Bridges between the theory of Bishop spaces and the theory of C-spaces

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Master's Thesis

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Eidesstattliche Versicherung

Hiermit erkläre ich, dass ich die vorliegende Masterarbeit eigenständig und ohne unerlaubte Beihilfe angefertigt habe. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

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Johanna Geins

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Abstract

The theory of C-spaces was analyzed in 2015 by Martín Escardó and Chuangjie Xu. The analysis included the embedding in Limits spaces and Kleene-Kreisel-spaces and the extraction of a computational content was studied. The construct of Bishop spaces was designed in 1967 by Errett Bishop and elaborated in 2012 by Douglas Bridges. In 2015 the theory of Bishop spaces was developed by Iosif Petrakis and Bishop topologies were further explored, which are defined as sets of functions.
The goal of this thesis is to compare these two theories. Analogies will be found by investigating properties regarding C-spaces, which are already known for Bishop spaces. Examples are the inductively definition of the least topology $\bigsqcup P_0$ for a given subbasis $P_0 \subseteq \mathbb{F}(2^{\mathbb{N}}, X)$ as well as explicit examples and the embedding of every C-space $\mathcal{P} = (X, P)$ as superset of $\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big)$ and as subset of $\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big)$. This thesis also includes the definition of the product $\mathcal{P} \times \mathcal{Q} = (X \times Y, P \times Q)$ for C-spaces $\mathcal{P} = (X, P)$ and $\mathcal{Q} = (Y, Q)$ as well as the relative C-space $\mathcal{P}|_Y = (Y, P|_Y)$ of the C-space $\mathcal{P} = (X, P)$ for $Y \subseteq X$. Also properties of morphisms between C-spaces like the Yoneda-lemma or the $\bigsqcup$-lifting of openness are explored.
After repeating those properties for Bishop-spaces the analysis focuses on relationships between C-spaces and Bishop spaces. First, C- and Bishop-continuous functions between $2^{\mathbb{N}}$ and $\mathbb{R}$ will be compared and examined for commonalities. Therefore we can establish wounding by means of this connections between general C-spaces and Bishop spaces. Bishop- respectively C-spaces are defined by
$\mathbf{F}_P = \big\{f : X \to \mathbb{R} \mid \forall p \in P\big(f \circ p \in Mor(C, R)\big)\big\}$ and
$P_{\mathbf{F}} = \big\{p : 2^{\mathbb{N}} \to X \mid \forall f \in \mathbf{F}\big(f \circ p \in Mor(C, R)\big)\big\}$, which satisfy useful properties like preserving products or relative spaces for a subset $Y \subseteq X$.

Zusammenfassung

Die Theorie der C-Räume wurde 2015 von Martín Escardó und Chuangjie Xu analysiert. Dabei wurde unter anderem die Einbettung in Limit spaces und Kleene-Kreisel-Räume und die Extrahierung eines rechnerischen Gehalts untersucht. Das Konstrukt der Bishop-Räume wurde 1967 von Errett Bishop entworfen und 2012 von Douglas Bridges ausgearbeitet. 2015 wurde daraus von Iosif Petrakis die Theorie der Bishop-Räume entwickelt und Bishop-Topologien, welche als Mengen von Funktionen definiert werden, untersucht.

Ziel dieser Masterarbeit ist der Vergleich dieser zwei Theorien. Dafür werden zunächst einige Eigenschaften, die für Bishop-Räume bereits bekannt sind, für C-Räume verifiziert und dadurch Analogien festgestellt. Als Beispiele sind die induktive Definition der kleinsten Topologie $\bigsqcup P_0$ zu einer gegebenen Subbasis $P_0 \subseteq \mathbb{F}(2^{\mathbb{N}}, X)$ sowie explizite Beispiele dazu und die Einbettung eines jeden C-Raumes $\mathcal{P} = (X, P)$ als Obermenge von $\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big)$ und als Teilmenge von $\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big)$ zu nennen. Außerdem werden das Produkt $\mathcal{P} \times \mathcal{Q} = (X \times Y, P \times Q)$ zweier C-Räume $\mathcal{P} = (X, P)$ und $\mathcal{Q} = (Y, Q)$ und der zugehörige C-Raum $\mathcal{P}|_Y = (Y, P|_Y)$ zum C-Raum $\mathcal{P} = (X, P)$ für $Y \subseteq X$ definiert und Eigenschaften von Morphismen zwischen C-Räumen, wie etwa das Yoneda-Lemma oder die Erhebung von Offenheit bzgl. $\bigsqcup$ erforscht.

Nachdem im Anschluss diese Eigenschaften für Bishop-Räume wiederholt werden, werden Wechselwirkungen zwischen C-Räumen und Bishop-Räumen gesucht. Dafür werden als erstes C- und Bishop-stetige Funktionen von $2^{\mathbb{N}}$ nach $\mathbb{R}$ miteinander verglichen und auf Gemeinsamkeiten untersucht, um anschließend mit deren Hilfe Verbindungen zwischen allgemeinen C- und Bishop-Räumen herzustellen. Durch die Mengen
$\mathbf{F}_P = \big\{ f : X \to \mathbb{R} \mid \forall p \in P\big(f \circ p \in Mor(C, R)\big)\big\}$ und
$P_{\mathbf{F}} = \big\{ p : 2^{\mathbb{N}} \to X \mid \forall f \in \mathbf{F}\big(f \circ p \in Mor(C, R)\big)\big\}$ werden Bishop- bzw. C-Topologien definiert, die nützliche Eigenschaften wie die Erhaltung von Produkten oder von zugehörigen Räumen für eine Teilmenge $Y \subseteq X$ besitzen.

Contents

1 Introduction

The concept of constructive mathematics has already been developed in the early 20^{th} century with its best-known proponent being L. E. J. Brouwer. Here mathematics is done with intuitionistic logic. But Brouwer could only convince a few companions of his view that classical mathematics would be insufficient for a numerical meaning. The difference between classical and intuitionistic logic is that while in classical logic we work with *stability* axioms $\forall\vec{x}(\neg\neg R\vec{x} \longrightarrow R\vec{x})$, where R is a relation symbol distinct from $\perp$, which allow the principle of indirect proofs, in intuitionistic logic we only work with *ex-falso-quodlibet* axioms $\forall\vec{x}(\perp \longrightarrow R\vec{x})$, where R again is a relation symbol distinct from $\perp$. Here $\perp$ denotes falsity and $\neg A$ is defined by $\neg A := (A \longrightarrow \perp)$. More information about classical and intuitionistic logic you find in [11].
Nowadays, many mathematicians have changed their opinion and constructive mathematics is practiced in many different fields of mathematics. A great deal to this E. Bishop had with his book [3] published in 1967. His achievement, in contrast to Brouwer, was that his system of constructive mathematics (BISH) does not contradict classical mathematics. His approach is that if p is a proof of A in BISH, then there exists a classical proof p_c of A_c, where A_c is the classical reading of A.
For an example let A be the following statement: "If $f : [0,1] \to \mathbb{R}$ is continuous, then $\sup f$ exists." In BISH this holds by definition of continuity: A function $f : [0,1] \to \mathbb{R}$ is continuous if and only if it is uniformly continuous and $\sup f$ is shown to exist.
In classical mathematics, continuity of f is defined by pointwise continuity, and it is shown that it is also uniformly continuous.
Based on Bishop's work and on D. S. Bridge's paper *Reflections on function spaces*, I. Petrakis developed in [10] the theory of Bishop spaces in 2015. Here a Bishop space consists of an inhabited set and a Bishop topology, which is defined as a set of functions. Almost simultaneously, the theory of C-spaces was investigated by M. Erscardó and C. Xu in [8], [13] and [14]. In their investigation, a C-space consists of an inhabited set and a C-topology defined by a set of functions.
Since the usual definition of topology is non-constructive, this new approach is quite interesting. These two definitions have similar approaches with the goal to construct a topology by functions. The difference between the two theories is that within the theory of Bishop spaces the critical set is the domain of the functions in the topology and the codomain is $\mathbb{R}$, while in the theory of C-spaces this set is the codomain and the domain is the Cantor space $2^{\mathbb{N}}$. Nevertheless, the respective dissertations already studied similar

aspects, which suggests a certain relationship.
This Thesis analyses wether there are further analogies in order to build a bridge between the two theories. For this, we work with Bishop's system of constructive mathematics BISH. First, analogical results like in the theory of Bishop spaces are found in the theory of C-spaces. After repeating some facts from [10], the morphisms between $2^{\mathbb{N}}$ and $\mathbb{R}$ are investigated, since here we have an interface by taking the critical set equal to $2^{\mathbb{N}}$ in the theory of Bishop spaces and equal to $\mathbb{R}$ in the theory of C-spaces. Afterwards, this connection will be expanded for arbitrary Bishop respectively C-spaces. This expansion will have some valuable properties w.r.t. the initial results.

2 C-Spaces

2.1 Basic Definitions

2.1.1 Definition Here $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$. The set $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}^+$.

2.1.2 Definition The *Cantor space* $2^{\mathbb{N}}$ is the set of all functions $\alpha : \mathbb{N} \to 2$. We denote an element α of $2^{\mathbb{N}}$ by $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$, where $\alpha_i \in \{0, 1\}$, for all $i \in \mathbb{N}$.

The standard metric ρ on the Cantor space is defined by

$$\rho(\alpha, \beta) := \inf\left\{2^{-n} \mid \alpha =_n \beta\right\}$$

for every $\alpha, \beta \in 2^{\mathbb{N}}$. For $n \in \mathbb{N}$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_0, \beta_1, \beta_2, \ldots)$ elements of the Cantor space $\alpha =_n \beta$ denotes, that it is $\alpha_i = \beta_i$, for all $i \in \{0, \ldots, n-1\}$.
This metric is constructively well-defined (see footnote in [10, page 146]).

2.1.3 Definition The set of all functions of type $2^{\mathbb{N}} \to X$, where X is a set, is denoted by $\mathbb{F}\left(2^{\mathbb{N}}, X\right)$. We denote the set of all constant functions of type $2^{\mathbb{N}} \to X$ by $Const\left(2^{\mathbb{N}}, X\right)$, the constant function on $2^{\mathbb{N}}$ with value $x \in X$ by $\overline{x} \in Const\left(2^{\mathbb{N}}, X\right)$.

2.1.4 Definition A function $f : (X, d) \to (Y, b)$ between inhabited metric spaces X and Y is called *uniformly continuous* if for every $\epsilon > 0$ there exists $\omega_f(\epsilon) > 0$ such that for every $x, y \in X$

$$d(x, y) \leq \omega_f(\epsilon) \longrightarrow b\big(f(x), f(y)\big) \leq \epsilon.$$

The function $\omega_f : \mathbb{R}^+ \to \mathbb{R}^+, \epsilon \mapsto \omega_f(\epsilon)$ is called the *modulus of uniformly continuity for f.*

2.1.5 Definition Let $C(X)$ be the set of uniformly continuous maps of an inhabited metric space X i.e.,

$$C(X) = \left\{ t : 2^{\mathbb{N}} \to X \mid \text{t is uniformly continuous} \right\}.$$

2.1.6 Lemma

$$C\left(2^{\mathbb{N}}\right) = \left\{ t : 2^{\mathbb{N}} \to 2^{\mathbb{N}} \,\middle|\, \begin{array}{c} \forall m \in \mathbb{N}\ \exists n_t(m) \in \mathbb{N}\ \forall \alpha, \beta \in 2^{\mathbb{N}} \\ \left(\alpha =_{n_t(m)} \beta \longrightarrow t(\alpha) =_m t(\beta)\right) \end{array} \right\}.$$

Proof. Let $t : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a uniformly continuous map. Hence, for every $\epsilon > 0$ there exists a $\omega_t(\epsilon) > 0$ such that for every $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\rho(\alpha, \beta) \leq \omega_t(\epsilon) \longrightarrow \rho\big(t(\alpha), t(\beta)\big) \leq \epsilon$$

or

$$\inf\left\{2^{-n} \mid \alpha =_n \beta\right\} \leq \omega_t(\epsilon) \longrightarrow \inf\left\{2^{-m} \mid t(\alpha) =_m t(\beta)\right\} \leq \epsilon.$$

This means that for every $m \in \mathbb{N}$ we can find $n_t(m) \in \mathbb{N}$ such that

$$\alpha =_{n_t(m)} \beta \longrightarrow t(\alpha) =_m t(\beta).$$

$\square$

2.1.7 Definition We also denote $C\left(2^{\mathbb{N}}\right)$ by C.

2.1.8 Corollary

$$C(\mathbb{R}) = \left\{ t : 2^{\mathbb{N}} \to \mathbb{R} \,\middle|\, \begin{array}{c} \forall \epsilon > 0\ \exists n_t(\epsilon) \in \mathbb{N}\ \forall \alpha, \beta \in 2^{\mathbb{N}} \\ \left(\alpha =_{n_t(\epsilon)} \beta \longrightarrow |t(\alpha) - t(\beta)| \leq \epsilon\right) \end{array} \right\}$$

and

$$C(2) = \left\{ t : 2^{\mathbb{N}} \to 2 \mid \exists n_t \in \mathbb{N}\ \forall \alpha, \beta \in 2^{\mathbb{N}} \big(\alpha =_{n_t} \beta \longrightarrow t(\alpha) = t(\beta)\big) \right\}.$$

Since $2 \subseteq \mathbb{R}$ *we get* $C(2) \subseteq C(\mathbb{R})$.

