assignment $\eta(x \mapsto a)$ agrees with η on each variable $\neq x$ and maps x to a. If $\mathcal{I}$ is an L-interpretation, then we denote by $\mathcal{I}(x \mapsto a)$ the interpretation

$$\mathcal{I}(x \mapsto a) := (\mathfrak{A}, \mathfrak{e}, \eta(x \mapsto a)).$$

Exercise 11. The object $t^{\mathcal{I}(x \mapsto a)} \in A$ and a replaces each occurrence of x in $t^{\mathcal{I}(x \mapsto a)}$.

Now we can extend trivially the above definition to the case where each one of the fixed pairwise distinct variables x_i are mapped to the fixed element $a_i \in A$, respectively: if $\vec{a}$ denotes a finite sequence of elements of A and $\vec{x}$ a finite sequence of pairwise distinct variables $x_i \in \mathrm{Var}$, such that $|\vec{a}| = |\vec{x}|$, where $|.|$ denotes the length of a finite sequence (how can we define it?), then we define for a given assignment η:

$$\eta(\emptyset \mapsto \emptyset) := \eta,$$
$$\eta((a_1, \ldots, a_m, a_{m+1}) \mapsto (x_1, \ldots, x_m, x_{m+1})) :=$$
$$:= [\eta((a_1, \ldots, a_m) \mapsto (x_1, \ldots, x_m))](a_{m+1} \mapsto x_{m+1}).$$

Exercise 12. If $\vec{x} = (x_1, \ldots, x_m)$ is a finite sequence of pairwise distinct variables and $\vec{a}$ is a finite sequence of elements of A of length m, then

$$\eta(\vec{x} \mapsto \vec{a})(v) := \begin{cases} \eta(v) & \text{, if } v \neq x_1 \wedge \ldots \wedge v \neq x_m \\ a_1 & \text{, if } v = x_1 \\ \ldots & \quad \ldots \\ a_m & \text{, if } v = x_m \end{cases}$$

It is with this assignment $\eta(\vec{x} \mapsto \vec{a})$ that the interpretation $t_{\vec{x}}^{\mathfrak{A}}[a_1, \ldots, a_m]$ of a term $t \in \mathrm{Term}$ in $\mathfrak{A}$ is defined in the lecture course notes. I.e., we have

$$t_{\vec{x}}^{\mathfrak{A}}[a_1, \ldots, a_m] = t^{\mathcal{I}(\vec{x} \mapsto \vec{a})}.$$

Next we define recursively the notion $\mathcal{I} \models \phi$, "the L-interpretation I *models* $\phi \in$ Form", or "ϕ is *true* under the L-interpretation $\mathcal{I}$". Note that we need to specify all the necessary information $\mathcal{I} = (\mathfrak{A}, \mathfrak{e}, \eta)$ on the left-hand side of $\mathcal{I} \models \phi$, and not just $\mathfrak{A}$, because the assignment η is necessary to define the interpretation $t^{\mathcal{I}}$ which appears in the case of prime formulas. The presence of the assignment η is also crucial in the case of an existential formula. What we can only fix, and therefore skip from our notation, is the L-semi interpretation $\mathfrak{e}$. Thus, we define:

1. $\mathcal{I} \models t_1 = t_2 :\leftrightarrow t_1^{\mathcal{I}} = t_2^{\mathcal{I}}$
2. $\mathcal{I} \models R(t_1, \ldots, t_n) :\leftrightarrow R^{\mathfrak{A}}(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}})$
3. $\mathcal{I} \models \neg\phi :\leftrightarrow$ not $(\mathcal{I} \models \phi)$
4. $\mathcal{I} \models \phi \vee \psi :\leftrightarrow \mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$
5. $\mathcal{I} \models \exists_x \phi :\leftrightarrow$ there exists $a \in A$ such that $\mathcal{I}(x \mapsto a) \models \phi$,

where the L-interpretation $\mathcal{I}(x \mapsto a)$ was defined above. The reason for the use of this interpretation in the last clause of the above definition has to do with avoiding "capture" in the direction $(\leftarrow)$ of clause 5. Intuitively, we want $\mathcal{I} \models \phi(a)$, for some $a \in A$, but we want to assure that each occurrence of x will be replaced by a, so that x does not occur in $\phi(a)$, and no capture occurs with the quantifier $\exists_x$.

A reformulation of clause 5 can be given through the equivalence (Blatt 2, exercise 3)

$$\mathcal{I}(x \mapsto a) \models \phi \leftrightarrow \mathcal{I} \models \phi_x(t),$$

where t is a closed term such that $t^{\mathcal{I}} = a$ and $\mathrm{FV}(\phi) \subseteq \{x\}$. Therefore for such formulas we could present the above definition without introducing the interpretation $\mathcal{I}(x \mapsto a)$ on the right-hand side. This equivalence expresses formally what we said in the previous paragraph intuitively.

Exercise 13. Show that the definition of the *Gültigkeitsrelation* in the lecture course notes can be written in the above terminology as

$$\mathcal{I}(\vec{x} \mapsto \vec{a}) \models \phi.$$

Of course, the definition given above of $\mathcal{I} \models \phi$ is derived from the definition of the lecture course notes by taking $\vec{x} = \emptyset$. Therefore, the two presentations are equivalent.

Exercise 14. If we consider the 1st-order language $L = (+, \cdot, 0, 1, <)$, the L-structure $\mathcal{N} = (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, <^{\mathbb{N}})$, the L-semi interpretation $\mathfrak{e}$ given by

$$+ \mapsto +^{\mathbb{N}}, \quad \cdot \mapsto \cdot^{\mathbb{N}}, \quad 0 \mapsto 0^{\mathbb{N}}, \quad 1 \mapsto 1^{\mathbb{N}}, \quad < \mapsto <^{\mathbb{N}},$$

and the L-assignment $\eta : \mathrm{Var} \to \mathbb{N}$ defined by

$$x_n \mapsto 2n,$$

for each $n \geq 0$, check if the following hold:

(a) $\mathcal{I} \models \cdot(x_2, +(x_1, x_2)) = x_4$,

(b) $\mathcal{I} \models \forall_{x_0} \exists_{x_1} (< (x_0, x_1))$,

where $\mathcal{I} = (\mathcal{N}, \mathfrak{e}, \eta)$.

Exercise 15. If we consider the 1st-order language $L = (\circ, e)$, the L-structure $\mathcal{R} = (\mathbb{R}, +, 0)$, the L-semi interpretation $\mathfrak{e}$ given by

$$\circ \mapsto +, \quad e \mapsto 0,$$

and the L-assignment $\eta : \mathrm{Var} \to \mathbb{R}$ in $\mathcal{R}$ defined as the constant function

$$x \overset{\eta}{\mapsto} 9,$$

show that

$$\mathcal{I} \models \forall_x (x \circ e = x),$$

where $\mathcal{I} = (\mathcal{R}, \mathfrak{e}, \eta)$.

