Logik - WS15/16

Iosif Petrakis

petrakis@mathematik.uni-muenchen.de

December 17, 2015

These notes include part of the material discussed in the Exercises that correspond to the lecture course "Logik" of Dr. habil. Josef Berger. Some extra exercises and questions can be found here.

Please feel free to send me your comments or your suggestions regarding these notes.

1. Inductive Definitions

The most characteristic example of an inductive (or recursive) definition is that of a *natural number*. It can be given using the following two rules

$$\frac{}{0 \in \mathbb{N}} \qquad \frac{n \in \mathbb{N}}{S(n) \in \mathbb{N}},$$

where $S(n)$ denotes the successor of n. Note though, that this definition alone does not determine a unique set; for example the rationals $\mathbb{Q}$ or the reals $\mathbb{R}$ satisfy the same rules. We determine $\mathbb{N}$ by postulating that $\mathbb{N}$ is the least set satisfying the above rules. We do so by stating the following induction axiom:

$$A(0) \to \forall_n (A(n) \to A(S(n))) \to \forall_n (A(n)),$$

where A is a formula representing a predicate on natural numbers. If A is any formula, then the above principle is the well-known full induction principle on natural numbers (if we restrict the range of A's we get other weaker induction principles). Its interpretation is the following: Suppose that A satisfies the two rules, $A(0)$ and $\forall_n (A(n) \to A(S(n)))$, i.e., as it is usually said, A is a "competitor" predicate to $\mathbb{N}$, then, if we view A as the set of all objects such that $A(n)$, $\mathbb{N} \subseteq A$, i.e., $\forall_n (A(n))$, or more precisely, $\forall_x (\mathbb{N}(x) \to A(x))$. A consequence of this inductive characterization of $\mathbb{N}$ is that if we want to define a function $f : \mathbb{N} \to X$, where X is a set, it suffices to define it on 0, and provide a rule G which gives the value of f on $S(n)$ through the value of f on n. I.e., we have the following recursive definition theorem:

Proposition 1. *If X is a set, $x_0 \in X$ and $G : X \to X$ is a function, then there exists a unique function $f : \mathbb{N} \to X$ such that $f(0) = x_0$ and $f(S(n)) = G(f(n))$.*

Proof. It is direct to see that the above defined f is a function with domain $\mathbb{N}$ and range in X. To show its uniqueness we suppose that there exists $g : \mathbb{N} \to X$ satisfying the above two conditions, and we apply the induction principle on $A(n) := (f(n) = g(n))$ to show that $\forall_n (A(n))$. $\square$

So, we cannot see an inductive definition without its corresponding induction principle, which in turn proves the corresponding recursive definition theorem. If we consider the inductive definition of a *term* of a first-order language $\mathcal{L}$

$$\frac{v \in \mathrm{FV}}{v \in \tau} \qquad \frac{c \in \mathrm{Konst}}{c \in \tau} \qquad \frac{f \in \mathrm{Funk}^{(n)}, \quad t_1, \ldots, t_n \in \tau}{f(t_1, \ldots, t_n) \in \tau},$$

the corresponding full induction principle is

$$\begin{aligned}
&\forall_{v \in \mathrm{FV}}(A(v)) \wedge \\
&\forall_{c \in \mathrm{Konst}}(A(c)) \wedge \\
&\forall_{n \in \mathbb{N}} \forall_{f \in \mathrm{Funk}^{(n)}} \forall_{t_1, \ldots, t_n \in \tau}(A(t_1) \wedge \ldots \wedge A(t_n) \rightarrow A(f(t_1, \ldots, t_n))) \ \rightarrow \\
&\forall_{t \in \tau}(A(t)),
\end{aligned}$$

where A is any formula of our meta-language, for example this could be the language of set theory. Some times in the bibliography one can find one more clause in the above inductive definition of the form "there are no other formulas except the ones determined by the previous rules", and then the induction principle is proved as a theorem. It is clear though, that the added rule is a disguised form of the induction principle. So we understand an inductive definition as a pair

$$(\text{Inductive Rules, Induction Principle}).$$

Again, a consequence of this inductive characterization of τ is the following recursive definition theorem.