The two sets C and $C(\mathbb{R})$ will have crucial roles in this thesis. C is needed for the definition of C-spaces, while $C(\mathbb{R})$ will be one of the analyzed sets in chapter 4.

2.1.9 Definition A *C-space* is a pair $\mathcal{P} = (X, P)$, where X is an inhabited set and P is a so called *C-topology* i.e., P is a set of functions of type $2^{\mathbb{N}} \to X$, called *probes*, with the following clauses, also called the *probe axioms*:
(CS_1) All constant maps are in P i.e., $\overline{x_0} \in P$ for every $x_0 \in X$.
(CS_2) $p \in P \longrightarrow t \in C \longrightarrow p \circ t \in P$,

$$\begin{tikzcd}
2^{\mathbb{N}} \arrow[r, "p\in P"] & X \\
2^{\mathbb{N}} \arrow[u, "C\ni t"] \arrow[ur, "p\circ t\in P"'] &
\end{tikzcd}$$

(CS_3) For all $n \in \mathbb{N}$ and for all families $\{p_s \in P \mid s \in 2^{\mathbb{N}}\}$ the unique map $p : 2^{\mathbb{N}} \to X, p(s\alpha) := p_s(\alpha)$, is in P.

For two C-spaces $\mathcal{P} = (X, P)$ and $\mathcal{Q} = (Y, Q)$ a map $f : X \to Y$ is called *C-continuous* or a *C-morphism* between X and Y, if $f \circ p$ is in Q, for every $p \in P$.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
2^{\mathbb{N}} \arrow[u, "P\ni p"] \arrow[ur, "f\circ p\in Q"'] &
\end{tikzcd}$$

We also write $f \in Mor(\mathcal{P}, \mathcal{Q})$.

2.1.10 Remark

- (CS_1) implies $Const\left(2^{\mathbb{N}}, X\right) \subseteq P$.
- In (CS_2) we see that for every probe $p \in P$ and for any uniformly contiuous map $t : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ the composition $p \circ t$ is again a probe of P.

2.1.11 Proposition *(CS_3) is equivalent to*
(CS_3') For all $p_0, p_1 \in P$ the function $p^* : 2^{\mathbb{N}} \to X$, defined by $p^*(i\alpha) = p_i(\alpha)$, for $i \in \{0, 1\}$, is in P.

The proof of this proposition and other properties of C-spaces, that will not be repeated in this thesis, are given in Chapter 3.3 of [13], Chapter 2 of [8] and Chapter 2.2 of [14].

The goal is to attain a category of C-spaces. For this, we give the definition from [1], before we search for a concrete category of C-spaces.

2.1.12 Definition The following data is required to produce a *category*:

- *Objects*: $A, B, C, \ldots$
- *Arrows*: $f, g, h, \ldots$
- For every object A there is an arrow $1_A : A \to A$, called the *identity arrow* of A.
- For each arrow f, there are given objects $dom(f)$ and $cod(f)$, called the *domain* and *codomain* of f. To indicate $A = dom(f)$ and $B = cod(f)$, we write $f : A \to B$.
- For arrows $f : A \to B$ and $g : B \to C$ with $cod(f) = dom(g)$ there is an arrow $g \circ f : A \to C$, called the *composite* of f and g.

This data needs to satisfy the following conditions:

- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B, g : B \to C, h : C \to D$.
- Unity: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \to B$.

2.1.13 Proposition *We take the C-spaces as the objects and the C-continuous maps as the arrows. Then we get the category of C-spaces, called* **CS**.

Proof.

- $1_{\mathcal{P}} = Id_X$.
- Let $\mathcal{P} = (X, P), \mathcal{Q} = (Y, Q)$ and $\mathcal{S} = (Z, S)$ be C-spaces, $q \in Mor(\mathcal{P}, \mathcal{Q})$ and $s \in Mor(\mathcal{Q}, \mathcal{S})$. Then the composition $s \circ q : X \to Z$ is in $Mor(\mathcal{P}, \mathcal{S})$, since if we fix some $p \in P$, we get that $q \circ p \in Q$, thus

$$(s \circ q) \circ p = s \circ (q \circ p) \in S.$$

$$\begin{tikzcd}
 & (Y,Q) \arrow[dr, "s \in Mor(\mathcal{Q},\mathcal{S})"] & \\
(X,P) \arrow[ur, "Mor(\mathcal{P},\mathcal{Q}) \ni q"] \arrow[rr, "s \circ q \in Mor(\mathcal{P},\mathcal{S})"] & & (Z,S) \\
2^{\mathbb{N}} \arrow[u, "P \ni p"] \arrow[urr, "(s \circ q) \circ p \in S"'] & &
\end{tikzcd}$$

$\square$

2.1.14 Proposition *The set* $C = \{t : 2^{\mathbb{N}} \to 2^{\mathbb{N}} \mid t \text{ is uniformly continuous}\}$ *is a C-topology on* $2^{\mathbb{N}}$.

Proof. We have to verify the three probe axioms:

(CS_1) Let $\overline{\alpha} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}, \alpha \mapsto \overline{\alpha}(\alpha) = (\overline{\alpha_0}, \overline{\alpha_1}, \overline{\alpha_2}, \dots)$ be a continuous function. For all $\alpha, \beta \in 2^{\mathbb{N}}$ it is $\overline{\alpha}(\alpha) = (\overline{\alpha_0}, \overline{\alpha_1}, \overline{\alpha_2}, \dots) = \overline{\alpha}(\beta)$, in particular $\overline{\alpha} \in C$.

(CS_2) Let $p \in C$ and $t \in C$ i.e., for all $m \in \mathbb{N}$ there exists a $n_p(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_p(m)} \beta \longrightarrow p(\alpha) =_m p(\beta)$$

and for all $m \in \mathbb{N}$ there exists a $n_t(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_t(m)} \beta \longrightarrow t(\alpha) =_m t(\beta).$$

Let now be $m \in \mathbb{N}$ arbitrary. We choose $n_{p \circ t}(m) = n_t\big(n_p(m)\big) \in \mathbb{N}$, then we first get for all $\alpha, \beta \in 2^{\mathbb{N}}$:

$$\alpha =_{n_{p \circ t}(m)} \beta \longrightarrow t(\alpha) =_{n_p(m)} t(\beta).$$

Moreover, we know

$$t(\alpha) =_{n_p(m)} t(\beta) \longrightarrow p\big(t(\alpha)\big) =_m p\big(t(\beta)\big).$$

Because of that we have found a $n_{p \circ t}(m) \in \mathbb{N}$, such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_{p \circ t}(m)} \beta \longrightarrow (p \circ t)(\alpha) =_m (p \circ t)(\beta).$$

(CS'_3) Let $p_0, p_1 \in C$ i.e., for all $m \in \mathbb{N}$ there exists a $n_{p_0}(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_{p_0}(m)} \beta \longrightarrow p_0(\alpha) =_m p_0(\beta)$$

and for all $m \in \mathbb{N}$ there exists a $n_{p_1}(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_{p_1}(m)} \beta \longrightarrow p_1(\alpha) =_m p_1(\beta).$$

We need to show that the function $p^* : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by $p^*(i\alpha) = p_i(\alpha)$, for $i \in \{0, 1\}$ is in C.
Let $m \in \mathbb{N}$ be arbitrary. We take $\tilde{n}_{p^*}(m) := \max\{n_{p_0}(m), n_{p_1}(m)\}$. For $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{\tilde{n}_{p^*}(m)} \beta$ we now get

$$\alpha =_{n_{p_0}(m)} \beta \text{ and } \alpha =_{n_{p_1}(m)} \beta$$

and therefore

$$p^*(0\alpha) = p_0(\alpha) =_m p_0(\beta) = p^*(0\beta) \text{ and } p^*(1\alpha) = p_1(\alpha) =_m p_1(\beta) = p^*(1\beta).$$

Thus for $n_{p^*}(m) = \max\{n_{p_0}(m), n_{p_1}(m)\} + 1$ we get the uniformly continuity of p^*. □

2.1.15 Definition We call $\mathcal{C} = (2^{\mathbb{N}}, C)$ the *Cantor C-space*.

2.1.16 Proposition *$C(\mathbb{R})$ is a C-topology on $\mathbb{R}$.*

Proof. (CS_1) Let $\overline{a} : 2^{\mathbb{N}} \to \mathbb{R}, \alpha \mapsto \overline{a}(\alpha) = a \in \mathbb{R}$ be a continuous function. Let $\epsilon > 0$. For all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\overline{a}(\alpha) = a = \overline{a}(\beta),$$

hence

$$|\overline{a}(\alpha) - \overline{a}(\beta)| = 0 \leq \epsilon.$$

Therefore $\overline{a} \in C(\mathbb{R})$.

(CS_2) Let $t \in C(\mathbb{R})$ and $s \in C$ i.e., for every $\epsilon > 0$ there exists a $n_t(\epsilon) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_t(\epsilon)} \beta \longrightarrow |t(\alpha) - t(\beta)| \leq \epsilon$$

and for every $m \in \mathbb{N}$ there is a $n_s(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_s(m)} \beta \longrightarrow s(\alpha) =_m s(\beta).$$

Let $\epsilon > 0$ be arbitrary. We choose $n_{t\circ s}(\epsilon) := n_s\big(n_t(\epsilon)\big) \in \mathbb{N}$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_{t\circ s}(\epsilon)} \beta$ it is

$$s(\alpha) =_{n_t(\epsilon)} s(\beta),$$

hence

$$|(t \circ s)(\alpha) - (t \circ s)(\beta)| \leq \epsilon.$$

On this account $t \circ s \in C(\mathbb{R})$.

(CS_3') Let $p_0, p_1 \in C(\mathbb{R})$ i.e., for every $\epsilon_0 > 0$ there exists a $n_{p_0}(\epsilon_0) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_{p_0}(\epsilon_0)} \beta \longrightarrow |p_0(\alpha) - p_0(\beta)| \leq \epsilon_0$$

and for every $\epsilon_1 > 0$ there is a $n_{p_1}(\epsilon_1) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_{p_1}(\epsilon_1)} \beta \longrightarrow |p_1(\alpha) - p_1(\beta)| \leq \epsilon_1.$$

Let $\epsilon > 0$ be arbitrary. We take $\tilde{n}_{p^*}(\epsilon) := \max\big\{n_{p_0}(\epsilon), n_{p_1}(\epsilon)\big\}$. For all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{\tilde{n}_{p^*}(\epsilon)} \beta$ we get

$$\alpha =_{n_{p_0}(\epsilon)} \beta \text{ and } \alpha =_{n_{p_1}(\epsilon)} \beta$$

and accordingly

$$|p^*(0\alpha) - p^*(0\beta)| = |p_0(\alpha) - p_0(\beta)| \leq \epsilon$$

and

$$|p^*(1\alpha) - p^*(1\beta)| = |p_1(\alpha) - p_1(\beta)| \leq \epsilon,$$

hence for $n_{p^*}(\epsilon) := \max\big\{n_{p_0}(\epsilon), n_{p_1}(\epsilon)\big\} + 1$ we get $p^* \in C(\mathbb{R})$. $\square$

2.1.17 Definition The C-space *Real* is defined by $\mathcal{R} = \big(\mathbb{R}, C\left(\mathbb{R}\right)\big)$.

2.1.18 Proposition *For every inhabited metric space X the space $\big(X, C(X)\big)$ is a C-space.*

Proof. We just give a sketch here, since we follow the same procedure as in the proofs before.

(CS_1) Obviously, $Const(2^{\mathbb{N}}, X) \subseteq C(X)$.

(CS_2) We take $\omega_{p\circ t}(\epsilon) := n_t\big(\omega_p(\epsilon)\big)$ as the modulus of uniformly continuity for $p \circ t$, for given moduli $\omega_p(\cdot)$ for $p \in C(X)$ and $n_t(\cdot)$ fot $t \in C$.

(CS_3') We choose $\omega_{p^*}(\epsilon) := \max\left\{\omega_{p_0}(\epsilon), \omega_{p_1}(\epsilon)\right\} + 1$ as the modulus for p^*, for given moduli $\omega_{p_0}(\cdot)$ for $p_0 \in C(X)$ and $\omega_{p_1}(\cdot)$ for $p_1 \in C(X)$, where $p^*(i\alpha) = p_i(\alpha)$, for $i \in \{0, 1\}$. $\square$

2.1.19 Definition For an inhabited metric space X we denote the *uniform C-space* by $\mathcal{U}_{C(X)} = \big(X, C(X)\big)$.

2.2 Inductively generated C-Spaces

As in chapter 3.4 of [10], we define the least C-topology generated by a given set $P_0 \subseteq \mathbb{F}\left(2^{\mathbb{N}}, X\right)$.

2.2.1 Definition Let $P_0 \subseteq \mathbb{F}\left(2^{\mathbb{N}}, X\right)$. The *least C-topology* $\bigsqcup P_0$ *generated by* P_0 is defined by the following clauses:

i) $p_0 \in P_0 \longrightarrow p_0 \in \bigsqcup P_0$.

ii) $x \in X \longrightarrow \overline{x} \in \bigsqcup P_0$.

iii) $p \in \bigsqcup P_0 \longrightarrow t \in C \longrightarrow p \circ t \in \bigsqcup P_0$.

iv) $p_0, p_1 \in \bigsqcup P_0 \longrightarrow p^* \in \bigsqcup P_0$, where p^* is defined by $p^*\left(i\alpha\right) = p_i\left(\alpha\right)$, for every $i \in \{0, 1\}$.