Exercise 16. If f is a binary function symbol, $L = (\emptyset, f, \emptyset)$, and

$$\phi := \forall_{x_1} (f(x_0, x_1) = x_0),$$

find L-interpretations $\mathcal{I}_1, \mathcal{I}_2$ such that

$$\mathcal{I}_1 \models \phi \quad \text{and} \quad \mathcal{I}_2 \not\models \phi.$$

5. Blatt 2, Aufgabe 3

Remark on the notation: The interpretation $t^{\mathfrak{A}}$ of an L-term t in the L-structure $\mathfrak{A}$ can be also denoted by

$$t^{\mathfrak{A}, \eta}$$

in order to specify the fixed assignment $\eta : \mathrm{Var} \to A$ we use (see section 4). In section 4 we have also used the notation

$$t^{\mathcal{I}},$$

where

$$\mathcal{I} = (\mathfrak{A}, \mathfrak{e}, \eta),$$

in order to give a full account of the information necessary in order to define the interpretation of a term. One can use any of these notations having in mind that whenever a simplified one is used then either $\mathfrak{e}$ or η are fixed. Similarly we can write

$$\mathfrak{A} \models \phi,$$

or

$$\mathfrak{A}, \eta \models \phi,$$

or the fully informative

$$\mathcal{I} \models \phi.$$

In order to solve this exercise we need to show the following lemma:

Lemma 5. *If* $\mathrm{Term}^{\mathrm{cl}}$ *denotes the closed terms of* $\mathcal{L}$ *(konstante Terme) and* $\mathcal{I} = (\mathfrak{A}, \mathfrak{e}, \eta)$ *is an L-interpretation, then*

$$\forall_{t \in \mathrm{Term}}(\forall_{s \in \mathrm{Term}^{\mathrm{cl}}} \forall_{a \in A}(s^{\mathfrak{A},\eta} = a \to t(s/x)^{\mathfrak{A},\eta} = t^{\mathfrak{A},\eta(x \mapsto a)})),$$

where our notational conventions w.r.t. the notation in the lecture course notes are

$$t(s/x) = t_x(s),$$

and the assignment $\eta(x \mapsto a)$ *is defined in section 4.*

Proof. We apply the induction principle corresponding to the inductive definition of Term on the formula

$$P(t) := \forall_{s \in \mathrm{Term}^{\mathrm{cl}}} \forall_{a \in A}(s^{\mathfrak{A},\eta} = a \to t(s/x)^{\mathfrak{A},\eta} = t^{\mathfrak{A},\eta(x \mapsto a)}).$$

$P(v)$: We fix $s \in \mathrm{Term}^{\mathrm{cl}}$ and $a \in A$ such that $s^{\mathfrak{A},\eta} = a$. Since

$$v(s/x) := \begin{cases} v & \text{, if } v \neq x \\ s & \text{, if } v = x, \end{cases}$$

we conclude

$$v(s/x)^{\mathfrak{A},\eta} := \begin{cases} \eta(v) & \text{, if } v \neq x \\ s^{\mathfrak{A},\eta} & \text{, if } v = x. \end{cases}$$

On the other hand, by the definition $v^{\mathcal{I}}$ of the interpretation of a variable we have that

$$v^{\mathfrak{A},\eta(x \mapsto a)} = \eta(x \mapsto a)(v) = \begin{cases} \eta(v) & \text{, if } v \neq x \\ a & \text{, if } v = x, \end{cases}$$

therefore by the hypothesis $s^{\mathfrak{A},\eta} = a$ we conclude the required equality.

$P(c)$: Since $c(s/x) = c$, we have that $c(s/x)^{\mathfrak{A},\eta} = c^{\mathfrak{A},\eta} = c^{\mathfrak{A}}$, while also $c^{\mathfrak{A},\eta(x \mapsto a)} = c^{\mathfrak{A}}$.

$P(t_1) \to \ldots \to P(t_n) \to P(f(t_1 \ldots t_n))$: Using the definition of substitution on complex terms we have that

$$\begin{aligned}
[(f(t_1 \ldots t_n))(s/x)]^{\mathfrak{A},\eta} &= [f(t_1(s/x) \ldots t_n(s/x))]^{\mathfrak{A},\eta} \\
&= f^{\mathfrak{A}}([t_1(s/x)]^{\mathfrak{A},\eta} \ldots [t_n(s/x)]^{\mathfrak{A},\eta}) \\
&\overset{(\mathrm{IV})}{=} f^{\mathfrak{A}}(t_1^{\mathfrak{A},\eta(x \mapsto a)} \ldots t_n^{\mathfrak{A},\eta(x \mapsto a)}) \\
&= [f(t_1 \ldots t_n)]^{\mathfrak{A},\eta(x \mapsto a)}.
\end{aligned}$$

$\square$

Next we rewrite the initial Exercise, using our initial notational conventions, as follows.

Proposition 6. *If $\mathcal{I} = (\mathfrak{A}, \mathfrak{e})$ is an L-semi interpretation and A^{Var} denotes the set of all L-assignments in $\mathfrak{A}$, then*

$$\forall_{\phi\in\mathrm{Form}}(\forall_{\eta\in A^{\mathrm{Var}}}(\forall_{t\in\mathrm{Term}^{\mathrm{cl}}}\forall_{a\in A}(t^{\mathfrak{A},\eta} = a \to [\mathfrak{A},\eta \models \phi(t/x) \leftrightarrow \mathfrak{A},\eta(x\mapsto a) \models \phi]))).$$

Proof. We apply the induction principle corresponding to the inductive definition of Form on the formula

$$Q(\phi) := \forall_{\eta\in A^{\mathrm{Var}}}(\forall_{t\in\mathrm{Term}^{\mathrm{cl}}}\forall_{a\in A}(t^{\mathfrak{A},\eta} = a \to [\mathfrak{A},\eta \models \phi(t/x) \leftrightarrow \mathfrak{A},\eta(x\mapsto a) \models \phi])).$$

$Q(t_1 = t_2)$: We fix $\eta \in A^{\mathrm{Var}}, s \in \mathrm{Term}^{\mathrm{cl}}$ and $a \in A$ such that $s^{\mathfrak{A},\eta} = a$. Since $[t_1 = t_2](t/x) = [t_1(t/x) = t_2(t/x)]$, we have that

$$\mathfrak{A},\eta \models t_1(t/x) = t_2(t/x) \leftrightarrow t_1(t/x)^{\mathfrak{A},\eta} = t_2(t/x)^{\mathfrak{A},\eta},$$

while

$$\mathfrak{A},\eta(x\mapsto a) \models t_1 = t_2 \leftrightarrow t_1^{\mathfrak{A},\eta(x\mapsto a)} = t_2^{\mathfrak{A},\eta(x\mapsto a)},$$

and we use the previous lemma to get the required equivalence. Actually, the lemma is forced to us by this very first case of our inductive proof.
$Q(R(t_1 \dots t_n))$: By the definition of substitution we get that

$$\begin{aligned}\mathfrak{A},\eta \models [R(t_1,\dots,t_n](t/x)) &\leftrightarrow \mathfrak{A},\eta \models R(t_1(t/x),\dots,t_n(t/x))\\ &\leftrightarrow R^{\mathfrak{A}}([t_1(t/x)]^{\mathfrak{A},\eta},\dots,[t_n(t/x)]^{\mathfrak{A},\eta})\\ &\leftrightarrow R^{\mathfrak{A}}(t_1^{\mathfrak{A},\eta(x\mapsto a)},\dots,t_n^{\mathfrak{A},\eta(x\mapsto a)})\\ &\leftrightarrow \mathfrak{A},\eta(x\mapsto a) \models R(t_1\dots,t_n).\end{aligned}$$