Proposition 2. *If X is any given set, $h : \mathrm{FV} \rightarrow X$, $g : \mathrm{Konst} \rightarrow X$ and $G_f : X \times \ldots \times X \rightarrow X$, for every $f \in \mathrm{Funk}^{(n)}$, then there exists a unique function $F : \tau \rightarrow X$ such that:*

$$F(v) = h(v),$$

$$F(c) = g(c),$$

$$F(f(t_1, \ldots, t_n)) = G_f(F(t_1) \ldots F(t_n)),$$

for every $v \in FV$, $c \in \mathrm{Konst}$, $f \in \mathrm{Funk}^{(n)}$ and $t_1, \ldots, t_n \in \tau$.

Question 1: If $s \in \tau$ and $v \in FV$ it is because of the previous proposition that we can define the function

$$[s/v] : \tau \to \tau,$$

$$t \mapsto [s/v](t) := t[s/v].$$

Which are the functions h, g, G_f in this case?

Question 2: Formulate the recursive definition theorem that corresponds to the inductive definition of formulas.

Question 3: There are many inductive definitions in mathematics. Try to think of some. If you take for example the definitional clauses of a topological space, which is the inductively defined notion that arises naturally from them?

2. Natural deduction

The second problem of Hilbert's famous 1900-list was to find a proof of the consistency of arithmetic. It took more than 30 years to understand in a concrete mathematical way all words appearing in the formulation of this problem. The standard understanding regarding its "solution" in [2] is the following:

> There is no consensus on whether results of Gödel and Gentzen give a solution to the problem as stated by Hilbert. Gödel's second incompleteness theorem, proved in 1931, shows that no proof of its consistency can be carried out within arithmetic itself. Gentzen proved in 1936 that the consistency of arithmetic follows from the well-foundedness of the ordinal ϵ_0.

The passage from proving mathematical theorems to treating mathematical **proofs as objects of mathematical study** is a major conceptual step that happened after 2.500 years of standard mathematical practice.
The Brouwer-Heyting-Kolmogoroff interpretation (BHK-interpretation for short) of intuitionistic logic appeared before Gentzen's definition and explains what it means to prove a logically compound statement in terms of

what it means to prove its components; the explanations use the notions of construction and constructive proof as *unexplained, primitive notions.* The notation

$$\Pi(p, \phi)$$

means that p is a proof of formula ϕ. For quantifier-free formulas the clauses of BHK are the following (see [1], p.55):

- For atomic formulas, except $\bot$, the notion of proof is supposed to be given.
- There is no p such that $\Pi(p, \bot)$.
- $\Pi(p, \phi \wedge \psi)$ if and only if $p = (p_1, p_2)$ and $\Pi(p_1, \phi)$, $\Pi(p_2, \psi)$.
- $\Pi(p, \phi \vee \psi)$ if and only if $p = (1, p_1)$ and $\Pi(p_1, \phi)$, or $p = (2, p_2)$ and $\Pi(p_2, \psi)$.
- $\Pi(p, \phi \to \psi)$ if and only if for every proof q, if $\Pi(q, \phi)$, then $\Pi(p(q), \psi)$.
- The clauses $\Pi(p, \forall x \phi[x/u])$ and $\Pi(p, \exists x \phi[x/u])$ are related to a given domain on which x varies, and we do not include them here.

Note that in the previous clauses the "if" corresponds to "introduction" and the "only if" to "elimination".

Gentzen went further and gave a recursive definition of a concrete notion of derivation based on the inductive definition of a formula.

We say that a formula ϕ is derivable in **minimal logic**, if there is a derivation of ϕ by no assumptions according to the definitional clauses of the main definition of the lecture course notes, without the use of any $\bot$-rule, and we write

$$\mathcal{H}_m(\mathcal{D}, \phi, \emptyset), \quad \text{or} \quad \vdash_m \phi.$$

Moreover, $\Sigma \vdash_m \phi$ if and only if $\mathcal{H}_m(\mathcal{D}, \phi, \Delta)$, for some $\Delta \subseteq \Sigma$. If we add to the previous rules the intuitionistic $\bot$-rule, we get **intuitionistic logic** with

$$\mathcal{H}_i(\mathcal{D}, \phi, \Delta), \quad \vdash_i \phi, \quad \Sigma \vdash_i \phi$$

defined accordingly, while if we add to the rules of minimal logic the classical $\bot$-rule, we get **classical logic** where

$$\mathcal{H}_c(\mathcal{D}, \phi, \Delta), \quad \vdash_c \phi, \quad \Sigma \vdash_c \phi$$