We call P_0 a *subbase* of $\bigsqcup P_0$.

Obviously, $\bigsqcup P_0$ is a C-topology. We also note that the rules of the inductive definition of $\bigsqcup P_0$ have finitely many premises.

These clauses induce the induction principle $Ind_{\sqcup}$ on $\bigsqcup P_0$:

$\forall p_0 \in P_0\big(A\left(p_0\right)\big) \longrightarrow$

$\forall x \in X\big(A\left(\bar{x}\right)\big) \longrightarrow$

$$\forall p \in \bigsqcup P_0 \, \forall t \in C\big(A\,(p) \longrightarrow A\,(p \circ t)\,\big) \longrightarrow$$
$$\forall p_0, p_1 \in \bigsqcup P_0\big(A\,(p_0) \longrightarrow A\,(p_1) \longrightarrow A\,(p^*)\,\big) \longrightarrow$$
$$\forall p \in \bigsqcup P_0\big(A\,(p)\,\big),$$

where A is a property on $\mathbb{F}\left(2^{\mathbb{N}}, X\right)$. Therefore, if we start with a constructively graspable subbase P_0, the generated least C-topology $\bigsqcup P_0 \subseteq \mathbb{F}\left(2^{\mathbb{N}}, X\right)$ is constructively graspable.

2.2.2 Proposition *Suppose $P_0, P_1 \subseteq \mathbb{F}\left(2^{\mathbb{N}}, X\right)$ and $\mathcal{P} = (X, P)$ a C-space.*

i) $\bigsqcup P_0 \subseteq P \longleftrightarrow P_0 \subseteq P$.

ii) $P_0 \subseteq P_1 \longrightarrow \bigsqcup P_0 \subseteq \bigsqcup P_1$.

iii) $\bigsqcup P_0 \cup \bigsqcup P_1 \subseteq \bigsqcup (P_0 \cup P_1)$.

iv) $\bigsqcup (\bigsqcup P_0) = \bigsqcup P_0$.

v) $\bigsqcup (P_0 \cap P_1) \subseteq \bigsqcup P_0 \cap \bigsqcup P_1$.

Proof. i) By $P_0 \subseteq \bigsqcup P_0 \subseteq P$, the $(\longrightarrow)$ direction follows immediately. The converse direction we show by induction. For $p_0 \in P_0$ we get $p_0 \in P$ by our hypothesis. Since P is a C-topology, $\overline{x} \in P$ for $x \in X$. If $p \in \bigsqcup P_0$, such that $p \in P$, then $p \circ t \in P$, for every $t \in C$ by definition. Suppose next that $p_0, p_1 \in \bigsqcup P_0$ such that $p_0, p_1 \in P$. By definition, $p^* \in P$.

ii) We use the $(\longleftarrow)$ implication of i) and get $P_0 \subseteq P_1 \subseteq \bigsqcup P_1 \longrightarrow \bigsqcup P_0 \subseteq \bigsqcup P_1$.

iii) Since $P_0, P_1 \subseteq P_0 \cup P_1$, the claim follows directly.

iv) $\bigsqcup P_0 \subseteq \bigsqcup (\bigsqcup P_0)$ by definition and $\bigsqcup (\bigsqcup P_0) \subseteq \bigsqcup P_0$ by i) and $\bigsqcup P_0 \subseteq \bigsqcup P_0$.

v) We use again ii) and the fact that $P_0 \cap P_1 \subseteq P_0, P_1$.

□

2.2.3 Proposition ($\bigsqcup$-lifting of morphisms) *If $h : (X, \bigsqcup P_0) \to (Y, Q)$, then h is C-continuous if and only if for all $p_0 \in P_0$ it is $h \circ p_0 \in Q$ i.e.,*

$$\forall p \in \bigsqcup P_0\,(h \circ p \in Q) \longleftrightarrow \forall p_0 \in P_0\,(h \circ p_0 \in Q)\,.$$

Proof. By definition of continuity we get $h \circ p \in Q$, for all $p \in \bigsqcup P_0$.
Let $p_0 \in P_0$, then since $p_0 \in \bigsqcup P_0$ by i) we also get $h \circ p_0 \in Q$.

$$\begin{tikzcd}
2^{\mathbb{N}} \arrow[r, "p_0 \in P_0"] \arrow[dr, "Q \ni h \circ p"'] & X \arrow[d, "h \text{ C-continuous}"] \\
& Y
\end{tikzcd}$$

The other direction is shown by induction. Our premise is that for all $p_0 \in P_0$ we have $h \circ p_0 \in Q$. To prove the continuity of h we have to show that $h \circ p$ is in Q, for all $p \in \bigsqcup P_0$. By the induction principle $Ind_{\sqcup}$ we need to prove:

i) $\forall p_0 \in P_0 \, (h \circ p_0 \in Q)$.

ii) $\forall x \in X \, (h \circ \overline{x} \in Q)$.

iii) $\forall p \in \bigsqcup P_0 \, \forall t \in C \big(h \circ p \in Q \longrightarrow h \circ (p \circ t) \in Q\big)$.

iv) $\forall p_0, p_1 \in \bigsqcup P_0 \, (h \circ p_0 \in Q \longrightarrow h \circ p_1 \in Q \longrightarrow h \circ p^* \in Q)$.

i) The claim follows exactly by our premise.

ii) Let $\overline{x} \in Const\left(2^{\mathbb{N}}, X\right)$. Then $h \circ \overline{x} \in Const\left(2^{\mathbb{N}}, Y\right)$, thus $h \circ \overline{x} \in Q$, since Q is a C-topology.

iii) Let $p \in \bigsqcup P_0$ and $t \in C$. If $h \circ p \in Q$, then $h \circ (p \circ t) = (h \circ p) \circ t \in Q$ by (CS_2) of Q.

iv) Let $p_0, p_1 \in \bigsqcup P_0$. If $h \circ p_0 \in Q$ and $h \circ p_1 \in Q$, then we get $(h \circ p^*)(i\alpha) = (h \circ p_i)(\alpha) \in Q$.

□

2.3 Examples of C-spaces

I We want to find the finest C-topology on X, in the sense of the smallest collection of probes. We work similarly to section 3.3.3 in [13].

2.3.1 Definition A function $p : 2^{\mathbb{N}} \to X$ is called *locally constant* if there is a $m_p \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{m_p} \beta \longrightarrow p(\alpha) = p(\beta).$$

m_p denotes the modulus of local constancy of p. We write $Const_{loc}\left(2^{\mathbb{N}}, X\right)$ for the set of locally constant functions $2^{\mathbb{N}} \to X$.

2.3.2 Corollary $Const_{loc}\left(2^{\mathbb{N}},2\right) = C\left(2\right).$

2.3.3 Lemma *$Const_{loc}\left(2^{\mathbb{N}},X\right)$ is a C-topology on X, for any inhabited set X.*

Proof. (CS_1) Obviously it is $Const\left(2^{\mathbb{N}},X\right) \subseteq Const_{loc}\left(2^{\mathbb{N}},X\right)$.

(CS_2) Let $p \in Const_{loc}\left(2^{\mathbb{N}},X\right)$ and $t \in C$ i.e., we find a $m_p \in \mathbb{N}$ such that for all $\alpha,\beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{m_p} \beta \longrightarrow p\left(\alpha\right) = p\left(\beta\right)$$

and for all $m \in \mathbb{N}$ there exists a $n_t\left(m\right) \in \mathbb{N}$ such that for all $\alpha,\beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_t(m)} \beta \longrightarrow t\left(\alpha\right) =_m t\left(\beta\right).$$

We choose $m_{p\circ t} = n_t\left(m_p\right) \in \mathbb{N}$, then for all $\alpha,\beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{m_{p\circ t}} \beta \longrightarrow t\left(\alpha\right) =_{m_p} t\left(\beta\right)$$

and

$$t\left(\alpha\right) =_{m_p} t\left(\beta\right) \longrightarrow p\big(t\left(\alpha\right)\big) = p\big(t\left(\beta\right)\big).$$

Therefore we have found a $m_{p\circ t} \in \mathbb{N}$ such that for all $\alpha,\beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{m_{p\circ t}} \beta \longrightarrow \left(p\circ t\right)\left(\alpha\right) = \left(p\circ t\right)\left(\beta\right),$$

hence $p\circ t$ is locally constant.

(CS_3') Let $p_0,p_1 \in Const_{loc}\left(2^{\mathbb{N}},X\right)$ i.e., there are m_{p_0} and $m_{p_1} \in \mathbb{N}$ such that for all $\alpha,\beta \in 2^{\mathbb{N}}$ we get

$$\Big(\alpha =_{m_{p_0}} \beta \longrightarrow p_0\left(\alpha\right) = p_0\left(\beta\right)\Big) \text{ and } \Big(\alpha =_{m_{p_1}} \beta \longrightarrow p_1\left(\alpha\right) = p_1\left(\beta\right)\Big).$$

We consider $m_p = \max\left\{m_{p_0},m_{p_1}\right\}$. Then for $\alpha =_{m_p} \beta$ we get $\alpha =_{m_{p_0}} \beta$ and $\alpha =_{m_{p_1}} \beta$. For $p^* : 2^{\mathbb{N}} \to X, p^*\left(i\alpha\right) = p_i\left(\alpha\right)$, for every $i \in \{0,1\}$, we see now

$$p^*\left(0\alpha\right) = p_0\left(\alpha\right) = p_0\left(\beta\right) = p^*\left(0\beta\right)$$

and

$$p^*\left(1\alpha\right) = p_1\left(\alpha\right) = p_1\left(\beta\right) = p^*\left(1\beta\right).$$

Consequently, with the modulus of constancy $m_{p^*} = \max\left\{m_{p_0},m_{p_1}\right\} + 1$ we get $p^* \in Const_{loc}\left(2^{\mathbb{N}},X\right)$. □

It is trivial that $Const_{loc}\left(2^{\mathbb{N}},2^{\mathbb{N}}\right) \subseteq C$, hence every locally constant map on $2^{\mathbb{N}}$ is also uniformly continuous.

2.3.4 Lemma *For any C-space $\mathcal{P} = (X, P)$, every locally constant map $2^{\mathbb{N}} \to X$ is in P.*

Proof. Let $p : 2^{\mathbb{N}} \to X$ be locally constant with $m \in \mathbb{N}$ its modulus of constancy. We take $s \in 2^m$ and define $p_s : 2^{\mathbb{N}} \to X, p_s(\alpha) := p(s\alpha)$, for $\alpha \in 2^{\mathbb{N}}$. Now

$$p_s(\alpha) = p(s\alpha) = p(s\beta) = p_s(\beta),$$

for any $\alpha, \beta \in 2^{\mathbb{N}}$. Thus, p_s is constant, hence in P. By (CS_3) we get $p \in P$. $\square$

These two lemmas show that $Const_{loc}(2^{\mathbb{N}}, X)$ is the finest C-topology on X. Because of that, for every C-topology P of X we have

$$Const_{loc}\left(2^{\mathbb{N}}, X\right) \subseteq P \subseteq \mathbb{F}\left(2^{\mathbb{N}}, X\right).$$

Since $\bigsqcup \emptyset \subseteq Const_{loc}(2^{\mathbb{N}}, X)$ by proposition 2.2.2 i), we get $\bigsqcup \emptyset = Const_{loc}(2^{\mathbb{N}}, X)$.

2.3.5 Definition A C-space $\mathcal{P} = (X, P)$ is called *discrete* if for every C-space $\mathcal{Q} = (Y, Q)$ all functions $X \to Y$ are continuous.

2.3.6 Proposition *A C-space $\mathcal{P} = (X, P)$ is discrete if and only if $P = Const_{loc}(2^{\mathbb{N}}, X)$ i.e., $Mor\left((X, Const_{loc}(2^{\mathbb{N}}, X)), \mathcal{Q}\right) = \mathbb{F}(X, Y)$.*

Proof. Let (X, P) be a discrete C-space. According to lemma 2.3.3, we also consider the C-space $(X, Const_{loc}(2^{\mathbb{N}}, X))$. Since (X, P) is discrete, the identity function $(X, P) \to (X, Const_{loc}(2^{\mathbb{N}}, X))$ is continuous. By the definition of continuity we get that every probe in P is also locally constant.
For the converse, let $\mathcal{Q} = (Y, Q)$ be a C-space and $f : X \to Y$ a map. For the discreteness of $(X, Const_{loc}(2^{\mathbb{N}}, X))$ we need to show the continuity of f. We prove this by induction on $p \in Const_{loc}(2^{\mathbb{N}}, X)$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ 2^{\mathbb{N}} \arrow[u, "Const_{loc}(X) \ni p"] \arrow[ur, "f \circ p \in Q"'] & \end{tikzcd}$$

(1) If p is constant, $f \circ p$ is also constant, therefore $f \circ p \in Q$.

(2) Let p be written as $\tilde{p} \circ t$, where $\tilde{p} \in Const_{loc}(2^{\mathbb{N}}, X)$ and $t \in C$, and with the induction hypothesis $f \circ \tilde{p} \in Q$.
Then $f \circ p = f \circ (\tilde{p} \circ t) = (f \circ \tilde{p}) \circ t \in Q$.