$Q(\phi) \to Q(\psi) \to Q(\phi \vee \psi)$: Straightforward.
$Q(\phi) \to Q(\neg\phi)$: Straightforward.
$Q(\psi) \to Q(\exists_y\psi)$: We fix $\eta \in A^{\mathrm{Var}}, s \in \mathrm{Term}^{\mathrm{cl}}$ and $a \in A$ such that $s^{\mathfrak{A},\eta} = a$. Since

$$[\exists_y\psi](t/x) = \begin{cases} \exists_y\psi & \text{, if } x = y\\ \exists_y\psi(t/x) & \text{, if } x \neq y,\end{cases}$$

we get

$$\mathfrak{A},\eta \models [\exists_y\psi](t/x) = \begin{cases} \mathfrak{A},\eta \models \exists_y\psi & \text{, if } x = y\\ \mathfrak{A},\eta \models \exists_y\psi(t/x) & \text{, if } x \neq y,\end{cases}$$

$$= \begin{cases} \text{there exists } b \in A:\ \mathfrak{A},\eta(y\mapsto b) \models \psi & \text{, if } x = y\\ \text{there exists } b \in A:\ \mathfrak{A},\eta(y\mapsto b) \models \psi(t/x) & \text{, if } x \neq y.\end{cases}$$

On the other hand,

$$\mathfrak{A},\eta(x\mapsto a) \models \exists_y\psi \leftrightarrow \text{there exists } c \in A:\ \mathfrak{A},[\eta(x\mapsto a)](y\mapsto c) \models \psi$$

i.e.,

$$\mathfrak{A}, \eta(x \mapsto a) \models \exists_y \psi \leftrightarrow \text{there exists } c \in A: \ \mathfrak{A}, \eta((x, y) \mapsto (a, c)) \models \psi.$$

If $x = y$, we get by the inductive definition of $\eta((x, y) \mapsto (a, c))$ (see Section 4) that $\eta((x, y) \mapsto (a, c)) = \eta(y \mapsto c)$ and the required equivalence is automatic. If $x \neq y$, we need to show that

$$(*) \quad \text{there exists } b \in A: \ \mathfrak{A}, \eta(y \mapsto b) \models \psi(t/x) \leftrightarrow$$

$$\leftrightarrow \text{there exists } c \in A: \ \mathfrak{A}, \eta((x, y) \mapsto (a, c)) \models \psi \quad (**).$$

But if we apply the inductive hypothesis $Q(\psi)$ on t, a and the assignment

$$\eta(y \mapsto b),$$

we get that

$$\begin{aligned} \mathfrak{A}, \eta(y \mapsto b) \models \psi(t/x) &\leftrightarrow \mathfrak{A}, [\eta(y \mapsto b)](x \mapsto a) \models \psi \\ &\leftrightarrow \mathfrak{A}, \eta((x, y) \mapsto (a, b)) \models \psi, \end{aligned}$$

which shows that $(*) \to (**)$. In a similar way we show that $(**) \to (*)$. $\square$

6. On the relation of logical consequence

If $T \subseteq S(L)$ and $\phi \in S(L)$, where $S(L)$ denotes the sentences of L (i.e., the L-formulas with no free variables), then we define the relation $T \models \phi$, "ϕ is a logical consequence of T" by

$$T \models \phi :\leftrightarrow \forall_{\mathfrak{A}} (\mathfrak{A} \models T \to \mathfrak{A} \models \phi).$$

Similarly we define $\Phi \models \phi$, where $\Phi \subseteq \text{Form}$ and $\phi \in \text{Form}$.

Exercise 17. If ϕ is a sentence of L, $\mathfrak{A}$ is an L-structure, and T is an L-theory, check the validity or not of the following propositions:

(i) not $(\mathfrak{A} \models \phi$ and $\mathfrak{A} \models \neg\phi)$.
(ii) $(\mathfrak{A} \models \phi) \vee (\mathfrak{A} \models \neg\phi)$.
(iii) not$(T \models \phi)$, then $T \models \neg\phi$.
(iv $(T \models \phi) \vee (T \models \neg\phi)$.

Exercise 18. Show that

$$\exists_x \forall_y \phi \models \forall_y \exists_x \phi,$$

but what about the converse? Try to find an example of such a logical consequence from standard mathematics.

Exercise 19. [Coincidence Lemma] Suppose that L_1, L_2 are 1st-order languages, $\mathcal{I}_1 = (\mathfrak{A}_1, \mathfrak{e}_1, \eta_1)$, $\mathcal{I}_2 = (\mathfrak{A}_2, \mathfrak{e}_2, \eta_2)$ are L_1 and L_2-interpretations, respectively, such that

$$\mathfrak{A}_1 = (A, \mathrm{Rel}_{\mathfrak{A}_1}, \mathrm{Funk}_{\mathfrak{A}_1}, c_{\mathfrak{A}_1}),$$

$$\mathfrak{A}_2 = (A, \mathrm{Rel}_{\mathfrak{A}_2}, \mathrm{Funk}_{\mathfrak{A}_2}, c_{\mathfrak{A}_2})$$

i.e., the two structures have a common carrier set, and

$$L := L_1 \cap L_2.$$

(i) If $t \in \mathrm{Term}_L$ such that the interpretations $\mathcal{I}_1, \mathcal{I}_2$ agree on the non-logical L-symbols occurring in t, and they also agree on $V(t)$, the set of variables occurring in t i.e.,

$$\forall_{x \in V(t)}(\eta_1(x) = \eta_2(x)),$$

then show that

$$t^{\mathcal{I}_1} = t^{\mathcal{I}_2}.$$

(ii) If $\phi \in \mathrm{Form}_L$ such that the interpretations $\mathcal{I}_1, \mathcal{I}_2$ agree on the non-logical L-symbols occurring in ϕ, and they also agree on $FV(\phi)$, the free variables in ϕ i.e.,

$$\forall_{x \in FV(\phi)}(\eta_1(x) = \eta_2(x)),$$

then show that

$$\mathcal{I}_1 \models \phi \leftrightarrow \mathcal{I}_2 \models \phi.$$

Exercise 20. Using the Coincidence Lemma show that $\mathcal{I} \models \phi$ depends only on

(a) the finitely many values $e_1(R)$, $e_2(f)$, $e_3(c)$, where R, f, c range over the finitely many non-logical symbols of L occurring in ϕ, and

(b) the finitely many values $\eta(x)$, where x ranges over the finitely many variables occurring freely in ϕ.

