# Logik - WS16/17 

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These notes include part of the material discussed in the Exercises that correspond to the lecture course "Logik" of Priv.-Doz. Dr. Josef Berger. Some extra exercises and questions can be found here.

Please feel free to send me your comments, or your suggestions regarding these notes.

## 1 Inductive sets

An inductive set $X$ is determined by three rules, or axioms.
(i) The formation rules Form $_{X}$ for $X$, which determine the way the elements of $X$ are formed (in the literature they are also called the introduction rules for $X$ ).
(ii) The induction principle $\operatorname{Ind}_{X}$ for $X$, which guarantees that $X$ is the least set satisfying its formation rules.
(iii) The recursion principle $\operatorname{Rec}_{X}$ for $X$, which determines the way functions of type $X \rightarrow Y$, where $Y$ is any other set, are defined.
The importance of an inductive set lies on the following facts:
(a) Through Form $_{X}$ we have a concrete way to grasp and manipulate the elements of $X$.
(b) Through $\operatorname{Ind}_{X}$ we have a powerful tool to prove the properties of $X$.
(c) Through $\operatorname{Rec}_{X}$ we have a concrete way to grasp and manipulate the functions defined on $X$.
Notational convention 1: If $X, Y, Z$ are sets, then the symbol

$$
X \rightarrow Y
$$

denotes the set of all functions from $X$ to $Y$, and the symbol

$$
X \rightarrow Y \rightarrow Z
$$

denotes the set

$$
X \rightarrow(Y \rightarrow Z)
$$

i.e., the functions from $X$ to the set of functions from $Y$ to $Z$. If $h \in X \rightarrow$ $Y \rightarrow Z$, then $h(x): Y \rightarrow Z$ and we write

$$
h(x)(y)=: h(x, y),
$$

for every $x \in X$ and $y \in Y$.
Notational convention 2: If $P, Q, R$ are propositions, the proposition

$$
P \rightarrow Q \rightarrow R
$$

means

$$
\text { if } P \text { and if } Q \text {, then } R \text {. }
$$

### 1.1 The inductive set $\mathbb{N}$ of naturals

The most fundamental example of an inductive set is that of the set of natural numbers $\mathbb{N}$.
(i) $\operatorname{Form}_{\mathbb{N}}$ :

$$
\overline{0 \in \mathbb{N}}, \quad \frac{n \in \mathbb{N}}{\operatorname{Succ}(n) \in \mathbb{N}} .
$$

According to Form $_{\mathbb{N}}$, the elements of $\mathbb{N}$ are formed by the element 0 and by the primitive, or given function Succ of type $\mathbb{N} \rightarrow \mathbb{N}$. The principle Form ${ }_{\mathbb{N}}$ alone does not determine a unique set; for example the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$ satisfy the same rules. We determine $\mathbb{N}$ by postulating that $\mathbb{N}$ is the least set satisfying the above rules. This we do with the induction principle for $\mathbb{N}$.
(ii) $\operatorname{Ind}_{\mathbb{N}}$ : If $A$ is any property, or predicate on $\mathbb{N}$, then

$$
A(0) \rightarrow \forall_{n}(A(n) \rightarrow A(\operatorname{Succ}(n))) \rightarrow \forall_{n}(A(n))
$$

The interpretation of $\operatorname{Ind}_{\mathbb{N}}$ is the following: The hypotheses of $\operatorname{Ind}_{\mathbb{N}}$ say that $A$ satisfies the two formation rules for $\mathbb{N}$ i.e., $A(0)$ and $\forall_{n}(A(n) \rightarrow A(\operatorname{Succ}(n)))$. In this case $A$ is a "competitor" predicate to $\mathbb{N}$. Then, if we view $A$ as the set of all objects such that $A(n)$, the conclusion of $\operatorname{Ind}_{\mathbb{N}}$ guarantees that $\mathbb{N} \subseteq A$, i.e., $\forall_{n}(A(n))$, or more precisely, $\forall_{x}(\mathbb{N}(x) \rightarrow A(x))$. In other words, $\mathbb{N}$ is "smaller" than $A$, and this is the case for any $A$.
(iii) $\operatorname{Rec}_{\mathbb{N}}$ : If $x_{0} \in X$ and $g: X \rightarrow X$, there exists $f: \mathbb{N} \rightarrow X$ such that

$$
\begin{gathered}
f(0)=x_{0} \\
f(\operatorname{Succ}(n))=g(f(n))
\end{gathered}
$$