are denoted without a subscript

$$\mathcal{H}(\mathcal{D}, \phi, \Delta), \quad \vdash \phi, \quad \Sigma \vdash \phi,$$

and they are defined accordingly. Note that

$$\Sigma \vdash_m \phi \Rightarrow \Sigma \vdash_i \phi,$$

$$\Sigma \vdash_m \phi \Rightarrow \Sigma \vdash \phi.$$

Question 4: Explain why

$$\Sigma \vdash_i \phi \Rightarrow \Sigma \vdash \phi.$$

3. Exercise 3(b), Blatt 3

We show by induction on n that

$$\forall_{n\in\mathbb{N}}(\forall_{\phi_1,\ldots,\phi_n,\phi\in\mathcal{F}}(\{\phi_1,\ldots,\phi_n\} \vdash \phi \Leftrightarrow \vdash \bigwedge_{i=1}^{n} \phi_i \to \phi)),$$

where

$$\bigwedge_{i=1}^{1} \phi_i := \phi_1,$$

$$\bigwedge_{i=1}^{n+1} \phi_i := (\bigwedge_{i=1}^{n} \phi_i) \wedge \phi_{n+1}.$$

For $n = 1$ our goal-formula becomes

$$\forall_{\sigma,\phi\in\mathcal{F}}(\{\sigma\} \vdash \phi \Leftrightarrow \vdash \sigma \to \phi).$$

We fix σ, ϕ and the equivalence

$$\{\sigma\} \vdash \phi \Leftrightarrow \vdash \sigma \to \phi$$

follows by Exercise 3(a), for $\Sigma = \emptyset$. Our inductive hypothesis is the following

$$\forall_{\phi_1,\ldots,\phi_n,\phi\in\mathcal{F}}(\{\phi_1,\ldots,\phi_n\} \vdash \phi \Leftrightarrow \vdash \bigwedge_{i=1}^{n} \phi_i \to \phi),$$

and we show

$$\forall_{\phi_1,\ldots,\phi_n,\phi_{n+1},\phi\in\mathcal{F}}(\{\phi_1,\ldots,\phi_n,\phi_{n+1}\}\vdash\phi\Leftrightarrow\vdash\bigwedge_{i=1}^{n+1}\phi_i\to\phi).$$

For that we fix $\phi_1,\ldots,\phi_n,\phi_{n+1},\phi$ and we have that

$$\begin{aligned}
&\{\phi_1,\ldots,\phi_n,\phi_{n+1}\}\vdash\phi\Leftrightarrow\\
&\{\phi_1,\ldots,\phi_n\}\cup\{\phi_{n+1}\}\vdash\phi\overset{3(a)}{\Leftrightarrow}\\
&\{\phi_1,\ldots,\phi_n\}\vdash\phi_{n+1}\to\phi\overset{(*)}{\Leftrightarrow}\\
&\vdash\bigwedge_{i=1}^{n}\phi_i\to(\phi_{n+1}\to\phi)\overset{(**)}{\Leftrightarrow}\\
&\vdash(\bigwedge_{i=1}^{n}\phi_i)\wedge\phi_{n+1}\to\phi\Leftrightarrow\\
&\vdash\bigwedge_{i=1}^{n+1}\phi_i\to\phi,
\end{aligned}$$

where $(*)$ is by the inductive hypothesis on $\phi_1,\ldots,\phi_n$ and $\phi_{n+1}\to\phi$, and $(**)$ is by Lemma 7(b) of the lecture course notes and the immediate to see fact that if $\vdash\sigma\leftrightarrow\psi$, then $\vdash\sigma\Leftrightarrow\vdash\psi$.

4. The Gödel-translation

The Gödel-translation is a translation of classical logic into intuitionistic (or minimal) logic. It was invented by Gödel and independently by Gentzen in 1933. For this reason it is also called the Gödel-Gentzen translation.

Question 5: Explain why the range of the Gödel-translation $^\circ$ is in the set of formulas. You need to show the following:

$$\forall_x\forall_\phi(x\notin\phi\Rightarrow x\notin\phi^\circ),$$

where $x\in\phi$ denotes that the symbol x occurs in the string ϕ.

Question 6: Explain why $^\circ$ is not onto $\mathcal{F}$.