Remark: Because of the above it is absolutely OK to use the suggestive notation of the lecture course notes

$$\mathfrak{A} \models \phi_{\vec{x}}[a_1, \dots, a_m],$$

where

$$\phi \in \mathrm{Var}^n \leftrightarrow \mathrm{FV}(\phi) \subseteq \{x_1, \dots, x_m\},$$

w.r.t. the fixed enumeration of Var, and $\eta(x_1) = a_1$, ..., $\eta(x_m) = a_m$.

Exercise 21. Suppose that $\overline{L}, L$ are 1st-order languages such that $\overline{L} \subseteq L$, and $T \subseteq S(\overline{L})$. Note then that also $T \subseteq S(L)$ (Why?). Then the following equivalence holds:

$$T \text{ is } \overline{L}\text{−satisfiable } \leftrightarrow T \text{ is } L\text{−satisfiable.}$$

7. On the relation of logical equivalence

If $\phi, \psi \in \mathrm{Form}$, we define $\phi \models\mathrel{\mkern-3mu}\dashv \psi$, "$\phi$ is logically equivalent to ψ", by

$$\begin{aligned}\phi \models\mathrel{\mkern-3mu}\dashv \psi &\leftrightarrow \phi \models \psi \text{ and } \psi \models \phi\\ &\leftrightarrow \forall_{\mathfrak{A}}(\mathfrak{A} \models \phi \text{ if and only if } \mathfrak{A} \models \psi).\end{aligned}$$

Exercise 22. Show the following:

(i) $\forall_x \forall_y \phi \models\mathrel{\mkern-3mu}\dashv \forall_y \forall_x \phi$.
(ii) $\exists_x \exists_y \phi \models\mathrel{\mkern-3mu}\dashv \exists_y \exists_x \phi$.
(iii) $\forall_x \phi \models\mathrel{\mkern-3mu}\dashv \phi$, if $x \notin \mathrm{FV}(\phi)$.
(iv) $\exists_x \phi \models\mathrel{\mkern-3mu}\dashv \phi$, if $x \notin \mathrm{FV}(\phi)$.
(v) $\forall_x (\phi \wedge \psi) \models\mathrel{\mkern-3mu}\dashv \forall_x \phi \wedge \forall_x \psi$.
(vi) $\exists_x (\phi \vee \psi) \models\mathrel{\mkern-3mu}\dashv \exists_x \phi \vee \exists_x \psi$.
(vii) $\forall_x (\phi \vee \psi) \models\mathrel{\mkern-3mu}\dashv \forall_x \phi \vee \psi$, if $x \notin \mathrm{FV}(\psi)$.
(viii) $\exists_x (\phi \wedge \psi) \models\mathrel{\mkern-3mu}\dashv \exists_x \phi \wedge \psi$, if $x \notin \mathrm{FV}(\psi)$.

Exercise 23. Show the following:

(i) If $\phi \models\mathrel{\mkern-3mu}\dashv \psi$, then $\neg\phi \models\mathrel{\mkern-3mu}\dashv \neg\psi$.
(ii) If $\phi \models\mathrel{\mkern-3mu}\dashv \psi$ and $\phi' \models\mathrel{\mkern-3mu}\dashv \psi'$, then $\phi \vee \phi' \models\mathrel{\mkern-3mu}\dashv \psi \vee \psi'$.
(iii) If $\phi \models\mathrel{\mkern-3mu}\dashv \psi$, then $\exists_x \phi \models\mathrel{\mkern-3mu}\dashv \exists_x \psi$.

We define the set $\mathrm{Sub}(\phi)$ of all subformulas of some formula ϕ recursively by:

$$\mathrm{Sub}(p) := \{p\},$$

$$\begin{aligned}\mathrm{Sub}(\neg\phi) &:= \mathrm{Sub}(\phi) \cup \{\neg\phi\},\\ \mathrm{Sub}(\phi \vee \psi) &:= \mathrm{Sub}(\phi) \cup \mathrm{Sub}(\psi) \cup \{\phi \vee \psi\},\\ \mathrm{Sub}(\exists_x \phi) &:= \mathrm{Sub}(\phi) \cup \{\exists_x \phi\},\end{aligned}$$

where $p \in \mathrm{Prim}$, the set of prime formulas in Form.

Exercise 24. If $\sigma \in \mathrm{Sub}(\phi)$ and $\sigma' \in \mathrm{Form}$, then

$$\text{if } \sigma \models\mathrel{\mkern-3mu}\dashv \sigma', \text{ then } \phi[\sigma'/\sigma] \models\mathrel{\mkern-3mu}\dashv \phi,$$

where $\phi[\sigma'/\sigma]$ is the formula resulting by substitution in ϕ of σ by σ'.

Exercise 25. Find the prenex normal form of the following formulas:

(i) $\forall_x (Rx \to \forall_y S(x, y))$.
(ii) $\forall_x (Rx \to \exists_y S(x, y))$.

(iii) $\neg\exists_x Rx \vee \forall_x Sx$.

Exercise 26. Find formulas ϕ, ψ which show the necessity of the variable condition in Exercise 22(iii) (iv) (vii), and (viii), respectively.

8. On first-order theories

If X is a set, and $\mathcal{P}(X)$ denotes the power set of X, a *closure operator* on X is a function C

$$C : \mathcal{P}(X) \to \mathcal{P}(X)$$
$$A \mapsto C(A),$$

satisfying the following properties:

(i) $A \subseteq C(A)$.
(ii) $A_1 \subseteq A_2 \to C(A_1) \subseteq C(A_2)$.
(iii) $C(C(A)) = C(A)$.

Exercise 27. Show that a closure operator C on X satisfies the following properties (actually the only necessary condition is (ii)):

(i) $\bigcup_{i\in I} C(A_i) \subseteq C(\bigcup_{i\in I} A_i)$.
(ii) $C(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} C(A_i)$,

where $(A_i)_{i\in I}$ is a family of subsets of X indexed by a set I.

If we fix a 1st-order language L, then we define the *logical closure operator* C on $S(L)$ (with respect to our fixed language L) as the function

$$C : \mathcal{P}(S(L)) \to \mathcal{P}(S(L))$$

$$T \mapsto C(T),$$

and

$$C(T) := \{\phi \in S(L) \mid T \models \phi\}.$$

As we know by exercise 3 of Blatt 4 the operator C is a closure operator on $S(L)$.
A theory T is called *closed*, if $C(T) = T$, which is equivalent, because of property (i) of a closure operator, to $C(T) \subseteq T$, in other words,

$$T \models \phi \to \phi \in T.$$

A special closure operator which is is fundamental in general topology is defined as follows: If X is a set, a *topological closure operator* on X is a function Cl

$$\mathrm{Cl} : \mathcal{P}(X) \to \mathcal{P}(X)$$

$$A \mapsto \mathrm{Cl}(A),$$

satisfying the following properties:

(i) Cl is a closure operator.
(ii) $\mathrm{Cl}(\emptyset) = \emptyset$.
(iii) $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$.