Note that the uniqueness of $f$ with the above properties can be proved by $\operatorname{Ind}_{\mathbb{N}}$; If $h: \mathbb{N} \rightarrow X$ is a function satisfying the above two conditions, we

As an example of a function defined through $\operatorname{Rec}_{\mathbb{N}}$, we define the function Double : $\mathbb{N} \rightarrow \mathbb{N}$ by

$$
\text { Double }(0)=0,
$$

$$
\operatorname{Double}(\operatorname{Succ}(n))=\operatorname{Succ}(\operatorname{Succ}(\operatorname{Double}(n)))
$$

i.e., $X=\mathbb{N}$, $x_{0}=0$ and $g=$ Succ $\circ$ Succ.

Task 1: Show that Double $(2)=\operatorname{Double}(\operatorname{Succ}(\operatorname{Succ}(0)))=4$.
Task 2: Use $\operatorname{Rec}_{\mathbb{N}}$ in order to define the function Add : $\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$, where $\operatorname{Add}(n, m)$ is the addition $n+m$ of $n, m$.

### 1.2 The inductive set $\mathcal{L}^{*}$ of $\mathcal{L}$-words

(i) $\operatorname{Form}_{\mathbb{N}}$ :

$$
\overline{\operatorname{Nil}_{\mathcal{L}^{*}} \in \mathcal{L}^{*}}, \quad \frac{W \in \mathcal{L}^{*}, s \in \mathcal{L}}{W \star s \in \mathcal{L}^{*}} .
$$

The word $W \star s$ denotes the concatenation of the word $W$ and the symbol $s$.
(ii) $\operatorname{Ind}_{\mathcal{L}^{*}}$ : If $A$ is any property, or predicate on $\mathcal{L}^{*}$, then

$$
A\left(\mathrm{Nil}_{\mathcal{L}^{*}}\right) \rightarrow \forall_{W \in \mathcal{L}^{*}} \forall_{s \in \mathcal{L}}(A(W) \rightarrow A(W \star s)) \rightarrow \forall_{W \in \mathcal{L}^{*}}(A(W))
$$

(iii) $\operatorname{Rec}_{\mathcal{L}^{*}}$ : If $x_{0} \in X$, and if $g_{s}: X \rightarrow X$, for every $s \in \mathcal{L}$, there is a function $f: \mathcal{L}^{*} \rightarrow X$ such that

$$
\begin{gathered}
f\left(\operatorname{Nil}_{\mathcal{L}^{*}}\right)=x_{0}, \\
f(W \star s)=g_{s}(f(W)),
\end{gathered}
$$

for every $W \in \mathcal{L}^{*}$ and $s \in \mathcal{L}$.
As an example of a function defined through $\operatorname{Rec}_{\mathcal{L}^{*}}$, if $W_{0} \in \mathcal{L}^{*}$ and if $g_{s}(W)=W \star s$, for every $s \in \mathcal{L}$, we define the function $f_{W_{0}}: \mathcal{L}^{*} \rightarrow \mathcal{L}^{*}$ by

$$
\begin{gathered}
f_{W_{0}}\left(\mathrm{Nil}_{\mathcal{L}^{*}}\right)=W_{0} \\
f_{W_{0}}(W \star s)=g_{s}\left(f_{W_{0}}(W)\right)
\end{gathered}
$$