(3) Let p be defined by $p\,(i\alpha) = p_i\,(\alpha)$, for $i \in \{0,1\}$, where for $p_0, p_1 \in Const_{loc}\left(2^{\mathbb{N}}, X\right)$ we have the induction hypothesis $f \circ p_i \in Q$, for $i \in \{0,1\}$. Since we get a $\alpha \in 2^{\mathbb{N}}$ for every $\beta \in 2^{\mathbb{N}}$ such that $\beta = i\alpha$, where i is either 0 or 1, we conclude $(f \circ p)\,(\beta) = (f \circ p)\,(i\alpha) = (f \circ p_i)\,(\alpha)$, hence $f \circ p \in Q$. □

Because of the previous proposition we call $Const_{loc}\left(2^{\mathbb{N}}, X\right)$ also the *discrete C-topology of X*.

II With the least C-topology $\bigsqcup P_0$ we can construct many examples. We already have seen that $\bigsqcup \emptyset = Const_{loc}\left(2^{\mathbb{N}}, X\right)$. The following propositions will also yield

- $\bigsqcup Id_{2^{\mathbb{N}}} = C$.
- $\bigsqcup i = C(\mathbb{R})$, where $i : 2^{\mathbb{N}} \to \mathbb{R}$ is defined by $i(\alpha) := \sum\limits_{n=0}^{\infty} \frac{\alpha_n}{2^n}$.
- $\bigsqcup \{\pi_n \mid n \in \mathbb{N}\} = C(2)$, where $\pi_n : 2^{\mathbb{N}} \to 2$ is defined by $\pi_n(\alpha) := \alpha_n$.

2.3.7 Proposition $\bigsqcup Id_{2^{\mathbb{N}}} = C$.

Proof. Let $P_0 = Id_{2^{\mathbb{N}}} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

i) We get that $Id_{2^{\mathbb{N}}} \in \bigsqcup Id_{2^{\mathbb{N}}}$.

ii) The constant maps $2^{\mathbb{N}} \to 2^{\mathbb{N}}$ are in $\bigsqcup Id_{2^{\mathbb{N}}}$.

iii) We have to test the combinations of elements in $\bigsqcup Id_{2^{\mathbb{N}}}$ and elements in C. Since $s \circ t$ is again in C, for $s, t \in C$, it suffices to look at simple compositions.
Let $t \in C$.

- $Id_{2^{\mathbb{N}}} \circ t = t \in C$, thus $C \subseteq \bigsqcup Id_{2^{\mathbb{N}}}$.
- $c \circ t = \tilde{c}$ for $c, \tilde{c}$ constant.

Since $Id_{2^{\mathbb{N}}} \in C$ and $Const\left(2^{\mathbb{N}}\right) \in C$, we pool i)-iii) and get $C \subseteq \bigsqcup Id_{2^{\mathbb{N}}}$.

iv) For $p_0, p_1 \in C$ we get $p^* \in C$, because C is a C-topology.

All together we get $\bigsqcup Id_{2^{\mathbb{N}}} = C$. □

2.3.8 Definition We define the function $\log_2 : \mathbb{R}^+ \to \mathbb{R}$ by

$$\log_2(x) := \frac{\ln(x)}{\ln(2)}, \text{ for } x > 0,$$

where $\ln : \mathbb{R}^+ \to \mathbb{R}$ is defined in [4, page 57], by $\ln(x) := \int\limits_1^x t^{-1}dt,$ for $x > 0$.

2.3.9 Proposition *The following conditions are satisfied:*

i) For the function $E : \mathbb{R} \to \mathbb{R}^+$ defined by $E(x) = 2^x$ we get

$$E\big(\log_2(x)\big) = x, \text{ for } x \in \mathbb{R}^+ \text{ and } \log_2\big(E(y)\big) = y, \text{ for } y \in \mathbb{R}.$$

ii) $\log_2(x \cdot y) = \log_2(x) + \log_2(y)$, for $x, y > 0$.

Proof. i) By the definitions of $\log_2$ and E it is

$$E\big(\log_2(x)\big) = E\left(\frac{\ln(x)}{\ln(2)}\right) = 2^{\frac{\ln(x)}{\ln(2)}} = exp\left(\frac{\ln(x)}{\ln(2)} \cdot \ln(2)\right) = \exp\big(\ln(x)\big) = x,$$

for $x > 0$, by [4, page 58] and

$$\log_2\big(E(y)\big) = \frac{\ln\big(E(y)\big)}{\ln(2)} = \frac{\ln(2^y)}{\ln(2)} = \frac{\ln\big(\exp\big(y \cdot \ln(2)\big)\big)}{\ln(2)} = \frac{y \cdot \ln(2)}{\ln(2)} = y,$$

for $y \in \mathbb{R}$.

ii) By definition of $\log_2$ we get

$$\log_2(x \cdot y) = \frac{\ln(x \cdot y)}{\ln(2)} = \frac{\ln(x) + \ln(y)}{\ln(2)} = \frac{\ln(x)}{\ln(2)} + \frac{\ln(y)}{\ln(2)} = \log_2(x) + \log_2(y).$$

□

2.3.10 Proposition *For the function $i : 2^{\mathbb{N}} \to \mathbb{R}$ defined by*

$$i(\alpha) := \sum_{n=0}^{\infty} \frac{\alpha_n}{2^n}.$$

we get $i \in C(\mathbb{R})$.

Proof. Let $\epsilon > 0$. We choose $n_i(\epsilon) := \log_2(\frac{1}{\epsilon}) + 1$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_i(\epsilon)} \beta$ we get

$$|i(\alpha) - i(\beta)| = \left|\sum_{n=0}^{\infty} \frac{\alpha_n}{2^n} - \sum_{n=0}^{\infty} \frac{\beta_n}{2^n}\right| = \left|\sum_{n=0}^{\infty} \frac{\alpha_n - \beta_n}{2^n}\right| = \left|\sum_{n=n_i(\epsilon)}^{\infty} \frac{\alpha_n - \beta_n}{2^n}\right| \leq$$

$$\leq \sum_{n=n_i(\epsilon)}^{\infty} \frac{\overbrace{|\alpha_n - \beta_n|}^{\leq 1}}{2^n} \leq \sum_{n=n_i(\epsilon)}^{\infty} \frac{1}{2^n} = \frac{1}{2^{n_i(\epsilon)-1}} = \epsilon,$$

hence $i \in C(\mathbb{R})$. □

2.3.11 Proposition $\bigsqcup i = C(\mathbb{R})$.

Proof. i) By definition, $i \in \bigsqcup i$.

ii) Obviously, $Const(2^{\mathbb{N}}, \mathbb{R}) \subseteq \bigsqcup i$.

iii) Let $t \in C$. Then

$$(i \circ t)(\alpha) = i\big(t(\alpha)\big) = \sum_{n=0}^{\infty} \frac{\big(t(\alpha)\big)_n}{2^n}.$$

Let $\epsilon > 0$. We choose $n_{i \circ t}(\epsilon) := n_t\big(\log_2(\frac{1}{\epsilon}) + 1\big)$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_{i\circ t}(\epsilon)} \beta$ we get

$$t(\alpha) =_{\log_2(\frac{1}{\epsilon})+1} t(\beta),$$

hence

$$|(i \circ t)(\alpha) - (i \circ t)(\beta)| = \big|i\big(t(\alpha)\big) - i\big(t(\beta)\big)\big| = \left|\sum_{n=0}^{\infty} \frac{\big(t(\alpha)\big)_n}{2^n} - \sum_{n=0}^{\infty} \frac{(t(\beta))_n}{2^n}\right| =$$

$$= \left|\sum_{n=0}^{\infty} \frac{\big(t(\alpha)\big)_n - \big(t(\beta)\big)_n}{2^n}\right| = \left|\sum_{n=\log_2(\frac{1}{\epsilon})+1}^{\infty} \frac{\big(t(\alpha)\big)_n - \big(t(\beta)\big)_n}{2^n}\right| \leq$$

$$\leq \sum_{n=\log_2(\frac{1}{\epsilon})+1}^{\infty} \frac{\left|\big(t(\alpha)\big)_n - \big(t(\beta)\big)_n\right|}{2^n} \leq \frac{1}{2^{\log_2(\frac{1}{\epsilon})+1-1}} = \frac{1}{2^{\log_2(\frac{1}{\epsilon})}} = \epsilon,$$

hence $i \circ t \in C(\mathbb{R})$. Therefore, by proposition 2.3.10 we conclude $C(\mathbb{R}) \subseteq \bigsqcup i$.

iv) Since $C(\mathbb{R})$ is a C-topology, we get no further elements of $\bigsqcup i$, hence $C(\mathbb{R}) = \bigsqcup i$.

□

2.3.12 Definition The function $\pi_n : 2^{\mathbb{N}} \to 2$ is defined by $\pi_n(\alpha) = \alpha_n$, for any $\alpha \in 2^{\mathbb{N}}$. $\bigsqcup\{\pi_n \mid n \in \mathbb{N}\}$ we also denote by $\bigsqcup_{n\in\mathbb{N}} \pi_n$.

2.3.13 Proposition $\bigsqcup_{n\in\mathbb{N}} \pi_n = C(2)$.

Proof. ($\supseteq$) Since $C(2) = Const_{loc}(2^{\mathbb{N}}, 2)$ by corollary 2.3.2 and since $Const_{loc}(2^{\mathbb{N}}, 2)$ is the finest C-topology on 2, it is $C(2) \subseteq \bigsqcup_{n\in\mathbb{N}} \pi_n$.

($\subseteq$) Let $f \in \bigsqcup_{n\in\mathbb{N}} \pi_n$. We use the induction principle $Ind_{\sqcup}$ with the property $A(f) : f \in C(2)$.

i) Let $f \in \{\pi_n \mid n \in \mathbb{N}\}$ i.e., there exists a $n_0 \in \mathbb{N}$ such that $f = \pi_{n_0}$. If $\alpha, \beta \in 2^{\mathbb{N}}$ such that $\alpha =_{n_0+1} \beta$, then

$$\pi_{n_0}(\alpha) = \alpha_{n_0} = \beta_{n_0} = \pi_{n_0}(\beta),$$

hence $f \in C(2)$.

ii) Let $f \in Const(2^{\mathbb{N}}, 2)$. Since $Const(2^{\mathbb{N}}, 2) \subseteq C(2)$, it is obviously $f \in C(2)$.

iii) Let $f = \tilde{f} \circ t$, where $\tilde{f} \in \bigsqcup_{n \in \mathbb{N}} \pi_n, \tilde{f} \in C(2)$ and $t \in C$ i.e., there is a $n_{\tilde{f}} \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_{\tilde{f}}} \beta \longrightarrow \tilde{f}(\alpha) = \tilde{f}(\beta)$$

and for every $m \in \mathbb{N}$ exists a $n_t(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_t(m)} \beta \longrightarrow t(\alpha) =_m t(\beta).$$

We choose $n_{\tilde{f} \circ t} := n_t(n_{\tilde{f}})$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_{\tilde{f} \circ t}} \beta$ we get

$$t(\alpha) =_{n_{\tilde{f}}} t(\beta),$$

hence

$$(\tilde{f} \circ t)(\alpha) = \tilde{f}\big(t(\alpha)\big) = \tilde{f}\big(t(\beta)\big) = (\tilde{f} \circ t)(\beta).$$

Because of that we get $f = \tilde{f} \circ t \in C(2)$.

iv) Let $f_0, f_1 \in \bigsqcup_{n \in \mathbb{N}} \pi_n, f_0, f_1 \in C(2)$ such that $f(i\alpha) = f_i(\alpha)$, for $i \in \{0, 1\}$ i.e., there are $n_{f_0}, n_{f_1} \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ and for $i \in \{0, 1\}$ we get

$$\alpha =_{n_{f_i}} \beta \longrightarrow f_i(\alpha) = f_i(\beta).$$

We choose $\tilde{n}_f := \max\left\{n_{f_0}, n_{f_1}\right\}$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{\tilde{n}_f} \beta$ and for $i \in \{0, 1\}$ we get

$$f(i\alpha) = f_i(\alpha) = f_i(\beta) = f(i\beta),$$

hence for $n_f := \max\left\{n_{f_0}, n_{f_1}\right\} + 1$ it is $f \in C(2)$.

$\square$

III *If (X, P_1) and (X, P_2) are C-spaces, then $(X, P_1 \cap P_2)$ is a C-space.*

(CS_1) Since $Const\left(2^{\mathbb{N}}, X\right) \subseteq P_1$ and $Const\left(2^{\mathbb{N}}, X\right) \subseteq P_2$ we get

$$Const\left(2^{\mathbb{N}}, X\right) \subseteq P_1 \cap P_2.$$

(CS_2) Let $p \in P_1 \cap P_2$ i.e., $p \in P_1$ and $p \in P_2$. Since P_1 and P_2 are C-topologies, we get $p \circ t \in P_1$ and $p \circ t \in P_2$, for all $t \in C$, hence $p \circ t \in P_1 \cap P_2$.

(CS_3') Let $p_0, p_1 \in P_1 \cap P_2$ i.e., $p_0, p_1 \in P_1$ and $p_0, p_1 \in P_2$. Now $p^* : 2^{\mathbb{N}} \to X$, defined by $p^*(i\alpha) = p_i(\alpha)$ for $i \in \{0, 1\}$, is in P_1 and P_2, hence $p^* \in P_1 \cap P_2$.