It is direct to see that the set

$$\mathcal{T}_{\mathrm{Cl}} := \{X \setminus \mathrm{Cl}(A) \mid A \subseteq X\}$$

is a topology on X, and that the sets of the form $A = \mathrm{Cl}(A)$ are the *closed* sets with respect to this topology. Conversely, if $(X, \mathcal{T})$ is a topological space, then the operator $\mathrm{Cl}_{\mathcal{T}}$ defined by

$$A \mapsto \overline{A},$$

where $A \subseteq X$ and $\overline{A}$ denotes the $\mathcal{T}$-closure of A i.e., the least $\mathcal{T}$-closed set including A, is a topological closure operator on X. We call the operator $\mathrm{Cl}_{\mathcal{T}}$ the topological closure operator *induced* by the topology $\mathcal{T}$.

Exercise 28. Find a (simple) topological space $(X, \mathcal{T})$ the induced topological closure operator $\mathrm{Cl}_{\mathcal{T}}$ of which does not satisfy the equalites in cases (i) and (ii) of Exercise 27, respectively.

Thus, although a topological closure operator satisfies more properties than the monotonicity condition (ii) of its definition, still these equalities do not hold in general.

A theory T is called *consistent*, if

$$\forall_{\phi \in S(L)}(T \not\models (\phi \wedge \neg\phi)) \leftrightarrow \nexists_{\phi \in S(L)}(T \models \phi \wedge \neg\phi).$$

A theory T is called *inconsistent*, if

$$\exists_{\phi \in S(L)}(T \models (\phi \wedge \neg\phi)).$$

A theory T is called *complete*, if

$$\forall_{\phi \in S(L)}(T \models \phi \vee T \models \neg\phi).$$

Obviously, a closed theory is complete, if

$$\forall_{\phi \in S(L)}(\phi \in T \vee \neg\phi \in T).$$

A theory T is called *incomplete*, if

$$\exists_{\phi \in S(L)}(T \not\models \phi \wedge T \not\models \neg\phi).$$

A theory T is called *finitely axiomatizable*, if

$$\exists_{F \subseteq^{\mathrm{fin}} T}(T = C(F)),$$

where $F \subseteq^{\text{fin}} T$ denotes that F is a finite subset of T.

Exercise 29. By the definition of the logical closure operator we have that

$$C(\emptyset) = \{\phi \in S(L) \mid \emptyset \models \phi\} = \{\text{valid sentences}\}.$$

(i) If T is a 1st-order theory, then

$$C(\emptyset) \subseteq C(T) \subseteq S(L),$$

and C does not satisfy condition (iv) of a topological closure operator.
(ii) Find theories T_1, T_2 satisfying

$$C(T_1) \cup C(T_2) \subsetneq C(T_1 \cup T_2),$$

i.e., C does not satisfy condition (v) of a topological closure operator.
(iii) The theory $S(L)$ is the maximum closed L-theory, and it is also inconsistent.
(iv) The theory $S(L)$ is the only inconsistent closed L-theory.
(v) Give an example of an incomplete theory.

Exercise 30. If $\mathfrak{A}$ is an L-structure, then we define *the theory of* $\mathfrak{A}$ by

$$\mathrm{Th}(\mathfrak{A}) := \{\phi \in S(L) \mid \mathfrak{A} \models \phi\}.$$

(i) $\mathrm{Th}(\mathfrak{A})$ is a closed theory.
(ii) $\mathrm{Th}(\mathfrak{A})$ is a complete theory.

A theory T is called *satisfiable* if there is some L-structure $\mathfrak{A}$ that satisfies T i.e.,

$$\exists_{\mathfrak{A}}(\mathfrak{A} \models T).$$

In this case we say that $\mathfrak{A}$ is a *model* of T. Clearly, T is satisfiable if and only if $\exists_\phi(T \not\models \phi)$.

Next exercise shows that a closed, satisfiable theory is complete if and only if it is the theory of some model of it.

Exercise 31. Suppose that T is a satisfiable and closed theory. Then the following are equivalent:

(i) T is complete.
(ii) If $\mathfrak{A}$ and $\mathfrak{B}$ are any models of T, then

$$\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{A}).$$

(iii) If $\mathfrak{A}$ is any model of T, then

$$T = \mathrm{Th}(\mathfrak{A}).$$

Note that you need the hypothesis "T is closed" in direction (ii) $\to$ (iii), but not in the direction (i) $\to$ (ii).

9. On isomorphism of L-structures

Exercise 32. Show that $<$ is not definable in $\mathfrak{A} = (\mathbb{R}, +, 0)$ i.e., there is no 1st-order formula of the language $L = (+, 0)$ with free variables x and y, such that

$$\forall_{a,b\in\mathbb{R}}(a < b \leftrightarrow \mathfrak{A} \models \phi_{x,y}[a, b]).$$

What about the definability of $<$ in $\mathfrak{B} = (\mathbb{R}, \cdot, 1)$?
Relate these facts with Exercise 3 of Blatt 5.

Exercise 33. If Pr is the set of prime numbers and

$$e : \mathrm{Pr} \to \mathrm{Pr}$$

is a function 1-1 and onto Pr, then

(i) There exists a unique automorphism $\hat{e}$ of the structure $(\mathbb{N}, \cdot, 1)$ extending e.
(ii) Show that addition is not definable within $(\mathbb{N}, \cdot, 1)$ (first formulate this question accordingly).

10. Some basic 1st-order theories

1. **Peano Arithmetic** PA.
Language: $L = (+, \cdot, S, 0)$.

Axioms:

1. $\neg S(x) = 0$.
2. $S(x) = S(y) \to x = y$.
3. $x + 0 = x$.
4. $x + S(y) = S(x + y)$.
5. $x \cdot 0 = 0$.
6. $x \cdot S(y) = x \cdot y + x$.

7_ϕ. $\phi(0) \to \forall_x(\phi(x) \to \phi(S(x))) \to \forall_x \phi(x)$.

2. **Partial Order** O.
Language: $L = (<)$.

Axioms:

1. $\neg x < x$.
2. $x < y \to y < z \to x < z$.
3. $x < y \to \neg y < x$.

3. **Linear Order** LO.
Language: $L = (<)$.

Axioms:

1. O.

2. $x < y \vee x = y \vee y < x$.

4. **Dense Linear Order** DLO.
Language: $L = (<)$.

Axioms:

1. LO.
2. $x < y \to \exists_z (x < z \wedge z < y)$.

5. **Dense Linear Order without Endpoints** DLO*.
Language: $L = (<)$.

Axioms:

1. DLO.
2. $\forall_x \exists_{y,z} (y < x \wedge x < z)$.

6. **Ring** R.
Language: $L = (+, \cdot, 0)$.

Axioms:

1. Abelian group w.r.t. $+$.
2. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
3. $x \cdot (y + z) = x \cdot y + x \cdot z$.
4. $(x + y) \cdot z = x \cdot z + y \cdot z$.

7. **Commutative Rings with Unit** CR.
Language: $L = (+, \cdot, 0, 1)$.

Axioms:

1. R.
2. $x \cdot y = y \cdot x$.
3. $1 \cdot x = x$.

8. **Integral Domain** ID.
Language: $L = (+, \cdot, 0, 1)$.