i.e., $f_{W_{0}}(W)=W_{0} \star W$ is the concatenation of the words $W_{0}$ and $W$ (for simplicity we use the same symbol for the concatenation of a word and a symbol and the concatenation of two words).
An important example of a function defined on $\mathcal{L}^{*}$ is the substitution function $\operatorname{Sub}_{\left[W^{\prime} / z\right]}: \mathcal{L}^{*} \rightarrow \mathcal{L}^{*}$, where its value

$$
\operatorname{Sub}_{\left[W^{\prime} / z\right]}=: W\left[W^{\prime} / z\right]
$$

expresses the substitution of the letter $z \in \mathcal{L}$ occurring in $W$ by the word $W^{\prime}$. Using $\operatorname{Rec}_{\mathcal{L}^{*}}$ we define $\operatorname{Sub}_{\left[W^{\prime} / z\right]}$ by

$$
\operatorname{Sub}_{\left[W^{\prime} / z\right]}\left(\operatorname{Nil}_{\mathcal{L}^{*}}\right)=\operatorname{Nil}_{\mathcal{L}^{*}}\left[W^{\prime} / z\right]=\operatorname{Nil}_{\mathcal{L}^{*}},
$$

$$
\operatorname{Sub}_{\left[W^{\prime} / z\right]}(W \star s)=(W \star s)\left[W^{\prime} / z\right]= \begin{cases}W\left[W^{\prime} / z\right] \star s & , \text { if } s \neq z \\ W\left[W^{\prime} / z\right] \star W^{\prime} & , s=z\end{cases}
$$

Note that $W\left[W^{\prime} / z\right] \star W^{\prime}$ is the concatenation of the words $W\left[W^{\prime} / z\right]$ and $W^{\prime}$. We can now easily show the following facts.

Fact 1: If $W_{1}, W_{2} \in \mathcal{L}^{*}$, then

$$
\left(W_{1} \star W_{2}\right)\left[W^{\prime} / z\right]=W_{1}\left[W^{\prime} / z\right] \star W_{2}\left[W^{\prime} / z\right] .
$$

Fact 2: If $W_{1}, W_{2}, W_{3} \in \mathcal{L}^{*}$, then

$$
W_{1} \star\left(W_{2} \star W_{3}\right)=\left(W_{1} \star W_{2}\right) \star W_{3} .
$$

We can define inductively the membership $z \dot{\in} W$ and the non-membership $z \dot{\notin W}$ of an $\mathcal{L}$-symbol $z$ in an $\mathcal{L}$-word $W$, respectively, as follows:

$$
\begin{array}{ll}
\overline{z \dot{\in} W \star z}, & \frac{z \dot{\in} W, s \in \mathcal{L}}{z \dot{\in} W \star s}, \\
\overline{z \dot{\notin \operatorname{Nil}_{\mathcal{L}^{*}}},}, \frac{z \notin W, z \neq s}{z \dot{\nexists W \star s}}
\end{array}
$$

Now we can easily prove the following expected fact.
Fact 3: $\forall_{W \in \mathcal{L}^{*}}\left(\forall_{z \in \mathcal{L}}\left(z \notin W \rightarrow \forall_{\Delta \in \mathcal{L}^{*}}(W[\Delta / z]=W)\right)\right)$.
Using the above facts one can show Lemma 1 of the lecture course notes.

### 1.3 The inductive set $\mathcal{T}$ of $\mathcal{L}$-terms

(i) $\operatorname{Form}_{\mathcal{T}}$ :

$$
\frac{v \in \mathrm{FV}}{v \in \mathcal{T}}, \quad \frac{c \in \text { Const }}{c \in \mathcal{T}}, \quad \frac{f \in \text { Funct }^{(n)}, t_{1}, \ldots, t_{n} \in \mathcal{T}, n \in \mathbb{N}}{f\left(t_{1}, \ldots, t_{n}\right) \in \mathcal{T}} .
$$