IV For two C-spaces $\mathcal{P} = (X, P)$ and $\mathcal{Q} = (Y, Q)$ we define their product $\mathcal{P} \times \mathcal{Q} := (X \times Y, P \times Q)$ by
$$P \times Q := \bigsqcup \{(p, q) \mid p \in P, q \in Q\}.$$

2.3.14 Proposition *The product $\mathcal{P} \times \mathcal{Q}$ satisfies the universal property for products i.e., for every C-space $\mathcal{T} = (S, T)$ and for every $f \in Mor\,(\mathcal{T}, \mathcal{P})$ and $g \in Mor\,(\mathcal{T}, \mathcal{Q})$ there exists a unique $h \in Mor\,(\mathcal{T}, \mathcal{P} \times \mathcal{Q})$, such that $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$ are satisfied, where π_1 and π_2 are the projections to X and Y, respectively.*

$$\begin{tikzcd}
& S \arrow[ld, "f"'] \arrow[d, dotted, "h"] \arrow[rd, "g"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

Proof. First we show that the projections are C-continuous. We prove this for π_1, and for π_2 we use the same approach.
It suffices to verify that $\pi_1 \circ (p, q) \in P$ for every $p \in P$ and $q \in Q$. This follows immediately, since $\pi_1 \circ (p, q) = p \in P$.

We define $h : S \to X \times Y$ by $h\,(s) = \big(f\,(s)\,, g\,(s)\,\big)$, for every $s \in S$. Then
$$f = \pi_1 \circ h \text{ and } g = \pi_2 \circ h.$$

It is obvious that h is the only function with these properties. Finally we have to show that h is C-continuous. First we see that $f \circ r \in P$ and $g \circ r \in Q$, for every $r \in T$, since f and g are C-continuous. For every $r \in T$ we get
$$(h \circ r)\,(\alpha) = h\big(r\,(\alpha)\,\big) = \big(f\big(r\,(\alpha)\,\big), g\big(r\,(\alpha)\,\big)\big) = \big(\,(f \circ r)\,(\alpha)\,, (g \circ r)\,(\alpha)\,\big) \in P \times Q,$$
hence $h \in Mor\,(\mathcal{T}, \mathcal{P} \times \mathcal{Q})\,.$ $\square$

2.4 Morphisms between C-spaces

In this section we investigate C-continuous functions. It will be demonstrated that every probe of a C-topology is C-continuous. We will explore the consequences to morphisms after restricting the domain or the codomain, before introducing important topological terms like isomorphism and openness for the theory of C-spaces later.

The following lemma is also proven in [8] and [14], but in a different way.

2.4.1 Lemma (Yoneda) *For any C-space $\mathcal{P} = (X, P)$ a map $2^{\mathbb{N}} \to X$ is a probe if and only if it is C-continuous i.e., if $\mathcal{P} = (X, P)$ is a C-space, then $P = Mor\,(\mathcal{C}, \mathcal{P})$.*

Proof. Let $\mathcal{P} = (X, P)$ be a C-space, $f : 2^{\mathbb{N}} \to X$ a map.
Let f be a probe on X i.e., $f \in P$. We consider an arbitrary $t \in C$. Since C is a C-topology and $f \circ t \in P$ with (CS_2) for $\mathcal{P}$, we get the C-continuity of f by definition.

$$\begin{tikzcd} 2^{\mathbb{N}} \arrow[r, "f \in P"] & X \\ 2^{\mathbb{N}} \arrow[u, "C \ni t"] \arrow[ur, "f \circ t \in P"'] & \end{tikzcd}$$

For the converse direction let f be C-continuous i.e., for $\mathcal{C} = \left(2^{\mathbb{N}}, C\right)$ and $\mathcal{P} = (X, P)$ it is $f \circ t$ in P, for every $t \in C$.

$$\begin{tikzcd} 2^{\mathbb{N}} \arrow[r, "f \text{ C-continuous}"] & X \\ 2^{\mathbb{N}} \arrow[u, "C \ni t"] \arrow[ur, "f \circ t \in P"'] & \end{tikzcd}$$

We have to verify that $f \in P$. This is easy, by taking $t = Id_{2^{\mathbb{N}}}$ as the identity map, which is clearly uniformly continuous. $\square$

2.4.2 Corollary The following hold:

$$C = Mor(\mathcal{C}, \mathcal{C}).$$
$$C(\mathbb{R}) = Mor(\mathcal{C}, \mathcal{R}).$$

The result of the previous corollary is that C is the set of all C-continuous functions between $2^{\mathbb{N}}$ and $2^{\mathbb{N}}$, and $C(\mathbb{R})$ the set of all C-continuous functions between $\mathbb{R}$ and $\mathbb{R}$.

2.4.3 Lemma *Suppose that $\mathcal{P} = (X, P)$, $\mathcal{Q} = (Y, Q)$, $\mathcal{P}_1 = (X, P_1)$, $\mathcal{P}_2 = (X, P_2)$, $\mathcal{Q}_1 = (Y, Q_1)$ and $\mathcal{Q}_2 = (Y, Q_2)$ are C-spaces. Then the following hold:*

i) $Const(X, Y) = \{\overline{y} \mid y \in Y\} \subseteq Mor(\mathcal{P}, \mathcal{Q})$.

ii) If $Q_1 \subseteq Q_2$, then $Mor(\mathcal{P}, \mathcal{Q}_1) \subseteq Mor(\mathcal{P}, \mathcal{Q}_2)$, while $Q_1 \subsetneq Q_2$ does not imply that $Mor(\mathcal{P}, \mathcal{Q}_1) \subsetneq Mor(\mathcal{P}, \mathcal{Q}_2)$. Moreover, we have that

$$Mor\Big(\mathcal{P}, \big(Y, Const_{loc}(2^{\mathbb{N}}, Y)\big)\Big) \subseteq Mor(\mathcal{P}, \mathcal{Q}) \subseteq Mor\Big(\mathcal{P}, \big(Y, \mathbb{F}(2^{\mathbb{N}}, Y)\big)\Big).$$

iii) If $P_1 \subseteq P_2$, then $Mor(\mathcal{P}_2, \mathcal{Q}) \subseteq Mor(\mathcal{P}_1, \mathcal{Q})$, while $P_1 \subsetneq P_1$ does not imply that $Mor(\mathcal{P}_2, \mathcal{Q}) \subsetneq Mor(\mathcal{P}_1, \mathcal{Q})$. Moreover, we have that

$$Mor\Big(\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big), \mathcal{Q}\Big) \subseteq Mor(\mathcal{P}, \mathcal{Q}) \subseteq Mor\Big(\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big), \mathcal{Q}\Big).$$

Proof. i) Since $\overline{y} \circ p \in Const(2^{\mathbb{N}}, Y)$, we get $\overline{y} \circ p \in Q$, for every $p \in P$.

ii) If $h \in Mor(\mathcal{P}, \mathcal{Q}_1)$, then we get $h \circ p \in Q_1 \subseteq Q_2$, for all $p \in P$ i.e., $h \in Mor(\mathcal{P}, \mathcal{Q}_2)$. Since we have derived the double inclusion $Const_{loc}(2^{\mathbb{N}}, Y) \subseteq Q \subseteq \mathbb{F}(2^{\mathbb{N}}, Y)$ for every C-topology Q on Y, we get

$$Mor\Big(\mathcal{P}, \big(Y, Const_{loc}(2^{\mathbb{N}}, Y)\big)\Big) \subseteq Mor(\mathcal{P}, \mathcal{Q}) \subseteq Mor\Big(\mathcal{P}, \big(Y, \mathbb{F}(2^{\mathbb{N}}, Y)\big)\Big).$$

Now we choose $Q_1 = Const_{loc}(2^{\mathbb{N}}, Y)$ and $Q_2 = \mathbb{F}(2^{\mathbb{N}}, Y)$. First we see that

$$Mor\Big(\mathcal{P}, \big(Y, \mathbb{F}(2^{\mathbb{N}}, Y)\big)\Big) = \mathbb{F}(X, Y), \qquad (*)$$

since $h \circ p \in \mathbb{F}(2^{\mathbb{N}}, Y)$, for every $p \in P$ and $h \in \mathbb{F}(X, Y)$.

$$\begin{tikzcd} X \arrow[r, "h\in\mathbb{F}(X,Y)"] & Y \\ 2^{\mathbb{N}} \arrow[u, "P\ni p"] \arrow[ur, "h\circ p\in\mathbb{F}(2^{\mathbb{N}},Y)"'] & \end{tikzcd}$$

On the other hand

$$h \in Mor\Big(\mathcal{P}, \big(Y, Const_{loc}(2^{\mathbb{N}}, Y)\big)\Big) \longleftrightarrow \forall p \in P\big(h \circ p \in Const_{loc}(2^{\mathbb{N}}, Y)\big).$$

If $P = Const_{loc}(2^{\mathbb{N}}, X)$, then

$$Mor\Big(\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big), \big(Y, Const_{loc}(2^{\mathbb{N}}, Y)\big)\Big) = \mathbb{F}(X, Y).$$

$$\begin{tikzcd} X \arrow[r, "h\in\mathbb{F}(X,Y)"] & Y \\ 2^{\mathbb{N}} \arrow[u, "Const_{loc}(2^{\mathbb{N}},X)\ni p"] \arrow[ur, "h\circ p\in Const_{loc}(2^{\mathbb{N}},Y)"'] & \end{tikzcd}$$

Hence, for $\mathcal{P} = \big(X, Const_{loc}(2^{\mathbb{N}}, X)\big)$ we get the required equality

$$Mor(\mathcal{P}, \mathcal{Q}_1) = Mor(\mathcal{P}, \mathcal{Q}_2).$$

iii) If $h \in Mor(\mathcal{P}_2, \mathcal{Q})$, then we get $h \circ p_2 \in Q$, for every $p_2 \in P_2$. Hence, $h \circ p_1 \in Q$ for every $p_1 \in P_1 \subseteq P_2$, thus $h \in Mor(\mathcal{P}_1, \mathcal{Q})$.
By the double inclusion $Const_{loc}(2^{\mathbb{N}}, X) \subseteq P \subseteq \mathbb{F}(2^{\mathbb{N}}, X)$ for every C-topology P on X, we get

$$Mor\Big(\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big), \mathcal{Q}\Big) \subseteq Mor(\mathcal{P}, \mathcal{Q}) \subseteq Mor\Big(\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big), \mathcal{Q}\Big).$$

Now we take $P_1 = Const_{loc}(2^{\mathbb{N}}, X)$ and $P_2 = \mathbb{F}(2^{\mathbb{N}}, X)$. By proposition 2.3.6 we know that

$$Mor\Big(\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big), \mathcal{Q}\Big) = \mathbb{F}(X, Y).$$

$$\begin{tikzcd} X \arrow[r, "h\in\mathbb{F}(X,Y)"] & Y \\ 2^{\mathbb{N}} \arrow[u, "Const_{loc}(2^{\mathbb{N}},X)\ni p"] \arrow[ur, "h\circ p\in Q"'] & \end{tikzcd}$$

On the other hand

$$h \in Mor\Big(\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big), \mathcal{Q}\Big) \longleftrightarrow \forall p \in \mathbb{F}(2^{\mathbb{N}}, X)(h \circ p \in Q).$$

If $Q = \mathbb{F}(2^{\mathbb{N}}, Y)$, then by $(*)$ we get

$$Mor\Big(\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big), \big(Y, \mathbb{F}(2^{\mathbb{N}}, Y)\big)\Big) = \mathbb{F}(X, Y),$$

hence for $\mathcal{Q} = \big(Y, \mathbb{F}(2^{\mathbb{N}}, Y)\big)$ we get the required equation.

□

2.4.4 Definition A function $f : X \to Y$ is an *injection*, written $1-1$, if for all $x, y \in X$ it is

$$f(x) = f(y) \longrightarrow x = y.$$

Obviously, then for every $y \in f(X)$ there exists an unique $x \in X$ such that $f(x) = y$, and the *inverse function* $f^{-1} : f(X) \to X$ is well-defined.
A function $f : X \to Y$ is *surjective* or *onto* if for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$.

2.4.5 Definition Let $\mathcal{P} = (X, P)$ and $\mathcal{Q} = (Y, Q)$ be C-spaces. A C-morphism $h \in Mor(\mathcal{P}, \mathcal{Q})$ is called *C-monomorphism*, or $h \in Mono(\mathcal{P}, \mathcal{Q})$ if for every C-space $\mathcal{S} = (Z, S)$ and for all $k, l \in Mor(\mathcal{S}, \mathcal{P})$ it is

$$(h \circ k = h \circ l) \longrightarrow (k = l).$$

$$\begin{tikzcd} Z \arrow[r, shift left, "k{,}l"] \arrow[r, shift right] & X \arrow[r, "h"] & Y \end{tikzcd}$$

We call h a *C-isomorphism* between $\mathcal{P}$ and $\mathcal{Q}$, if there is some $j \in Mor(\mathcal{Q}, \mathcal{P})$ such that $j \circ h = Id_X$ and $h \circ j = Id_Y$.
A C-isomorphism between $\mathcal{P}$ and $\mathcal{P}$ is called *C-automorphism* of $\mathcal{P}$.
We call h a *C-epimorphism*, if for every C-space $\mathcal{S} = (Z, S)$ and for every $k, l \in Mor(\mathcal{Q}, \mathcal{S})$ it is

$$(k \circ h = l \circ h) \longrightarrow (k = l).$$

$$\begin{tikzcd} & & Z \\ X \arrow[r, "h"] \arrow[urr, "k\circ h"] \arrow[drr, "l\circ h"'] & Y \arrow[ur, "k"] \arrow[dr, "l"] & \\ & & Z \end{tikzcd}$$

We denote the set of the C-epimorphisms between $\mathcal{P}$ and $\mathcal{Q}$ by $Epi(\mathcal{P}, \mathcal{Q})$.
If a C-epimorphism h is onto Y, we call it a *C-set-epimorphism*. The set of the C-set-epimorohisms is denoted by $setEpi(\mathcal{P}, \mathcal{Q})$.