Axioms:

1. CR.
2. $x \cdot y = 0 \to x = 0 \vee y = 0$.

9. **Field** F.
Language: $L = (+, \cdot, 0, 1)$.

Axioms:

1. ID.
2. $x \neq 0 \to \exists_y (x \cdot y = 1)$.

10. **Field of characteristic** p F(p).
Language: $L = (+, \cdot, 0, 1)$.

Axioms:

1. F.
2. $p \cdot 1 = 0 \wedge (p-1) \cdot 1 \neq 0 \wedge \ldots \wedge 2 \cdot 1 \neq 0$.

11. **Field of characteristic** 0 F(0).
Language: $L = (+, \cdot, 0, 1)$.

Axioms:

1. F.
2_p. $p \cdot 1 \neq 0$.

12. **Ordered Field** OF.
Language: $L = (<, +, \cdot, 0, 1)$.

Axioms:

1. F.
2. LO.
3. $x < y \to x + z < y + z$.
4. $0 < x \to 0 < y \to 0 < x \cdot y$.

13. **Real closed Field** RCF.
Language: $L = (<, +, \cdot, 0, 1)$.

Axioms:

1. OF.
2. $0 < x \to \exists_y (x = y \cdot y)$.
3_{2n+1}. $x_{2n+1} \neq 0 \to \exists_y (x_{2n+1} y^{2n+1} + x_{2n} y^{2n} + \ldots + x_1 y + x_0 = 0)$.

14. **Algerbraic Closed Field** ACF.
Language: $L = (+, \cdot, 0, 1)$.

Axioms:

1. F.
2_n. $x_n \neq 0 \to \exists_y (x_n y^n + x_{n-1} y^{n-1} + \ldots + x_1 y + x_0 = 0)$.

11. On the theory DLO*

Exercise 34. Suppose that $(A, <)$ and $(B, <)$ are are two dense linear orders without endpoints (we keep for simplicity the same symbol for the order). Then the following hold:

(i) There cannot be a finite dense linear order without endpoints i.e., the carrier set cannot be finite.
(ii) Give an example of a countable dense linear order without endpoints.
(iii) Give an example of an uncountable dense linear order without endpoints.
(iv) If A, B are countable, then there exists $e : A \to B$ which is 1-1 and onto B satisfying

$$\forall_{a_1, a_2 \in A} (a_1 < a_2 \leftrightarrow e(a_1) < e(a_2)).$$

(v) If A, B are countable $a \in A$ and $b \in B$, then there exists $e_{ab} : A \to B$ which is 1-1 and onto B satisfying

$$\forall_{a_1, a_2 \in A}(a_1 < a_2 \leftrightarrow e_{ab}(a_1) < e_{ab}(a_2)),$$

and

$$e_{ab}(a) = b.$$

12. On the Compactness Theorem

Compactness Theorem: If T is a finitely satisfiable L-theory, then T is satisfiable.

The converse of the compactness theorem hods trivially, while its name is due to its equivalence to the compactness of a suitable topological space (see e.g., [1]).

Completeness Theorem: If T is an L-theory, and $\phi \in S(L)$ then

$$T \vdash \phi \leftrightarrow T \models \phi,$$

where $T \vdash \phi$ means that ϕ is derivable from T w.r.t. an appropriate concept of derivation. Actually, the simpler direction $(\to)$ is called the Soundness Theorem and the non-trivial direction $(\leftarrow)$ is Gödel's completeness theorem.

An L-theory T is *syntactically consistent*, if

$$\nexists_{\phi \in S(L)}(T \vdash \phi \wedge \neg\phi).$$

The next simple exercise shows that completeness is stronger than compactness.

Exercise 35. (i) If T is an L-theory, then

$$T \text{ is syntactically consistent } \leftrightarrow T \text{ is satisfiable.}$$

(ii) Show that the completeness theorem implies the compactness theorem.

It is not an accident that in the lecture course the compactness theorem is proved independently from the completeness theorem. The compactness theorem is a fundamental result of model theory, a branch of mathematical logic not concerned with formal provability but only with satisfiability (see [1], Chapter 2). On the other hand, the completeness theorem requires a fixed formal proof system in order to be even formulated, while we have seen that there is a purely semantic proof of the compactness theorem.

The compactness theorem has many important model-theoretic consequences.

Exercise 36. (i) If C is the logical closure operator on $S(L)$, then

$$\begin{aligned} C(T) &= \{\phi \in S(L) \mid T \models \phi\} \\ &= \bigcup_{F \subseteq^{\text{fin}} T} \{\phi \in S(L) \mid F \models \phi\}. \end{aligned}$$

(ii) If T is satisfiable, then $C(T)$ is satisfiable.
(iii) If T is satisfiable, then

$$T \text{ is complete } \leftrightarrow C(T) \text{ is } \subseteq -\text{maximal},$$

where $\subseteq$ is the inclusion relation on the set of satisfiable L-theories.
(ii) If $L = (+, 0, \cdot, 1)$ is the 1st-order language of fields, and $F(0)$, $F(p)$ are the theories of fields of characteristic 0 and of characteristic p, respectively (see section 10), then show that, if $\phi \in S(L)$

$$F(0) \models \phi \to \exists_{n \in \mathbb{N}} \forall_{p > n} (F(p) \models \phi).$$

Exercise 37. Suppose that T is an L-theory.
(i) If T has arbitrary large finite models, then T has an infinite model.
(ii) Show that if all models of T are finite, then the set of their cardinalities is bounded.
(iii) Show that there is no T having models exactly all L-structures with a finite carrier.
(iv) Show that if $\mathrm{Funk}_L \neq \emptyset$, there is some T having models only with infinite carrier.
(v) Show that there is no finite theory T having models exactly all L-structures with infinite carrier, since if some $\phi \in S(L)$ holds in every infinite L-structure, then there is some $m \in \mathbb{N}$ such that ϕ holds in every finite L-structure of cardinality $> m$.

Exercise 38. Suppose that L is a first-order language with a single constant symbol and a binary relation symbol $\leq$. Recall that a well-ordering $(D, \leq_D)$ is a total ordering such that each non-empty subset of D has a $\leq_D$-least element.
Show that there is no L-theory T such that

$$\mathfrak{A} = (A, \leq^{\mathfrak{A}}) \models T \leftrightarrow \mathfrak{A} \text{ is a well-ordering}.$$

(Hint: Assume that exists such a theory T, adjoin countably many new constants, extend T with appropriate infinitely many new L-sentences, and use the compactness theorem to reach a contradiction.)

Exercise 39. Suppose that κ is a cardinal such that $\kappa \geq \max(\aleph_0, |L|)$, and $T \subseteq S(L)$ with an infinite model. Then T has a model $\mathfrak{A}$ such that $|A| \geq \kappa$.