(ii) $\operatorname{Ind}_{\mathcal{T}}$ :

$$
\begin{aligned}
& \forall_{v \in \mathrm{FV}}(A(v)) \rightarrow \\
& \forall_{c \in \text { Const }}(A(c)) \rightarrow \\
& \forall_{n \in \mathbb{N}} \forall_{f \in \text { Funct }^{(n)}} \forall_{t_{1}, \ldots, t_{n} \in \tau}\left(A\left(t_{1}\right) \rightarrow \ldots \rightarrow A\left(t_{n}\right) \rightarrow A\left(f\left(t_{1}, \ldots, t_{n}\right)\right)\right) \rightarrow \\
& \forall_{t \in \mathcal{T}}(A(t)) .
\end{aligned}
$$

(iii) $\operatorname{Rec}_{\mathcal{T}}:$ If $g: \mathrm{FV} \rightarrow X, h:$ Const $\rightarrow X$ and $F_{f, n}: X \times \ldots \times X \rightarrow X$, for every $f \in$ Funct $^{(n)}$ and $n \in \mathbb{N}$, there is $F: \mathcal{T} \rightarrow X$ such that

$$
\begin{gathered}
F(v)=g(v), \\
F(c)=h(c), \\
F\left(f\left(t_{1}, \ldots, t_{n}\right)\right)=F_{f, n}\left(F\left(t_{1}\right) \ldots F\left(t_{n}\right)\right) .
\end{gathered}
$$

Through $\operatorname{Rec}_{\mathcal{T}}$ we define the complexity $\|t\|$ of a term $t$ as a function $\|$.$\| :$ $\mathcal{T} \rightarrow \mathbb{N}$ by the following conditions:

$$
\begin{gathered}
\|u\|=\|c\|=0 \\
\left\|f\left(t_{1}, \ldots, t_{n}\right)\right\|=1+\sum_{i=1}^{n}\left\|t_{i}\right\| .
\end{gathered}
$$

### 1.4 The inductive set $\mathcal{F}$ of $\mathcal{L}$-formulas

(i) Form $_{\mathcal{F}}$ :

$$
\begin{aligned}
& \overline{\perp \in \mathcal{F}}, \quad \frac{R \in \operatorname{Rel}^{(n)}, t_{1}, \ldots, t_{n} \in \mathcal{T}, n \in \mathbb{N}}{R\left(t_{1}, \ldots, t_{n}\right) \in \mathcal{F}}, \quad \frac{s, t \in \mathcal{T}}{s \doteq t \in \mathcal{F}}, \\
& \frac{\phi, \psi \in \mathcal{F}}{\phi \square \psi \in \mathcal{F}}, \quad \square \in\{\wedge, \vee, \rightarrow\} . \\
& \frac{\phi \in \mathcal{F}, v \in \mathrm{FV}, x \in \mathrm{BV}, x \notin \phi}{\triangle x \phi[x / v] \in \mathcal{F}}, \quad \triangle \in\{\forall, \exists\} .
\end{aligned}
$$

(ii) $\operatorname{Ind}_{\mathcal{F}}$ :

$$
\begin{aligned}
& A(\perp) \rightarrow \\
& \forall_{n \in \mathbb{N}} \forall_{R \in \operatorname{Rel}^{(n)}} \forall_{t_{1}, \ldots, t_{n} \in \mathcal{T}}\left(A\left(R\left(t_{1}, \ldots, t_{n}\right)\right)\right) \rightarrow \\
& \forall_{s, t \in \mathcal{T}}(A(s \doteq t)) \rightarrow \\
& \forall_{\phi, \psi \in \mathcal{F}}(A(\phi) \rightarrow A(\psi) \rightarrow A(\phi \square \psi)) \rightarrow \\
& \forall_{\phi \in \mathcal{F}} \forall_{v \in \mathrm{FV}} \forall_{x \in \operatorname{BV}}(x \notin \phi \rightarrow A(\phi) \rightarrow A(\triangle x \phi[x / v])) \rightarrow \\
& \forall_{\phi \in \mathcal{F}}(A(\phi)) .
\end{aligned}
$$