2.4.6 Proposition *$h \in Mono(\mathcal{P}, \mathcal{Q})$ if and only if $h \in Mor(\mathcal{P}, \mathcal{Q})$ and h is $1 - 1$.*

Proof. If h is $1 - 1$, then for every $k, l \in Mor(\mathcal{S}, \mathcal{P})$ with $h \circ k = h \circ l$ we get

$$h\big(k(z)\big) = (h \circ k)(z) = (h \circ l)(z) = h\big(l(z)\big),$$

hence $k(z) = l(z)$ i.e., $h \in Mono(\mathcal{P}, \mathcal{Q})$.
If $h \in Mono(\mathcal{P}, \mathcal{Q})$ and if $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$, then for $k = \overline{x_1}$ and $l = \overline{x_2} \in Mor(\mathcal{P}, \mathcal{Q})$ we get

$$h \circ \overline{x_1} = h \circ \overline{x_2},$$

hence $\overline{x_1} = \overline{x_2}$ i.e., $x_1 = x_2$. □

2.4.7 Proposition *A C-morphism $h \in Mor(\mathcal{P}, \mathcal{Q})$ is a C-isomorphism if and only if h is a C-monomorphism onto Y and $h^{-1} \in Mor(\mathcal{Q}, \mathcal{P})$.*

Proof. If $h \in Mor(\mathcal{P}, \mathcal{Q})$ is a C-isomorphism, then there exists a $j \in Mor(\mathcal{Q}, \mathcal{P})$ such that $j \circ h = Id_X$ and $h \circ j = Id_Y$. If $k, l \in Mor(\mathcal{S}, \mathcal{P})$ such that $h \circ k = h \circ l$, then we get

$$k = Id_X \circ k = (j \circ h) \circ k = j \circ (h \circ k) = j \circ (h \circ l) = (j \circ h) \circ l = Id_X \circ l = l,$$

hence $h \in Mono(\mathcal{P}, \mathcal{Q})$.
If $y \in Y$, then for $x := j(y) \in X$ we get

$$h(x) = h\big(j(y)\big) = (h \circ j)(y) = Id_Y(y) = y,$$

hence h is onto Y.
If $q \in Q$, then, since $j \in Mor(\mathcal{Q}, \mathcal{P})$,

$$h^{-1} \circ q = Id_X \circ (h^{-1} \circ q) = (j \circ h) \circ (h^{-1} \circ q) = j \circ (h \circ h^{-1}) \circ q = j \circ Id_Y \circ q = j \circ q \in P,$$

hence $h^{-1} \in Mor(\mathcal{Q}, \mathcal{P})$.
The converse direction is obvious by choosing $j := h^{-1} \in Mor(\mathcal{Q}, \mathcal{P})$. □

2.4.8 Proposition *Let $\mathcal{P}$ and $\mathcal{Q}$ be C-spaces. If $h \in Mor(\mathcal{P}, \mathcal{Q})$ such that h is $1-1$ and onto Y, then $h^{-1} \in Mor(\mathcal{Q}, \mathcal{P})$ if and only if for every $q \in Q$ there exists a $p \in P$ such that $q = h \circ p$.*

Proof. By definition, $h^{-1} \in Mor(\mathcal{Q}, \mathcal{P})$ if and only if $h^{-1} \circ q \in P$, for every $q \in Q$.

$$\begin{tikzcd}
Y \arrow[r, "h^{-1} \in Mor(\mathcal{Q},\mathcal{P})"] & X \\
2^{\mathbb{N}} \arrow[u, "\mathcal{Q} \ni q"] \arrow[ur, "h^{-1} \circ q \in P"'] &
\end{tikzcd}$$

If $q \in Q$, then we define $p := h^{-1} \circ q$ and we get that $q = h \circ p$.
For the converse we have that $h^{-1} \circ q = h^{-1} \circ (h \circ p) = p$. □

Obviously, $setEpi(\mathcal{P}, \mathcal{Q}) \subseteq Epi(\mathcal{P}, \mathcal{Q})$. Classically we can show, that there is no C-epimorphism in **CS** which is not a C-set-epimorphism:
Suppose there are C-spaces $\mathcal{Q} = (Y, Q), \mathcal{Q}' = (Y', Q')$ and a function $k : Y \to Y' \in Epi(\mathcal{Q}, \mathcal{Q}')$ which is not onto Y'. Because of that there exists some $y_0' \in Y'$ such that $y_0' \notin k(Y)$. Let X be a set containing at least two points and $\mathcal{P} = \big(X, Const_{loc}(2^{\mathbb{N}}, X)\big)$. If $l \in Mor(\mathcal{Q}', \mathcal{P})$, then we define $j : Y' \to X$ as follows: if $y \in k(Y)$, then $j(y) = l(y)$, and $j(y_0) \neq l(y_0)$. Since $Mor\bigg(\mathcal{Q}', \big(X, Const_{loc}(2^{\mathbb{N}}, X)\big)\bigg) = \mathbb{F}(Y', X)$ we get that

$j \in Mor(\mathcal{Q}', \mathcal{P})$. By the definition of k we have that $l \circ k = j \circ k$ and at the same time $l \neq j$.

Based on proposition 2.4.8 we define the openness of a C-morphism.

2.4.9 Definition Let $\mathcal{P}$ and $\mathcal{Q}$ be C-spaces. A C-morphism $h \in Mor(\mathcal{P}, \mathcal{Q})$ is called *open*, if for every $q \in Q$ there exists a $p \in P$ such that $q = h \circ p$.

$$\begin{tikzcd} X \arrow[r, "h"] & Y \\ 2^{\mathbb{N}} \arrow[u, "P \ni p"] \arrow[ur, "q \in Q"'] & \end{tikzcd}$$

2.4.10 Proposition ($\bigsqcup$-lifting of openness) *If $\mathcal{P} = (X, P)$, $\mathcal{Q} = (Y, \bigsqcup Q_0)$ are C-spaces and $h \in setEpi(\mathcal{P}, \mathcal{Q})$, then*

$$\forall q_0 \in Q_0 \ \exists p \in P(q_0 = h \circ p) \longrightarrow \forall q \in \bigsqcup Q_0 \ \exists p \in P(q = h \circ p).$$

Proof. i) If $q = q_0 \in Q_0$, then we just use our premise.

ii) For a constant function $\overline{q} : 2^{\mathbb{N}} \to Y$, we find, by the surjectivity of h, a $x_0 \in X$ such that $h(x_0) = \overline{q}(\alpha)$ for any $\alpha \in 2^{\mathbb{N}}$. Hence, if we define $p : 2^{\mathbb{N}} \to X, p(\alpha) = x_0$ for every $\alpha \in 2^{\mathbb{N}}$, we get $\overline{q} = h \circ p$.

iii) If $q = q_0 \circ t$, where $t \in C$ and $q_0 \in \bigsqcup Q_0$ such that $q_0 = h \circ p_0$ for some $p_0 \in P$, then
$$q = q_0 \circ t = (h \circ p_0) \circ t = h \circ (p_0 \circ t),$$
where $p_0 \circ t \in P$ by (CS_2).

iv) If $q(i\alpha) = q_i(\alpha)$ for $i \in \{0, 1\}$, where $q_i \in \bigsqcup Q_0$ such that $q_i = (h \circ p_i)$ for some $p_i \in P, i \in \{0, 1\}$, then for $p : 2^{\mathbb{N}} \to X$ defined by $p(i\alpha) = p_i(\alpha)$, for $i \in \{0, 1\}$, we get
$$q(i\alpha) = q_i(\alpha) = (h \circ p_i)(\alpha) = h\big(p_i(\alpha)\big) = h\big(p(i\alpha)\big) = (h \circ p)(i\alpha),$$
hence $q = h \circ p$.

□

2.4.11 Remark In the theory of C-spaces the $\bigsqcup$-lifting of openness can be shown directly, while in the theory of Bishop spaces this takes more effort (see [10, page 56]). The well-definability lemma and the $\mathcal{U}$-lifting of openness need to be proven first, before the $\bigvee$-lifting of openness can be shown.

2.4.12 Proposition *Let $\mathcal{P} = (X, P)$ be a C-space and $e : X \to Y$ a bijection. Then there is a unique C-topology Q_P on Y such that e is a C-isomorphism between $\mathcal{P}$ and $\mathcal{Q}_P = (Y, Q_P)$.*

Proof. We define $Q_P := \{e_p \mid p \in P\}$, where $e_p = e \circ p$.

$$\begin{tikzcd} X \arrow[r, "e"] & Y \\ 2^{\mathbb{N}} \arrow[u, "P \ni p"] \arrow[ur, "e_p"'] & \end{tikzcd}$$

Obviously, $e_{\overline{p}} \in Const(2^{\mathbb{N}}, Y)$, for $\overline{p} \in Const(2^{\mathbb{N}}, X)$.
For any $p \in P$ and $t \in C$ it is

$$e_p \circ t = (e \circ p) \circ t = e \circ (p \circ t) = e_{p \circ t}.$$

For $p^* \in P$ defined by $p^*(i\alpha) = p_i(\alpha)$, for $i \in \{0, 1\}$, where $p_i \in P$ such that $e_{p_i} \in Q_P$, we get for $i \in \{0, 1\}$

$$e_{p^*}(i\alpha) = (e \circ p^*)(i\alpha) = (e \circ p_i)(\alpha) = e_{p_i}(\alpha).$$

Hence, Q_P is a C-topology on Y.
Since for every $p \in P$ it is $e_p = e \circ p \in Q_P$, we conclude $e \in Mor(\mathcal{P}, \mathcal{Q}_P)$.
Since for every $e_p \in Q_P$ there exists a $p \in P$ such that $e_p = e \circ p$, we have that e is an open C-morphism, hence by proposition 2.4.8 a C-isomorphism between $\mathcal{P}$ and $\mathcal{Q}_P$.
Suppose now that Q is a C-topology on Y such that e is C-isomorphism between $\mathcal{P}$ and $\mathcal{Q} = (Y, Q)$. Since $e \in Mor(\mathcal{P}, \mathcal{Q})$, if we fix some $p \in P$, then there is a $q \in Q$ such that $q = e \circ p = e_p$ i.e., $Q_P \subseteq Q$.
Since e is open, we have that for every $q \in Q$ there exists a $p \in P$ such that $q = e \circ p$, hence, if we fix some $q \in Q$, we have that $q = e \circ p = e_p$ i.e., $Q \subseteq Q_P$. □

2.5 Relative C-spaces

In this section we will study what happens if our set X is restricted to a subset $Y \subseteq X$. Is there a C-space on Y, depending on a given C-space on X? And if there is one, which properties does it satisfy? How does it, for example, react with building products?

2.5.1 Definition Let $\mathcal{P} = (X, P)$ be a C-space. For $Y \subseteq X$ inhabited, we define the C-space $\mathcal{P}|_Y = (Y, P|_Y)$ by the topology

$$\begin{aligned} P|_Y &:= \bigsqcup Q_{0,Y}, \\ Q_{0,Y} &:= \left\{ p \in P \mid \forall \alpha \in 2^{\mathbb{N}} \big(p(\alpha) \in Y\big) \right\}. \end{aligned}$$

$P|_Y$ is called the *relative C-topology of $\mathcal{P}$ on Y*, $\mathcal{P}|_Y$ the *relative C-space of $\mathcal{P}$ on Y*. We denote $r : (X, P) \to (Y, P|_Y)$.

2.5.2 Example $C(\mathbb{R})|_2 = \bigsqcup \big\{c \in C(\mathbb{R}) \mid \forall \alpha \in 2^{\mathbb{N}}\big(c(\alpha) \in 2\big)\big\} = C(2)$.

2.5.3 Lemma *Let $\mathcal{P} = (X, P)$ be a C-space and $\mathcal{P}|_Y = (Y, P|_Y)$ the relative C-space, for $Y \subseteq X$ inhabited. If $p|_Y \in P|_Y$, then there exists a $p \in P$ such that $Id_Y \circ p|_Y = p$, where $Id_Y : Y \to X, Id_Y(y) = y$. In particular, we get $Id_Y \in Mor(\mathcal{P}|_Y, \mathcal{P})$.*

$$\begin{tikzcd}
2^{\mathbb{N}} \arrow[r, "p\in P"] \arrow[dr, "P|_Y \ni p|_Y"'] & X \\
& Y \arrow[u, "Id_Y"']
\end{tikzcd}$$

Proof. We use the induction principle $Ind_{\sqcup}$ with the property $A(p|_Y) : \exists p \in P(Id_Y \circ p|_Y = p)$.

i) If $p|_Y \in \big\{p \in P \mid \forall \alpha \in 2^{\mathbb{N}}\big(p(\alpha) \in Y\big)\big\}$, then $(Id_Y \circ p|_Y)(\alpha) = p|_Y(\alpha) \in P$.

ii) If $p|_Y \in Const(2^{\mathbb{N}}, Y)$, then we get $Id_Y \circ p|_Y \in Const(2^{\mathbb{N}}, Y) \subseteq Const(2^{\mathbb{N}}, X) \subseteq P$.

iii) Let $q|_Y \in P|_Y$ with $Id_Y \circ q|_Y = q \in P$ and $t \in C$, then

$$Id_Y \circ (q|_Y \circ t) = (Id_Y \circ q|_Y) \circ t = q \circ t \in P,$$

since P is a C-topology.

iv) Let $p_0|_Y, p_1|_Y \in P|_Y$ with $Id_Y \circ p_i|_Y = p_i \in P$ for $i \in \{0, 1\}$, then we get

$$(Id_Y \circ p^*|_Y)(i\alpha) = Id_Y\big(p^*|_Y(i\alpha)\big) = Id_Y\big(p_i|_Y(\alpha)\big) = (Id_Y \circ p_i|_Y)(\alpha) \in P,$$

where $p^*|_Y : 2^{\mathbb{N}} \to Y$ is defined by $p^*|_Y(i\alpha) = p_i|_Y(\alpha)$, for $i \in \{0, 1\}$.