Question 7: The negative formulas $\mathcal{F}^-$ are defined by the following clauses:

$$P \to \bot \mid \bot \mid \phi \wedge \psi \mid \phi \to \psi \mid \forall x \phi[x/v],$$

where P denotes a prime formula. Check that the range of $^\circ$ is in $\mathcal{F}^-$. One can show that all negative formulas are *i-stable* i.e.,

$$\vdash_i \neg\neg\theta \to \theta,$$

for every $\theta \in \mathcal{F}^-$. Hence Exercise 1 of Blatt 4, expressing that all formulas ϕ° in the range of $^\circ$ are i-stable, is a special case of this fact.

The most important feature of the Gödel-translation is that it is a *proof translation*. This is expressed by the main theorem on the Gödel-translation:

$$\Gamma \vdash \phi \;\Leftrightarrow\; \Gamma^\circ \vdash_i \phi^\circ.$$

Question 8: Use the above equivalence to show that classical and intuitionistic logic are equiconsistent i.e., if $\vdash$ is consistent, then $\vdash_i$ is consistent, and if $\vdash_i$ is consistent, then $\vdash$ is consistent

$$\nvdash \bot \;\Leftrightarrow \nvdash_i \bot.$$

Question 9: Explain why

$$\Gamma \vdash \phi^\circ \;\Rightarrow\; \Gamma^\circ \vdash_i \phi^\circ,$$

for every $\phi \in \mathcal{F}$.

5. Blatt 6

Aufgabe 2. (d) We suppose that $\Sigma \cup \{\phi\} \vDash \psi$ i.e., for all $\mathcal{A}$ and b we have that $\mathcal{A} \vDash (\Sigma \cup \{\phi\})[b] \Rightarrow \mathcal{A} \vDash \psi[b]$, and we show that $\Sigma \vDash \phi \to \psi$ i.e., for all $\mathcal{A}$ and b we show that $\mathcal{A} \vDash \Sigma[b] \Rightarrow \mathcal{A} \vDash (\phi \to \psi)[b]$. For that we fix $\mathcal{A}$ and b, we suppose that $\mathcal{A} \vDash \Sigma[b]$ and $\mathcal{A} \vDash \phi[b]$, and we show that $\mathcal{A} \vDash \psi[b]$. Since $\mathcal{A} \vDash \Sigma[b]$ and $\mathcal{A} \vDash \phi[b]$, we get that $\mathcal{A} \vDash (\Sigma \cup \{\phi\})[b]$, therefore by our initial hypothesis we conclude that $\mathcal{A} \vDash \psi[b]$.

For the converse implication we suppose that $\Sigma \vDash \phi \to \psi$ i.e., for all $\mathcal{A}$ and b we have that $\mathcal{A} \vDash \Sigma[b] \Rightarrow \mathcal{A} \vDash (\phi \to \psi)[b]$, and we show that $\Sigma \cup \{\phi\} \vDash \psi$

i.e., for all $\mathcal{A}$ and b we show that $\mathcal{A} \vDash (\Sigma \cup \{\phi\})[b] \Rightarrow \mathcal{A} \vDash \psi[b]$. For that we fix $\mathcal{A}$ and b, we suppose that $\mathcal{A} \vDash (\Sigma \cup \{\phi\})[b]$ and we show that $\mathcal{A} \vDash \psi[b]$. Since $\mathcal{A} \vDash (\Sigma \cup \{\phi\})[b]$, we get that $\mathcal{A} \vDash \Sigma[b]$ and $\mathcal{A} \vDash \phi[b]$, therefore by our initial hypothesis we conclude that $\mathcal{A} \vDash \psi[b]$.

(e) First we show that $\vDash \phi \vee \psi \rightarrow \neg(\neg\phi \wedge \neg\psi))$ i.e., for every $\mathcal{A}$ and b, if $\mathcal{A} \vDash (\phi \vee \psi)[b]$, then $\mathcal{A} \vDash (\neg(\neg\phi \wedge \neg\psi)))[b]$. For that it suffices to show that $\mathcal{A} \nvDash (\neg\phi \wedge \neg\psi)[b]$. Suppose next that $\mathcal{A} \vDash (\neg\phi \wedge \neg\psi)[b]$ i.e., $\mathcal{A} \vDash \neg\phi[b]$ and $\mathcal{A} \vDash \neg\psi[b]$, or equivalently $\mathcal{A} \nvDash \phi[b]$ and $\mathcal{A} \nvDash \psi[b]$, which contradicts our initial hypothesis that $\mathcal{A} \vDash \phi[b]$ or $\mathcal{A} \vDash \psi[b]$.