(iii) $\operatorname{Rec}_{\mathcal{F}}$ : If $x_{0} \in X$,

$$
\begin{gathered}
\phi_{\text {Rel }}:\left\{R\left(t_{1}, \ldots, t_{n}\right) \mid R \in \operatorname{Rel}^{(n)}, t_{1}, \ldots, t_{n} \in \mathcal{T}, n \in \mathbb{N}\right\} \rightarrow X, \\
\phi_{\mathcal{T}}:\{s \doteq t \mid s, t \in \mathcal{T}\} \rightarrow X, \\
\phi_{\square}: X \times X \rightarrow X \\
\phi_{x, v, \Delta}: X \rightarrow X,
\end{gathered}
$$

there is a function $\Phi: \mathcal{F} \rightarrow X$ such that

$$
\begin{gathered}
\Phi(\perp)=x_{0} \\
\Phi\left(R\left(t_{1}, \ldots, t_{n}\right)\right)=\phi_{\operatorname{Rel}}\left(R\left(t_{1} \ldots t_{n}\right)\right) \\
\Phi(s \doteq t)=\phi_{\mathcal{T}}(s \doteq t) \\
\Phi(\phi \square \psi)=\phi_{\square}(\Phi(\phi), \Phi(\psi)) \\
\Phi(\triangle x \phi[x / v])=\phi_{x, v, \Delta}(\Phi(\phi)) .
\end{gathered}
$$

Through $\operatorname{Rec}_{\mathcal{F}}$ we define the complexity $\|\phi\|$ of a formula $\phi$ as a function $\|\|:. \mathcal{F} \rightarrow \mathbb{N}$ by the following conditions:

$$
\begin{gathered}
\|\perp\|=\left\|R\left(t_{1}, \ldots, t_{n}\right)\right\|=\|s \doteq t\|=0 \\
\|\phi \square \psi\|=\|\phi\|+\|\psi\|+1, \\
\|\triangle x \phi[x / v]\|=1+\|\phi\| .
\end{gathered}
$$

## 2 Natural deduction

The second problem of Hilbert's famous 1900-list was to find a proof of the consistency of arithmetic. That is to show that there can be no derivation of the absurdity $\perp$ from the Peano axioms. It took more than 30 years to understand in a concrete mathematical way all words appearing in the formulation of this problem. The standard understanding regarding its "solution" in [3] is the following:

There is no consensus on whether results of Gödel and Gentzen give a solution to the problem as stated by Hilbert. Gödel's second incompleteness theorem, proved in 1931, shows that no proof of its consistency can be carried out within arithmetic itself. Gentzen proved in 1936 that the consistency of arithmetic follows from the well-foundedness of the ordinal $\epsilon_{0}$.

The passage from proving mathematical theorems to treating mathematical proofs as objects of mathematical study is a major conceptual step that happened after 2.500 years of standard mathematical practice.
The Brouwer-Heyting-Kolmogoroff interpretation (BHK-interpretation for short) of intuitionistic logic appeared before Gentzen's definition and explains what it means to prove a logically compound statement in terms of what it means to prove its components; the explanations use the notions of construction and constructive proof as unexplained, primitive notions. The notation

$$
\Pi(p, \phi)
$$

means that $p$ is a proof of formula $\phi$. For quantifier-free formulas ${ }^{1}$ the clauses of BHK are the following (see [2], p.55):

- For atomic formulas, except $\perp$, the notion of proof is supposed to be given.
- There is no $p$ such that $\Pi(p, \perp)$.
- $\Pi(p, \phi \wedge \psi)$ if and only if $p=\left(p_{1}, p_{2}\right)$ and $\Pi\left(p_{1}, \phi\right), \Pi\left(p_{2}, \psi\right)$.
- $\Pi(p, \phi \vee \psi)$ if and only if $p=\left(1, p_{1}\right)$ and $\Pi\left(p_{1}, \phi\right)$, or $p=\left(2, p_{2}\right)$ and $\Pi\left(p_{2}, \psi\right)$.
- $\Pi(p, \phi \rightarrow \psi)$ if and only if for every proof $q$ such that $\Pi(q, \phi)$, then $\Pi(p(q), \psi)$.