13. On the relation of elementary equivalence

If $\mathfrak{A}, \mathfrak{B}$ are L-structures we say that they are are *elementarily equivalent*, $\mathfrak{A} \equiv \mathfrak{B}$, if they satisfy the same L-sentences. Actually the following hold:

$$\begin{aligned}\mathfrak{A} \equiv \mathfrak{B} &\leftrightarrow \mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})\\ &\leftrightarrow \forall_{\phi \in S(L)}(\mathfrak{A} \models \phi \leftrightarrow \mathfrak{B} \models \phi)\\ &\leftrightarrow \mathfrak{A} \models \mathrm{Th}(\mathfrak{B})\\ &\leftrightarrow \mathfrak{B} \models \mathrm{Th}(\mathfrak{A}).\end{aligned}$$

Exercise 40. (i) If $|L| = \kappa \geq \aleph_0$, and $\mathfrak{A}, \mathfrak{B}$ range over L-structures, then

$$\forall_{\mathfrak{A}}(|\mathfrak{A}| \geq \aleph_0 \to \forall_{\lambda \geq \kappa} \exists_{\mathfrak{B}}(|\mathfrak{B}| = \lambda \ \wedge \ \mathfrak{A} \equiv \mathfrak{B})).$$

(ii) Show that $\mathfrak{A} \cong \mathfrak{B} \to \mathfrak{A} \equiv \mathfrak{B}$, and give a counterexample to the converse implication.

Exercise 41. If $\mathbb{F} = (F, +, 0, \cdot, 1, <)$ is an ordered field, an $x \in F$ is called *positive*, if $0 < x$. A positive x is called an *infinitesimal*, if

$$\forall_{n \in \mathbb{N}}(x < \underline{n}^{-1}),$$

where $\underline{n} = \underbrace{1 + 1 + \ldots + 1}_{n}$. A positive x is called *infinitely large*, if

$$\forall_{n \in \mathbb{N}}(\underline{n} < x).$$

Obviously, x is an infinitesimal iff x^{-1} is infinitely large. An ordered field $\mathbb{F}$ is called *Archimedean*, if it contains no infinitesimals, while it is called *non-Archimedean*, if it contains infinitesimals.

(i) Using the compactness theorem show that there exists a non-Archimedean ordered field $\mathcal{R}$ which is elementarily equivalent to the Archimedean ordered field $\mathbb{R}$ i.e., $\mathcal{R} \equiv \mathbb{R}$.
(ii) Since the existence of an infinitely large positive element can be given by the sentence

$$\exists_x(x > 0 \ \wedge \ \forall_{n \in \mathbb{N}}(\underline{n} < x))$$

why this doesn't contradict the result $\mathcal{R} \equiv \mathbb{R}$?
(iii) Show that the 1st-order theory DLO* (section 11) is not c-categorical i.e., not all models of DLO* of cardinality c (the cardinality of $\mathbb{R}$) are isomorphic.

Exercise 42. If $\mathfrak{N} = (\mathbb{N}, +, 0, \cdot, 1)$ we define a *non-standard model of arithmetic* to be a structure $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{N}$ and $\mathfrak{A} \not\cong \mathfrak{N}$. Then the following hold:

(i) There exist uncountable non-standard models of arithmetic.
(ii) There exist countable non-standard models of arithmetic.

Exercise 43. If $\mathfrak{N}^{<} = (\mathbb{N}, +, 0, \cdot, 1, <)$ we define a *non-standard model of arithmetic with order* to be a structure $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{N}^{<}$ and $\mathfrak{A} \not\cong \mathfrak{N}^{<}$. Then the following hold:

(i) There exist uncountable non-standard models of arithmetic with order.
(ii) There exist countable non-standard models of arithmetic with order.
(iii) If $\mathfrak{A}$ is a countable non-standard models of arithmetic with order, then the following hold:

(a) $<^{\mathfrak{A}}$ is a total ordering.
(b) $<^{\mathfrak{A}}$ is not a complete ordering, i.e., there is a $<^{\mathfrak{A}}$-bounded subset of $\mathfrak{A}$ without having a $<^{\mathfrak{A}}$-least upper bound in $\mathfrak{A}$.
(c) There is an infinite $<^{\mathfrak{A}}$-decreasing sequence in $\mathfrak{A}$.
(d) An element of $\mathfrak{A}$ outside the copy of $\mathbb{N}$ in $\mathfrak{A}$ is called a *non-standard number* of $\mathfrak{A}$, while an element of the copy of $\mathbb{N}$ in $\mathfrak{A}$ is called a *standard number* of $\mathfrak{A}$. Show that a non-standard number of $\mathfrak{A}$ generates a countable set of copies of $\mathbb{Z}$ in $\mathfrak{A}$.
(e) If a is a non-standard number of $\mathfrak{A}$ and if m, n are fixed positive numbers, there is a non-standard number b of $\mathfrak{A}$ such that $n \cdot a$ and $m \cdot b$ differ by a standard number of $\mathfrak{A}$.
(iv) Show that between the standard numbers of $\mathfrak{A}$ and any copy of $\mathbb{Z}$ in $\mathfrak{A}$ there exists another copy of $\mathbb{Z}$ $<^{\mathfrak{A}}$-between them.
(v) Show that between any two copies of $\mathbb{Z}$ in $\mathfrak{A}$ there exists another copy of $\mathbb{Z}$ $<^{\mathfrak{A}}$-between them.
(vi) Using the main idea of Cantor's proof of the $\aleph_0$-categoricity of DLO* (section 11) show that there is a unique, up to isomorphism, countable non-standard model of arithmetic with order.
(vii) Use Exercise 1, Blatt 8, to show that there are uncountably many non-isomorphic countable non-standard models of arithmetic (alone). I.e., there is a big difference in the amount of countable non-standard models between the structures $\mathfrak{N}$ and $\mathfrak{N}^{<}$.

14. On Sequent Calculus

Exercise 44. Which of the following rules is correct? If so find a derivation of it.

$$\frac{\Gamma\phi_1\psi_1 \quad \Gamma\phi_2\psi_2}{\Gamma(\phi_1 \vee \phi_2)(\psi_1 \vee \psi_2)},$$

$$\frac{\Gamma\phi_1\psi_1 \quad \Gamma\phi_2\psi_2}{\Gamma(\phi_1 \vee \phi_2)(\psi_1 \wedge \psi_2)}.$$

Exercise 45. Show that the following rules are derivable:

$$\frac{}{\phi \vee \neg\phi},$$

$$\frac{\Gamma\phi\psi}{\Gamma\neg\psi\neg\phi},$$

$$\frac{\Gamma(\phi \vee \psi) \quad \Gamma\neg\phi}{\Gamma\psi},$$

$$\frac{\Gamma(\phi \to \psi) \quad \Gamma\phi}{\Gamma\psi},$$

$$\frac{\Gamma\neg\neg\phi}{\Gamma\phi},$$

$$\frac{\Gamma\phi \quad \Gamma\psi}{\Gamma(\phi \wedge \psi)},$$

$$\frac{\Gamma\ \forall_x\phi}{\Gamma\phi_x(t)}.$$

Exercise 46. Derive the following sequents:

$$\phi\ (\phi \vee \psi),$$
$$(\phi \vee \psi)\ \neg\phi\ \psi.$$

15. On Recursive functions

Exercise 47. Show that the following functions are recursive:

(i) The predecessor of n.
(ii) $f(n, m) = |n - m|$.
(iii) $\min(n, m)$.
(iv) $\max(n, m)$.