Next we show that $\vDash \neg(\neg\phi \wedge \neg\psi)) \rightarrow \phi \vee \psi$ i.e., for every $\mathcal{A}$ and b, if $\mathcal{A} \vDash (\neg(\neg\phi \wedge \neg\psi)))[b]$, then $\mathcal{A} \vDash (\phi \vee \psi)[b]$. The hypothesis is equivalent to $\mathcal{A} \nvDash (\neg\phi \wedge \neg\psi)[b]$ i.e., it is impossible that $\mathcal{A} \vDash \neg\phi[b]$ and $\mathcal{A} \vDash \neg\psi[b]$, or equivalently it is impossible that $\mathcal{A} \nvDash \phi[b]$ and $\mathcal{A} \nvDash \psi[b]$. Therefore, $\mathcal{A} \vDash \phi[b]$ or $\mathcal{A} \vDash \psi[b]$.

Aufgabe 3. We show by induction on t that

$$\forall_{t \in \tau}(\forall_{s \in \tau}\forall_{v \in \mathrm{FV}}\forall_b((t[s/v])^{\mathcal{A}}_b = t^{\mathcal{A}}_{b^{\hat{s}}_v})).$$

If $t = w$, we have that

$$(w[s/v])^{\mathcal{A}}_b = \begin{cases} s^{\mathcal{A}}_b & \text{, if } w = v \\ b(w) & \text{, if } w \neq v, \end{cases}$$

and

$$w^{\mathcal{A}}_{b^{\hat{s}}_v} = b^{\hat{s}}_v(w) = \begin{cases} \hat{s} = s^{\mathcal{A}}_b & \text{, if } w = v \\ b(w) & \text{, if } w \neq v, \end{cases}$$

If $t = c$, then $c^{\mathcal{A}}_{b^{\hat{s}}_v} = \hat{c} = c^{\mathcal{A}}_b = (c[s/v])^{\mathcal{A}}_b$.

If $t = f(t_1, \ldots, t_n)$ and $(t_1[s/v])^{\mathcal{A}}_b = t_1{}^{\mathcal{A}}_{b^{\hat{s}}_v}, \ldots, (t_n[s/v])^{\mathcal{A}}_b = t_n{}^{\mathcal{A}}_{b^{\hat{s}}_v}$, then

$$\begin{aligned}
(f(t_1, \ldots, t_n)[s/v])^{\mathcal{A}}_b &= (f(t_1[s/v], \ldots, t_n[s/v]))^{\mathcal{A}}_b \\
&= \hat{f}((t_1[s/v])^{\mathcal{A}}_b, \ldots, (t_n[s/v])^{\mathcal{A}}_b) \\
&= \hat{f}(t_1{}^{\mathcal{A}}_{b^{\hat{s}}_v}, \ldots, t_n{}^{\mathcal{A}}_{b^{\hat{s}}_v}) \\
&= (f(t_1, \ldots, t_n))^{\mathcal{A}}_{b^{\hat{s}}_v}.
\end{aligned}$$

6. Blatt 8

Aufgabe 4. We define $\#t \in \mathbb{N}$ recursively as follows:

$$\#v := 0 =: \#c,$$

$$\#f(t_1, \ldots, t_n) := (\sum_{i=1}^{n} \#t_i) + 1,$$

where $\#t$ expresses the number of function symbols occuring in t. In analogy to Proposition 6 we show the following:

Let $T : \tau \to 2 = \{0, 1\}$ such that

$$\forall_{t \in \tau}(\forall_{\sigma \in \tau}(\#\sigma < \#t \to T(\sigma) = 1) \to T(t) = 1).$$

Then

$$\forall_{t \in \tau}(T(t) = 1).$$

Proof We fix some $t \in \tau$. If $\#t = 0$, then we get directly by our hypothesis that $T(t) = 1$, since the implication $\forall_{\sigma \in \tau}(\#\sigma < \#t \to T(\sigma) = 1)$ holds trivially (it is impossible that $\#\sigma < 0$, therefore the implication is true).