Note that in the previous clauses the "if" corresponds to "introduction" and the "only if" to "elimination".
Gentzen went further and gave an inductive definition of a concrete notion of derivation based on the inductive definition of a first-order formula.

## 3 The Gödel-translation

The Gödel-translation is a translation of classical logic into intuitionistic (or minimal) logic. It was invented by Gödel and independently by Gentzen in 1933. For this reason it is also called the Gödel-Gentzen translation.

[^0]Task 1: Explain why the range of the Gödel-translation ${ }^{\circ}$ is in the set of formulas. You need to show the following:

$$
\forall_{x} \forall_{\phi}\left(x \notin \phi \Rightarrow x \notin \phi^{\circ}\right)
$$

Task 2: Explain why ${ }^{\circ}$ is not onto $\mathcal{F}$.
Task 3: The negative formulas $\mathcal{F}^{-}$are defined by the following clauses:

$$
P \rightarrow \perp|\perp| \phi \wedge \psi|\phi \rightarrow \psi| \forall x \phi[x / v]
$$

where $P$ denotes a prime formula. Check that the range of ${ }^{\circ}$ is in $\mathcal{F}^{-}$.
The most important feature of the Gödel-translation is that it is a proof translation. This is expressed by the main theorem on the Gödel-translation:

$$
\Gamma \vdash \phi \Leftrightarrow \Gamma^{\circ} \vdash_{m} \phi^{\circ} .
$$

Task 4: Use the above equivalence to show that classical, intuitionistic and minimal logic are equiconsistent i.e., $\vdash$ is consistent if and only if $\vdash_{i}$ is consistent if and only if $\vdash_{m}$ is consistent

$$
\nvdash \perp \Leftrightarrow \vdash_{i} \perp \Leftrightarrow \vdash_{m} \perp
$$

Task 5: Explain why

$$
\Gamma \vdash \phi^{\circ} \Rightarrow \Gamma^{\circ} \vdash_{m} \phi^{\circ},
$$

for every $\phi \in \mathcal{F}$.

## 4 Recursive functions

Using the principle of the excluded middle the characteristic function $1_{A}$ of some $A \subseteq \mathbb{N}^{k}$, where $k \geq 1$, is defined by

$$
1_{A}(\vec{n}):= \begin{cases}0 & , \text { if } A(\vec{n}) \\ 1 & , \text { ow } .\end{cases}
$$

Also, the projection functions $\operatorname{pr}_{i}^{k}: \mathbb{N}^{k} \rightarrow \mathbb{N}$ are defined by

$$
\operatorname{pr}_{i}^{k}(\vec{n})=\operatorname{pr}_{i}^{k}\left(n_{1}, \ldots, n_{k}\right)=n_{i}
$$

for every $\vec{n} \in \mathbb{N}^{k}$ and $i \in\{1, \ldots, k\}$.

Definition 1. The sets of recursive functions $\operatorname{Rec}^{(k)}$ of type $\mathbb{N}^{k} \rightarrow \mathbb{N}$, where $k \geq 1$, are defined simultaneously by the following inductive rules:
(I)

$$
\begin{gathered}
\overline{+\in \operatorname{Rec}^{(2)}}, \quad \overline{\cdot \in \operatorname{Rec}^{(2)}}, \quad \overline{1_{<} \in \operatorname{Rec}^{(2)}} \\
\overline{\operatorname{pr}_{i}^{k} \in \operatorname{Rec}^{(k)}}, \quad 1 \leq i \leq k
\end{gathered}
$$