$\square$

2.5.4 Proposition *If $\mathcal{P} = (X, \bigsqcup P_0)$ is a C-space, then the relative C-topology on some inhabited $Y \subseteq X$ is given by*

$$\begin{aligned}
(\bigsqcup P_0)|_Y &:= \bigsqcup Q_{00,Y}, \\
Q_{00,Y} &:= \Big\{p_0 \in P_0 \mid \forall \alpha \in 2^{\mathbb{N}}\big(p_0(\alpha) \in Y\big)\Big\}.
\end{aligned}$$

Proof. We need to show that $\bigsqcup Q_{0,Y} = \bigsqcup Q_{00,Y}$, where $Q_{0,Y} = \{p \in \bigsqcup P_0 \mid \forall \alpha \in 2^{\mathbb{N}} \big(p(\alpha) \in Y\big)\}$.

$(\supseteq)$ Since $Q_{00,Y} \subseteq Q_{0,Y}$, we get by proposition 2.2.2 ii) that $\bigsqcup Q_{00,Y} \subseteq \bigsqcup Q_{0,Y}$.

$(\subseteq)$ By proposition 2.2.2 i) it suffices to show that $Q_{0,Y} \subseteq \bigsqcup Q_{00,Y}$. For this, let $p \in Q_{0,Y}$ i.e., $p \in \bigsqcup P_0$ such that for all $\alpha \in 2^{\mathbb{N}}$ it is $p(\alpha) \in Y$.

i) If $p \in P_0$, then obviously $p \in Q_{00,Y}$, hence $p \in \bigsqcup Q_{00,Y}$.

ii) If $p \in Const(2^{\mathbb{N}}, X)$ such that for all $\alpha \in 2^{\mathbb{N}}$ it is $p(\alpha) \in Y$, then we get $p \in Const(2^{\mathbb{N}}, Y) \subseteq \bigsqcup Q_{00,Y}$.

iii) If $q \in Q_{0,Y}$ such that $q \in \bigsqcup Q_{00,Y}$ and if $t \in C$ is arbitrary, then $q \circ t \in \bigsqcup Q_{00,Y}$ by (CS_2) for $\bigsqcup Q_{00,Y}$.

iv) Let $p_0, p_1 \in Q_{0,Y}$ such that $p_0, p_1 \in \bigsqcup Q_{00,Y}$. For $p^* : 2^{\mathbb{N}} \to X$ defined by $p^*(i\alpha) = p_i(\alpha)$, for $i \in \{0,1\}$, we get $p^* \in \bigsqcup Q_{00,Y}$ by (CS_3') for $\bigsqcup Q_{00,Y}$.

$\square$

This proposition yields the $\bigsqcup$-lifting of relativity. The next step is an investigation of what happens to morphisms between restricted sets.

2.5.5 Proposition *For C-spaces $\mathcal{P} = (X, P), \mathcal{S} = (Y, S)$ and for inhabited sets $A \subseteq X, B \subseteq Y$ the following hold:*

i) $P|_A$ is the smallest C-topology Q on A satisfying the property $Id_A \in Mor(\mathcal{Q}, \mathcal{P})$.

ii) If $e : X \to B$, then $e \in Mor(\mathcal{P}, \mathcal{S}) \longleftrightarrow e \in Mor(\mathcal{P}, \mathcal{S}|_B)$.

iii) If $e : X \to Y$, then $e \in Mor(\mathcal{P}, \mathcal{S}) \longrightarrow e|_A \in Mor(\mathcal{P}|_A, \mathcal{S})$, where $e|_A : A \to Y$ is defined by $e|_A(a) = e(a)$ for every $a \in A$.

Proof. i) By lemma 2.5.3, $Id_A \in Mor(\mathcal{P}|_A, \mathcal{P})$.
Let Q be a C-topology on A such that $Id_A \in Mor(\mathcal{Q}, \mathcal{P})$. This means that $Id_A \circ q \in P$ for every $q \in Q$. Since we have already seen that $Id_A \circ p|_A \in P$, for every $p|_A \in P|_A$, we conclude $P|_A \subseteq Q$.

ii) By definition,

$$\begin{aligned} e \in Mor(\mathcal{P}, \mathcal{S}) &\longleftrightarrow \forall p \in P(e \circ p \in S) \longleftrightarrow \\ &\longleftrightarrow \forall p \in P(e \circ p \in S|_B) \longleftrightarrow \\ &\longleftrightarrow e \in Mor(\mathcal{P}, \mathcal{S}|_B). \end{aligned}$$

$$\begin{tikzcd} X \arrow[r, "e"] & B \subseteq Y \\ 2^{\mathbb{N}} \arrow[u, "P \ni p"] \arrow[ur, "e \circ p \in S|_B"'] & \end{tikzcd}$$

iii) By definition,

$$\begin{aligned} e \in Mor(\mathcal{P}, \mathcal{S}) &\longleftrightarrow \forall p \in P(e \circ p \in S) \longrightarrow \\ &\longrightarrow \forall p|_A \in P|_A(e|_A \circ p|_A \in S) \longleftrightarrow \\ &\longleftrightarrow e|_A \in Mor(\mathcal{P}|_A, \mathcal{S}). \end{aligned}$$

$$\begin{tikzcd} X \arrow[r, "e"] & Y \\ 2^{\mathbb{N}} \arrow[u, "P \ni p"] \arrow[ur, "s \in S"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e|_A"] & Y \\ 2^{\mathbb{N}} \arrow[u, "P|_A \ni p|_A"] \arrow[ur, "s \in S"'] & \end{tikzcd}$$

□

This demonstrates that the restriction of the codomain does not affect the continuity of a map, while the restriction of the domain preserves the continuity, but the expansion backwards does not do so.

2.5.6 Proposition *For C-spaces $\mathcal{P} = (X, P), \mathcal{S} = (Y, S)$ and $A \subseteq X, B \subseteq Y$ inhabited sets we get $(P \times S)|_{A \times B} = P|_A \times S|_B$.*

Proof. By definition,

$$\begin{aligned} (P \times S)|_{A \times B} &= \bigsqcup \left\{ h \in P \times S \mid \forall \alpha \in 2^{\mathbb{N}} \big(h(\alpha) \in A \times B \big) \right\} = \\ &= \bigsqcup \left\{ (p,s) \in P \times S \mid \forall \alpha \in 2^{\mathbb{N}} \big((p,s)(\alpha) \in A \times B \big) \right\} = \\ &= \bigsqcup \left\{ (p,s) \in P \times S \mid \forall \alpha \in 2^{\mathbb{N}} \big(p(\alpha) \in A, s(\alpha) \in B \big) \right\} = \\ &= \bigsqcup \{ (p|_A, s|_B) \mid p|_A \in P|_A, s|_B \in S|_B \} = \\ &= P|_A \times S|_B. \end{aligned}$$

□

The result of this proposition is that relativity preserve products.

3 Bishop spaces

The facts summed up in this chapter have already been explored in [10], therefore there will be no proof here. We will merely be giving an overview of facts that are of interest for this thesis, since analogical results for the theory of C-spaces were part of the previous chapter. For a better reading, we use the same structure of chapter 2, even if this may differ from the sequence of the items in [10].

3.1 Basic Definitions

3.1.1 Definition We denote the set of functions of type $X \to \mathbb{R}$ by $\mathbb{F}(X, \mathbb{R})$. The constant function on X with value $a \in \mathbb{R}$ is denoted by $\overline{a}$, the set of all constant functions of type $X \to \mathbb{R}$ by $Const(X, \mathbb{R})$.

3.1.2 Definition If a function $\Phi : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on every bounded subset B of $\mathbb{R}$, then Φ is called *Bishop continuous* i.e., for every bounded subset B of $\mathbb{R}$ and for every $\epsilon > 0$ there exists $\omega_{\Phi,B}(\epsilon) > 0$ such that for all $x, y \in B$

$$|x - y| \leq \omega_{\Phi,B}(\epsilon) \longrightarrow |\Phi(x) - \Phi(y)| \leq \epsilon.$$

The function $\omega_{\Phi,B} : \mathbb{R}^+ \to \mathbb{R}^+, \epsilon \mapsto \omega_{\Phi,B}(\epsilon)$ is called a *modulus of continuity for* Φ *on* B. A continuous function can also be written as a pair $\left(\Phi, (\omega_{\Phi,B})_{B \subseteq^b \mathbb{R}}\right)$. The set of Bishop continuous functions is denoted by $Bic(\mathbb{R})$. Similarly, by $Bic(Y)$ we denote the set of real-valued continuous functions defined on some $Y \subseteq \mathbb{R}$ such that they are uniformly continuous on every bounded subset of Y.

3.1.3 Definition An inhabited metric space (X, d) is *locally compact* if each bounded subset can be included in some compact subset of X. It is *compact* if it is complete and totally bounded.

3.1.4 Definition If (X,d) is a locally compact metric space, a function $f : X \to \mathbb{R}$ is called *Bishop continuous* if it is uniformly continuous on every bounded subset of X i.e., for every bounded subset B of X and for every $\epsilon > 0$ there exists $\omega_{f,B}(\epsilon) > 0$ such that for all $x, y \in B$

$$d(x,y) \leq \omega_{f,B}(\epsilon) \longrightarrow |f(x) - f(y)| \leq \epsilon.$$

The function $\omega_{f,B} : \mathbb{R}^+ \to \mathbb{R}^+, \epsilon \mapsto \omega_{f,B}(\epsilon)$ is called the *modulus of continuity of f on B*. The set of all Bishop continuous functions from X to $\mathbb{R}$ is denoted by $Bic(X)$.

3.1.5 Definition If $f, g \in \mathbb{F}(X, \mathbb{R}), \epsilon > 0$, and $\Phi \subseteq \mathbb{F}(X, \mathbb{R})$, we define

$$U(g,f,\epsilon) :\longleftrightarrow \forall x \in X\left(|g(x) - f(x)| \leq \epsilon\right),$$
$$U(\Phi, f) :\longleftrightarrow \forall \epsilon > 0\, \exists g \in \Phi\big(U(g,f,\epsilon)\big).$$

3.1.6 Definition A *Bishop space* is a pair $\mathcal{F} = (X, \mathbf{F})$, where X is an inhabited set and $\mathbf{F}$ is a so called *Bishop topology* i.e., $\mathbf{F}$ is a set of functions of type $X \to \mathbb{R}$ with the following clauses

(BS_1) All constant maps are in $\mathbf{F}$ i.e., $\overline{a} \in \mathbf{F}$ for $a \in \mathbb{R}$.
(BS_2) $f \in \mathbf{F} \longrightarrow g \in \mathbf{F} \longrightarrow f + g \in \mathbf{F}$.
(BS_3) $f \in \mathbf{F} \longrightarrow \Phi \in Bic(\mathbb{R}) \longrightarrow \Phi \circ f \in \mathbf{F}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\mathbf{F} \ni \Phi \circ f"'] & \mathbb{R} \arrow[d, "\Phi \in Bic(\mathbb{R})"] \\ & \mathbb{R} \end{tikzcd}$$

(BS_4) $f \in \mathbb{F}(X, \mathbb{R}) \longrightarrow U(\mathbf{F}, f) \longrightarrow f \in \mathbf{F}$.

3.1.7 Definition For two Bishop spaces $\mathcal{F} = (X, \mathbf{F})$ and $\mathcal{G} = (Y, \mathbf{G})$ a map $h : X \to Y$ is called a *Bishop morphism* from $\mathcal{F}$ to $\mathcal{G}$ if $g \circ h$ is in $\mathbf{F}$, for every $g \in \mathbf{G}$.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "\mathbf{F} \ni g \circ h"'] & Y \arrow[d, "g \in \mathbf{G}"] \\ & \mathbb{R} \end{tikzcd}$$

We write $h \in Mor(\mathcal{F}, \mathcal{G})$.

3.1.8 Definition The Bishop spaces as the objects together with the Bishop morphisms as the arrows build the category of Bishop spaces $\mathbf{BS}$.

3.1.9 Remark (BS_1) implies $Const\,(X,\mathbb{R}) \subseteq \mathbf{F}$.

3.1.10 Definition For $(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{F}\,(X,\mathbb{R})$ we denote by $f_n \overset{\text{u}}{\to} f$ that f_n *converges uniformly* to f i.e., for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in X$

$$|f_n\,(x) - f\,(x)| \leq \epsilon.$$

3.1.11 Proposition (BS_4) *is equivalent to*
(BS_4') $f_n \subseteq \mathbf{F} \longrightarrow f \in \mathbb{F}\,(X,\mathbb{R}) \longrightarrow f_n \overset{u}{\to} f \longrightarrow f \in \mathbf{F}$.