Suppose next that $\#t > 0$, and let's assume that $T(t) = 0$. Hence the hypothesis $\forall_{\sigma \in \tau}(\#\sigma < \#t \to T(\sigma) = 1)$ cannot hold, therefore

$$\exists_{\sigma \in \tau}(\#\sigma < \#t \wedge T(\sigma) = 0).$$

If $\#\sigma = 0$, then we reach the contradiction $T(\sigma) = 0 = 1$. If $\#\sigma > 0$, we repeat the previous step and we get

$$\exists_{\sigma_1 \in \tau}(\#\sigma_1 < \#\sigma \wedge T(\sigma_1) = 0).$$

Again, either we reach the contradiction $T(\sigma_1) = 0 = 1$, or we repeat the procedure. It is clear that after at most a finite number of steps n (can you determine n?) we reach a contradiction of the form $T(\sigma_n) = 0 = 1$.

7. Recursive functions

Using the principle of the excluded middle the characteristic function 1_A of some $A \subseteq \mathbb{N}^k$, where $k \geq 1$, is defined by

$$1_A(\vec{n}) := \begin{cases} 0 & \text{, if } A(\vec{n}) \\ 1 & \text{, ow.} \end{cases}$$

Also, the projection functions $\mathrm{pr}_i^k : \mathbb{N}^k \to \mathbb{N}$ are defined by

$$\mathrm{pr}_i^k(\vec{n}) = \mathrm{pr}_i^k(n_1, \ldots, n_k) = n_i,$$

for every $\vec{n} \in \mathbb{N}^k$ and $i \in \{1, \ldots, k\}$.

Definition 3. *The sets of recursive functions* $\mathrm{Rec}^{(k)}$ *of type* $\mathbb{N}^k \to \mathbb{N}$*, where* $k \geq 1$*, are defined* ***simultaneously*** *by the following inductive rules:*
(**I**)

$$\frac{}{+ \in \mathrm{Rec}^{(2)}}, \quad \frac{}{\cdot \in \mathrm{Rec}^{(2)}}, \quad \frac{}{1_< \in \mathrm{Rec}^{(2)}},$$

$$\frac{}{\mathrm{pr}_i^k \in \mathrm{Rec}^{(k)}}, \qquad 1 \leq i \leq k$$

(**II**)

$$\frac{g \in \mathrm{Rec}^{(n)}, \quad h_1, \ldots, h_n \in \mathrm{Rec}^{(k)}}{\mathrm{Comp}(g, h_1, \ldots, h_n) \in \mathrm{Rec}^{(k)}},$$

where

$$\mathrm{Comp}(g, h_1, \ldots, h_n)(\vec{n}) = g(h_1(\vec{n}), \ldots, h_n(\vec{n})).$$

(**III**)

$$\frac{g \in \mathrm{Adm}^{(k+1)}}{g_\mu \in \mathrm{Rec}^{(k)}},$$

where

$$\mathrm{Adm}^{(k+1)} := \{g \in \mathrm{Rec}^{(k+1)} \mid \forall_{\vec{n} \in \mathbb{N}^k} \exists_m (g(\vec{n}, m) = 0)\}$$

and

$$g_\mu(\vec{n}) := \mu m : g(\vec{n}, m) = 0.$$

Note that:

1. $\mathrm{pr}_1^1 = \mathrm{id}_{\mathbb{N}}$, where $\mathrm{id}_{\mathbb{N}}$ denotes the identity function on $\mathbb{N}$.

2. If we combine (I) and (II) we get e.g., that $h_1 + h_2, h_1 \cdot h_2 \in \mathrm{Rec}^{(k)}$. I.e., the case $k = 2$ is essential to the formation of new elements of $\mathrm{Rec}^{(k)}$, where $k \geq 1$.

3. Using (III) we get new elements of $\mathrm{Rec}^{(2)}$ i.e.,
$$\frac{g \in \mathrm{Adm}^{(3)}}{g_\mu \in \mathrm{Rec}^{(2)}}.$$

4. Because of this interaction between the distinct $\mathrm{Rec}^{(k)}$'s we say that the sets $\mathrm{Rec}^{(k)}$ are defined simultaneously.

5. Every recursive function $f \in \mathrm{Rec}^{(k)}$ is *total* i.e., its domain is the whole set $\mathbb{N}^k$. If we drop the admissiblity condition in (III), we get the *partial* recursive functions.

6. Exactly because the recursive functions are defined inductively, if we want to show that for every $f \in \mathrm{Rec}$ we have $P(f)$, where P is any formula on functions and
$$\mathrm{Rec} := \bigcup_{k=1}^{\infty} \mathrm{Rec}^{(k)},$$
we can use the induction axiom that corresponds to the above definition.