(II)

$$
\frac{g \in \operatorname{Rec}^{(n)}, \quad h_{1}, \ldots, h_{n} \in \operatorname{Rec}^{(k)}}{\operatorname{Comp}\left(g, h_{1}, \ldots, h_{n}\right) \in \operatorname{Rec}^{(k)}}
$$

where

$$
\operatorname{Comp}\left(g, h_{1}, \ldots, h_{n}\right)(\vec{n})=g\left(h_{1}(\vec{n}), \ldots, h_{n}(\vec{n})\right)
$$

(III)

$$
\frac{g \in \operatorname{Adm}^{(k+1)}}{g_{\mu} \in \operatorname{Rec}^{(k)}}
$$

where

$$
\operatorname{Adm}^{(k+1)}:=\left\{g \in \operatorname{Rec}^{(k+1)} \mid \forall_{\vec{n} \in \mathbb{N}^{k}} \exists_{m}(g(\vec{n}, m)=0)\right\}
$$

and

$$
g_{\mu}(\vec{n}):=\mu m: g(\vec{n}, m)=0 .
$$

## Note that:

1. $\operatorname{pr}_{1}^{1}=\mathrm{id}_{\mathbb{N}}$, where $\mathrm{id}_{\mathbb{N}}$ denotes the identity function on $\mathbb{N}$.
2. If we combine (I) and (II) we get e.g., that $h_{1}+h_{2}, h_{1} \cdot h_{2} \in \operatorname{Rec}^{(k)}$. I.e., the case $k=2$ is essential to the formation of new elements of $\operatorname{Rec}^{(k)}$, where $k \geq 1$.
3. Using (III) we get new elements of $\operatorname{Rec}^{(2)}$ i.e.,

$$
\frac{g \in \operatorname{Adm}^{(3)}}{g_{\mu} \in \operatorname{Rec}^{(2)}}
$$

4. Because of this interaction between the distinct $\operatorname{Rec}^{(k)}$ 's we say that the sets $\operatorname{Rec}^{(k)}$ are defined simultaneously.
5. Every recursive function $f \in \operatorname{Rec}^{(k)}$ is total i.e., its domain is the whole set $\mathbb{N}^{k}$. If we drop the admissiblity condition in (III), we get the partial recursive functions.
6. Exactly because the recursive functions are defined inductively, if we want to show that for every $f \in$ Rec we have $P(f)$, where $P$ is any formula on functions and

$$
\operatorname{Rec}:=\bigcup_{k=1}^{\infty} \operatorname{Rec}^{(k)}
$$

we can use the induction axiom that corresponds to the above definition.
7. Verify that every rule of the main definition expresses an algorithm for finding the output $f(\vec{n})$, given the input $\vec{n}$. The non-trivial issue is that every algorithmic function (this is an intuitive notion) is recursive (Church-Turing Thesis)!!!
8. Because of the clauses used in the main definition each set $\operatorname{Rec}^{(k)}$ is countable (but there is no algorithmic enumeration of it). Although it is not trivial to exhibit a non-recursive function, because of cardinality issues most of the functions $\mathbb{N}^{k} \rightarrow \mathbb{N}$ are not recursive

If $A \subseteq \mathbb{N}^{k}$ is recursive, we write $A \in \operatorname{REC}^{(k)}$.

## We can use the following:

1. From a recursive set and some recursive functions we can define a new recursive set, their composition. Namely, if $A \in \operatorname{REC}^{(n)}$ and $h_{1}, \ldots, h_{n} \in \operatorname{Rec}^{(k)}$, then

$$
B=\operatorname{Comp}\left(A, h_{1}, \ldots, h_{n}\right) \in \operatorname{REC}^{(k)}
$$

where

$$
B(\vec{n}) \leftrightarrow A\left(h_{1}(\vec{n}), \ldots, h_{n}(\vec{n})\right)
$$