Exercise 48. (i) Using the primitive recursion scheme (p.42 of the lecture course notes) show that if $f : \mathbb{N}^{n+1} \to \mathbb{N}$ is recursive, then the functions

$$g(m, \vec{a}) = \sum_{i \leq m} f(i, \vec{a})$$

$$h(m, \vec{a}) = \prod_{i \leq m} f(i, \vec{a})$$

are recursive.
(ii) If $G : \mathbb{N}^{n+1} \to \mathbb{N}$ is a recursive function, then the function $F : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$F(\vec{a}, m) = \begin{cases} \mu n \leq m.G(\vec{a}, n) = 0 & , \text{if } \exists_{n \leq m}(G(\vec{a}, n) = 0) \\ 0 & , \text{ow} \end{cases}$$

is recursive (do not use the μ-operator scheme).

Exercise 49. Show that the if f is recursive, then the function on $\mathbb{N}^2$ defined by

$$f^0(m) = m$$
$$f^{n+1}(m) = f(\underbrace{f(\ldots f}_{n}(m)\ldots))$$

is recursive.

Exercise 50. Let $f(n) = g_1(n)$, if n is a perfect cube, and $f(n) = g_2(n)$ otherwise. Show that if g_1, g_2 are recursive, then f is also recursive.

Exercise 51. Let

$$h_1(0, n) = f_1(n)$$
$$h_2(0, n) = f_2(n)$$
$$h_1(m + 1, n) = g_1(h_1(m, n), h_2(m, n), n)$$
$$h_2(m + 1, n) = g_2(h_1(m, n), h_2(m, n), n)$$

Show that if f_1, f_2, g_1, g_2 are recursive, then h_1, h_2 are recursive.

Exercise 52. Show that the function of the Fibonacci numbers

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n + 2) = f(n) + f(n + 1)$$

is recursive.

16. On Recursively enumerable sets

An $R \subseteq \mathbb{N}^n$ is called *recursively enumerable* (r.e.) iff

$$\exists_{Q \subseteq \mathbb{N}^{n+1}}(Q \in \mathrm{Rek}^{n+1} \wedge \forall_{\vec{a}}(R(\vec{a}) \leftrightarrow \exists_{b \in \mathbb{N}}(Q(\vec{a}, b)))).$$

Verify that from the above definition we have an algorithm for answering only Yes in the question if $R(\vec{a})$, and not generally one for answering No.

Exercise 53. Show the following:

(i) $\mathbb{N}$ is recursive.
(ii) If A, B are recursive sets, then $A \times B$ is recursive.
(iii) If R is recursive, then $R, \mathbb{N}^n \setminus R$ are recursively enumerable.
(iv) If A, B are recursively enumerable sets, then $A \times B$ is recursively enumerable.

Exercise 54. Show that the following are equivalent:

(i) R is recursive and infinite.
(ii) There exists $f : \mathbb{N} \to \mathbb{N}$ recursive and strictly monotone such that $R = \mathrm{rng}(f)$.

Exercise 55. (i) Suppose that $R \subseteq \mathbb{N}$ is recursively enumerable and $k \in \mathbb{N}$. Find a recursive function $g_{R,k} : \mathbb{N} \to \{0, 1\}$ such that

$$\forall_n(g_{R,k}(n) = 0) \leftrightarrow k \notin R.$$

(ii) Suppose that $g : \mathbb{N} \to \{0, 1\}$ is recursive. Then, there is no decision procedure to show

$$\forall_n(g(n) = 0) \ \vee \ \exists_n(g(n) = 1).$$

(iii) One can use the above to show that $a = 0$ is not decidable in the set of computable reals.

17. On representable sets

Consider the definition of a representable set in a theory T as it is given in p.47 of the lecture notes and extend the definition of a weakly representable subset of $\mathbb{N}$ (p.51 of the lecture notes) to a subset of $\mathbb{N}^n$ in the obvious way.

Exercise 56. If T is a consistent theory and $A \subseteq \mathbb{N}^n$ is representable in T, then A and A^c (the complement of A) are weakly representable in T.

Exercise 57. An $R \subseteq \mathbb{N}^n$ is called *definable* iff there is an L_0-formula (p.48) ϕ such that

$$(m_1, \ldots, m_k) \in R \leftrightarrow \mathbb{N} \models \phi(s_{m_1}, \ldots, s_{m_k}).$$

Show that if A is representable, then A is definable.

Exercise 58. A formula ϕ is called *bounded*, or Σ_0, if its quantifiers (if there are any) are all bounded, and it is called Σ_1, if it is of the form $\exists_{\vec{x}}\phi$, where ϕ is bounded.
(i) Show that if ϕ is a Σ_1-sentence of L_0, then

$$\mathbb{N} \models \phi \to N \vdash \phi,$$

i.e., a Σ_1-sentence is true iff it is provable.
(ii) Each Σ_1-set i.e., definable by some Σ_1-formula, is weakly representable.

18. Miscellaneous

Exercise 59. If $\mathfrak{A} = (\mathbb{Q}, +, <)$, show that there is no $(+, <)$-formula with x as free variable such that

$$\forall_{a \in \mathbb{Q}}(a \in \mathbb{N} \leftrightarrow \mathfrak{A} \models \phi_x[a]).$$

In other words, we cannot define $\mathbb{N}$ from $\mathbb{Q}$ by a 1st-order formula.

Exercise 60. Let $L = L_0 \cup \{c\}$, where c is a new constant symbol, and T is the L-theory which includes exactly the following formulas:

$$0 < c$$

$$S(0) < c$$

$$S(S(0)) < c$$

$$\ldots$$

Is T complete? If not, do we need Gödel's incompleteness theorem for that?

Exercise 61. (i) Show that if $K \subseteq \mathbb{N}$ is recursively enumerable and not recursive, then K and $\mathbb{N} \setminus K$ are infinite.
(ii) If $Q_1, Q_2 \subseteq \mathbb{N}$ are recursively enumerable, is $Q_1 \setminus Q_2$ recursively enumerable?
(iii) If $Q \subseteq \mathbb{N}$ is recursively enumerable and $A \subseteq \mathbb{N}$ is recursive, then $Q \setminus A$ is recursively enumerable.
(iv) How many recursive, recursively enumerable, and not recursively enumerable sets are there?
(v) Show that there are infinitely many recursively enumerable subsets of $\mathbb{N}$ which are not recursive.

Exercise 62. (i) If $A \subseteq \mathbb{N}$ is recursive and $f : \mathbb{N} \to \mathbb{N}$ is a recursive bijection, then $f(A)$ is recursive.

(ii) Is the set $\{n^3 \mid n \in \mathbb{N}\}$ recursive?

Exercise 63. If $\mathfrak{N} = (\mathbb{N}, 0, +, S, \cdot, <)$, then show that $\mathrm{Th}(\mathfrak{N})$ is not recursive.

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