7. Verify that every rule of the main definition expresses an **algorithm** for finding the output $f(\vec{n})$, given the input $\vec{n}$. The non-trivial issue is that every algorithmic function (this is an intuitive notion) is recursive (**Church-Turing Thesis**)!!!

8. Because of the clauses used in the main definition each set $\mathrm{Rec}^{(k)}$ is countable (but there is no algorithmic enumeration of it). Although it is not trivial to exhibit a non-recursive function, because of cardinality issues most of the functions $\mathbb{N}^k \to \mathbb{N}$ are **not** recursive

If $A \subseteq \mathbb{N}^k$ is recursive, we write $A \in \mathrm{REC}^{(k)}$.

We can use the following:

1. From a recursive set and some recursive functions we can define a new recursive set, their *composition*. Namely, if $A \in \mathrm{REC}^{(n)}$ and $h_1, \ldots, h_n \in \mathrm{Rec}^{(k)}$, then
$$B = \mathrm{Comp}(A, h_1, \ldots, h_n) \in \mathrm{REC}^{(k)}$$
where
$$B(\vec{n}) \leftrightarrow A(h_1(\vec{n}), \ldots, h_n(\vec{n})).$$

2. From an appropriate recursive relation we can define a recursive function. Namely, if $A \subseteq \mathbb{N}^{k+1}$ is an *admissible recursive relation* i.e.,
$$\forall_{\vec{n} \in \mathbb{N}^k} \exists_m A(\vec{n}, m),$$
and in this case we write $A \in \mathrm{ADM}^{(k+1)}$, then the function a_μ defined by
$$a_\mu(\vec{n}) := \mu m : A(\vec{n}, m)$$
is in $\mathrm{Rec}^{(k)}$.

3. Recursiveness is closed under arbitrary compositions.

4. The constant functions $\overline{m}_k : \mathbb{N}^k \to \mathbb{N}$, defined by $\vec{n} \mapsto m$, are recursive.

5. Recursiveness is closed under complements, $\cap$ and $\cup$.

6. The relations $\geq, \leq, =, >, <$ are recursive.

7. Bounded quantification preserves recursiveness i.e., if $A \in \mathrm{REC}^{(k+1)}$, then
$$B(\vec{n}, m) \leftrightarrow \exists_{k<m} A(\vec{n}, k)$$
$$C(\vec{n}, m) \leftrightarrow \forall_{k<m} A(\vec{n}, k)$$
are in $\mathrm{REC}^{(k+1)}$. By the same argument we get that if $A \in \mathrm{REC}^{(2)}$, then
$$B(m) \leftrightarrow \exists_{k<m} A(m, k)$$
$$C(m) \leftrightarrow \forall_{k<m} A(m, k)$$
are in $\mathrm{REC}^{(1)}$.

8. The definition by cases preserves recursiveness e.g., if $A \in \mathrm{REC}^{(k)}$ and $g_1, g_2 \in \mathrm{Rec}^{(k)}$, then

$$f(\vec{n}) := \begin{cases} g_1(\vec{n}) & \text{, if } A(\vec{n}) \\ g_2(\vec{n}) & \text{, ow} \end{cases}$$

is in $\mathrm{Rec}^{(k)}$. This is a way to get a new recursive function from a given recursive relation and two recursive functions, where the *order* (the number k) of the functions and the recursive relations is the same.

9. $\dot{-} \in \mathrm{Rec}^{(2)}$.

10. $\pi \in \mathrm{Rec}^{(2)}$, $\pi_1, \pi_2 \in \mathrm{Rec}^{(1)}$.

We shall use the following:

1. The successor function $S(n) = n + 1$ is in $\mathrm{Rec}^{(1)}$.

2. If $A \in \mathrm{REC}^{(2)}$ and $g \in \mathrm{Rec}^{(1)}$, then

$$B(m) \leftrightarrow \exists_{k<g(m)} A(m, k)$$

$$C(m) \leftrightarrow \forall_{k<g(m)} A(m, k)$$

are in $\mathrm{REC}^{(1)}$.

3. If $A = \{n_1, \ldots, n_k\}$ is a finite subset of naturals, then $A \in \mathrm{REC}^{(1)}$.

References

[1] A.S. Troelstra and H. Schwichtenberg: *Basic Proof Theory*, 2nd edition, Cambridge, 2000.

[2] https://en.wikipedia.org/wiki/Hilbert's_second_problem