2. From an appropriate recursive relation we can define a recursive function. Namely, if $A \subseteq \mathbb{N}^{k+1}$ is an admissible recursive relation i.e.,

$$
\forall_{\vec{n} \in \mathbb{N}^{k}} \exists_{m} A(\vec{n}, m),
$$

and in this case we write $A \in \mathrm{ADM}^{(k+1)}$, then the function $a_{\mu}$ defined by

$$
a_{\mu}(\vec{n}):=\mu m: A(\vec{n}, m)
$$

is in $\operatorname{Rec}^{(k)}$.
3. Recursiveness is closed under arbitrary compositions.
4. The constant functions $\bar{m}_{k}: \mathbb{N}^{k} \rightarrow \mathbb{N}$, defined by $\vec{n} \mapsto m$, are recursive.
5. Recursiveness is closed under complements, $\cap$ and $\cup$.
6. The relations $\geq, \leq,=,>,<$ are recursive.
7. Bounded quantification preserves recursiveness i.e., if $A \in \operatorname{REC}^{(k+1)}$, then

$$
\begin{aligned}
& B(\vec{n}, m) \leftrightarrow \exists_{k<m} A(\vec{n}, k) \\
& C(\vec{n}, m) \leftrightarrow \forall_{k<m} A(\vec{n}, k)
\end{aligned}
$$

are in $\mathrm{REC}^{(k+1)}$. By the same argument we get that if $A \in \mathrm{REC}^{(2)}$, then

$$
\begin{aligned}
& B(m) \leftrightarrow \exists_{k<m} A(m, k) \\
& C(m) \leftrightarrow \forall_{k<m} A(m, k)
\end{aligned}
$$

are in $\mathrm{REC}^{(1)}$.
8. The definition by cases preserves recursiveness e.g., if $A \in \mathrm{REC}^{(k)}$ and $g_{1}, g_{2} \in \operatorname{Rec}^{(k)}$, then

$$
f(\vec{n}):= \begin{cases}g_{1}(\vec{n}) & , \text { if } A(\vec{n}) \\ g_{2}(\vec{n}) & , \text { ow }\end{cases}
$$

is in $\operatorname{Rec}^{(k)}$. This is a way to get a new recursive function from a given recursive relation and two recursive functions, where the order (the number $k$ ) of the functions and the recursive relations is the same.
9. $-\in \operatorname{Rec}^{(2)}$.
10. $\pi \in \operatorname{Rec}^{(2)}, \pi_{1}, \pi_{2} \in \operatorname{Rec}^{(1)}$.

## We shall use the following:

1. The successor function $S(n)=n+1$ is in $\operatorname{Rec}^{(1)}$.
2. If $A \in \operatorname{REC}^{(2)}$ and $g \in \operatorname{Rec}^{(1)}$, then

$$
\begin{aligned}
& B(m) \leftrightarrow \exists_{k<g(m)} A(m, k) \\
& C(m) \leftrightarrow \forall_{k<g(m)} A(m, k)
\end{aligned}
$$

are in $\mathrm{REC}^{(1)}$.
3. If $A=\left\{n_{1}, \ldots, n_{k}\right\}$ is a finite subset of naturals, then $A \in \operatorname{REC}^{(1)}$.

## References

[1] Homotopy Type Theory: Univalent Foundations of Mathematics, The Univalent Foundations Program, Institute for Advanced Study, Princeton, 2013.
[2] A.S. Troelstra and H. Schwichtenberg: Basic Proof Theory, 2nd edition, Cambridge, 2000.
[3] https://en.wikipedia.org/wiki/Hilbert's_second_problem
[4] H. Schwichtenberg and S. Wainer: Proofs and Computations, Perspectives in Logic. Assoc. Symb. Logic and Cambridge University Press, 2012.


[^0]:    ${ }^{1}$ The clauses $\Pi(p, \forall x \phi[x / u])$ and $\Pi(p, \exists x \phi[x / u])$ are related to a given domain on which $x$ varies, and we do not include them here.