3.1.12 Proposition *If* $\mathbf{F}$ *is a Bishop topology on* X*, then*

- $fg = \frac{(f+g)^2-f^2-g^2}{2}$,
- λf,
- $-f$,
- $\max\{f,g\} = \frac{f+g+|f-g|}{2}$,
- $\min\{f,g\} = -\max\{-f,-g\}$ *and*
- $|f|$

are in $\mathbf{F}$*, for every* $f,g \in \mathbf{F}$ *and* $\lambda \in \mathbb{R}$.

3.1.13 Definition The *Cantor space* is defined by the Bishop space $\boldsymbol{\mathcal{C}} = \big(2^{\mathbb{N}}, Bic(2)^{\mathbb{N}}\big)$.

3.1.14 Definition We call $\boldsymbol{\mathcal{R}} = \big(\mathbb{R}, Bic\,(\mathbb{R})\,\big)$ the *Bishop reals*.

3.1.15 Proposition *For every compact metric space* X*, the set* $C_u(X,\mathbb{R})$ *of all uniformly continuous functions of type* $X \to \mathbb{R}$ *is a topology, called by Bishop the uniform topology on* X*. We denote this space by*

$$\mathcal{U}(X) = \big(X, C_u(X,\mathbb{R})\big).$$

3.2 Inductively generated Bishop spaces

3.2.1 Definition Let $\mathbf{F}_0 \subseteq \mathbb{F}(X, \mathbb{R})$. The *least Bishop topology* $\bigvee \mathbf{F}_0$ *generated by* $\mathbf{F}_0$ is defined by the following clauses:

i) $f_0 \in \mathbf{F}_0 \longrightarrow f_0 \in \bigvee \mathbf{F}_0$.

ii) $a \in \mathbb{R} \longrightarrow \overline{a} \in \bigvee \mathbf{F}_0$.

iii) $f \in \bigvee \mathbf{F}_0 \longrightarrow g \in \bigvee \mathbf{F}_0 \longrightarrow f + g \in \bigvee \mathbf{F}_0$.

iv) $f \in \bigvee \mathbf{F}_0 \longrightarrow \Phi \in Bic(\mathbb{R}) \longrightarrow \Phi \circ f \in \bigvee \mathbf{F}_0$.

v) $\forall \epsilon > 0 \big(g \in \bigvee \mathbf{F}_0 \longrightarrow U(g, f, \epsilon)\big) \longrightarrow f \in \bigvee \mathbf{F}_0$.

We call $\mathbf{F}_0$ a *subbase* of $\bigvee \mathbf{F}_0$.

These clauses induce the induction principle $Ind_{\vee}$ on $\bigvee \mathbf{F}_0$:

$$\forall f_0 \in \bigvee \mathbf{F}_0 \big(P(f_0)\big) \longrightarrow$$

$$\forall a \in \mathbb{R}\big(P(\overline{a})\big) \longrightarrow$$

$$\forall f, g \in \bigvee \mathbf{F}_0 \big(P(f) \longrightarrow P(g) \longrightarrow P(f+g)\big) \longrightarrow$$

$$\forall f \in \bigvee \mathbf{F_0}\ \forall \Phi \in Bic(\mathbb{R})\ \big(P(f) \longrightarrow P(\Phi \circ f)\big) \longrightarrow$$

$$\forall f \in \bigvee \mathbf{F}_0 \Big(\forall \epsilon > 0\ \exists g \in \bigvee \mathbf{F}_0 \big(P(g) \wedge U(g, f, \epsilon)\big) \longrightarrow P(f)\Big) \longrightarrow$$

$$\forall f \in \bigvee \mathbf{F}_0 \big(P(f)\big),$$

where P is any property on $\mathbb{F}(X, \mathbb{R})$.

3.2.2 Proposition *Suppose that* $\mathbf{F}_0, \mathbf{F}_1 \subseteq \mathbb{F}(X, \mathbb{R})$ *and* $\boldsymbol{\mathcal{F}} = (X, \mathbf{F})$ *a Bishop space.*

i) $\bigvee \mathbf{F}_0 \subseteq \mathbf{F} \longleftrightarrow \mathbf{F}_0 \subseteq \mathbf{F}$.

ii) $\mathbf{F}_0 \subseteq \mathbf{F}_1 \longrightarrow \bigvee \mathbf{F}_0 \subseteq \bigvee \mathbf{F}_1$.

iii) $\bigvee \mathbf{F}_0 \cup \bigvee \mathbf{F}_1 \subseteq \bigvee (\mathbf{F}_0 \cup \mathbf{F}_1)$.

iv) $\bigvee (\bigvee \mathbf{F}_0) = \bigvee \mathbf{F}_0$.

v) $\bigvee (\mathbf{F}_0 \cap \mathbf{F}_1) \subseteq \bigvee \mathbf{F}_0 \cap \bigvee \mathbf{F}_1$.

3.3 Examples of Bishop spaces

I *Const* $(X, \mathbb{R})$ and $\mathbb{F}(X, \mathbb{R})$ are Bishop topologies on X. We call $\mathbb{F}(X, \mathbb{R})$ the *discrete* Bishop topology on X.
For every Bishop topology $\mathbf{F}$ on some inhabited set X we get

$$Const\,(X, \mathbb{R}) \subseteq \mathbf{F} \subseteq \mathbb{F}\,(X, \mathbb{R})\,.$$

II *Through the least Bishop topology $\bigvee \mathbf{F}_0$ we can construct Bishop topologies:*

$$\bigvee \emptyset = Const\,(X, \mathbb{R})$$
$$\bigvee Id_{\mathbb{R}} = Bic\,(\mathbb{R})$$
$$\bigvee_{n \in \mathbb{N}} \pi_n = Bic(2)^{\mathbb{N}}, \text{where } \pi_n\,(\alpha) = \alpha_n \text{for every } \alpha \in 2^{\mathbb{N}}.$$

III *If $(X, \mathbf{F}_1)$ and $(X, \mathbf{F}_2)$ are Bishop spaces, then $(X, \mathbf{F}_1 \cap \mathbf{F}_2)$ is a Bishop space.*

IV For two Bishop spaces $\mathcal{F} = (X, \mathbf{F})$ and $\mathcal{G} = (Y, \mathbf{G})$ we define their product $\mathcal{F} \times \mathcal{G} = (X \times Y, \mathbf{F} \times \mathbf{G})$ by

$$\mathbf{F} \times \mathbf{G} := \bigvee \left(\{f \circ \pi_1 \mid f \in \mathbf{F}\} \cup \{g \circ \pi_2 \mid g \in \mathbf{G}\}\right).$$

3.4 Morphisms between Bishop spaces

3.4.1 Lemma (Yoneda) *If $\mathcal{F} = (X, \mathbf{F})$ is a Bishop space, then $\mathbf{F} = Mor(\mathcal{F}, \mathcal{R})$.*

3.4.2 Corollary The following hold:

$Bic(\mathbb{R}) = Mor(\mathcal{R}, \mathcal{R})$.
$Bic(2)^{\mathbb{N}} = Mor(\mathcal{C}, \mathcal{R})$.

3.4.3 Lemma *Suppose that $\mathcal{F} = (X, \mathbf{F}), \mathcal{G} = (Y, \mathbf{G}), \mathcal{F}_1 = (X, \mathbf{F}_1), \mathcal{F}_2 = (X, \mathbf{F}_2), \mathcal{G}_1 = (Y, \mathbf{G}_1)$ and $\mathcal{G}_2 = (Y, \mathbf{G}_2)$ are Bishop spaces. Then the following hold:*

i) $Const(X, Y) \subseteq Mor(\mathcal{F}, \mathcal{G})$.

ii) $\mathbf{G}_1 \subseteq \mathbf{G}_2 \longrightarrow Mor(\mathcal{F}, \mathcal{G}_2) \subseteq Mor(\mathcal{F}, \mathcal{G}_1)$, *while* $\mathbf{G}_1 \subsetneq \mathbf{G}_2$ *does not imply that* $Mor(\mathcal{F}, \mathcal{G}_2) \subsetneq Mor(\mathcal{F}, \mathcal{G}_1)$. *Moreover, we have that*

$$Mor\Big(\mathcal{F}, \big(Y, \mathbb{F}(Y, \mathbb{R})\big)\Big) \subseteq Mor(\mathcal{F}, \mathcal{G}) \subseteq Mor\Big(\mathcal{F}, \big(Y, Const(Y, \mathbb{R})\big)\Big).$$

iii) $\mathbf{F}_1 \subseteq \mathbf{F}_2 \longrightarrow Mor(\mathcal{F}_1, \mathcal{G}) \subseteq Mor(\mathcal{F}_2, \mathcal{G})$, *while* $\mathbf{F}_1 \subsetneq \mathbf{F}_2$ *does not imply that* $Mor(\mathcal{F}_1, \mathcal{G}) \subsetneq Mor(\mathcal{F}_2, \mathcal{G})$. *Moreover, we have that*

$$Mor\Big(\big(X, Const(X, \mathbb{R})\big), \mathcal{G}\Big) \subseteq Mor(\mathcal{F}, \mathcal{G}) \subseteq Mor\Big(\big(X, \mathbb{F}(X, \mathbb{R})\big), \mathcal{G}\Big).$$

3.4.4 Remark In the proof of this lemma it is used that

$$Mor\Big(\mathcal{F}, \big(Y, Const(Y, \mathbb{R})\big)\Big) = \mathbb{F}(X, Y),$$

for any Bishop space $\mathcal{F} = (X, \mathbf{F})$.

3.4.5 Proposition ($\bigvee$-lifting of morphisms) *Suppose that* $\mathcal{F} = (X, \mathbf{F})$ *and* $\mathcal{G}_0 = (Y, \bigvee \mathbf{G}_0)$ *are Bishop spaces. A function* $h : X \to Y \in Mor(\mathcal{F}, \mathcal{G}_0)$ *if and only if for all* $g_0 \in \mathbf{G}_0$ *it is* $g_0 \circ h \in \mathbf{F}$.

3.4.6 Definition Suppose that $\mathcal{F}, \mathcal{G}$ are Bishop spaces and $h \in Mor(\mathcal{F}, \mathcal{G})$. We call h a *Bishop-monomorphism*, or $h \in Mono(\mathcal{F}, \mathcal{G})$, if

$$\forall \mathcal{H}\big(\forall e, j \in Mor(\mathcal{H}, \mathcal{F})(h \circ e = h \circ j \longrightarrow e = j)\big).$$

We call h a *Bishop-isomorphism* between $\mathcal{F}$ and $\mathcal{G}$ if there is some $e \in Mor(\mathcal{G}, \mathcal{F})$ such that $e \circ h = Id_X$ and $h \circ e = Id_Y$. A Bishop-isomorphism between $\mathcal{F}$ and $\mathcal{F}$ is called a *Bishop-automorphism* of $\mathcal{F}$. We call h a *Bishop-epimorphism* if

$$\forall \mathcal{H}\big(\forall e, j \in Mor(\mathcal{G}, \mathcal{H})(e \circ h = j \circ h \longrightarrow e = j)\big).$$

We denote the set of the Bishop-epimorphisms between $\mathcal{F}$ and $\mathcal{G}$ by $Epi(\mathcal{F}, \mathcal{G})$. We call h a *Bishop-set-epimorphism* if it is onto Y. We denote the set of the Bishop-set-epimorphisms between $\mathcal{F}$ and $\mathcal{G}$ by $setEpi(\mathcal{F}, \mathcal{G})$.

3.4.7 Proposition

i) $h \in Mono(\mathcal{F}, \mathcal{G})$ *if and only if* $h \in Mor(\mathcal{F}, \mathcal{G})$ *and* h *is* $1-1$

ii) $h \in Mor(\mathcal{F}, \mathcal{G})$ *is a Bishop-isomorphism if and only if* h *is a Bishop-monomorphism onto* Y *and* $h^{-1} \in Mor(\mathcal{G}, \mathcal{F})$.

3.4.8 Proposition *Suppose that* $\mathcal{F}$ *and* $\mathcal{G}$ *are Bishop spaces. If* $h \in Mor(\mathcal{F}, \mathcal{G})$ *such that* h *is* $1-1$ *and onto* Y, *then* $h^{-1} \in Mor(\mathcal{G}, \mathcal{F})$ *if and only if for every* $f \in \mathbf{F}$ *there exists a* $g \in \mathbf{G}$ *such that* $f = g \circ h$.

3.4.9 Definition We call a Bishop-morphism $h \in Mor(\mathcal{F}, \mathcal{G})$ *open* if

$$\forall f \in \mathbf{F}\, \exists g \in \mathbf{G}(f = g \circ h).$$

3.4.10 Lemma (Well-definability lemma) *Suppose that X, Y are inhabited sets, $h : X \to Y$ is onto Y, $\Theta \subseteq \mathbb{F}(Y, \mathbb{R})$ and $f : X \to \mathbb{R}$. If $f \in \mathcal{U}(\Theta \circ h)$, then the function*

$$f^{\#} : Y \to \mathbb{R},$$

$$f^{\#}(y) = f^{\#}\big(h(x)\big) := f(x),$$

for every $y \in Y$, is well-defined i.e.,

$$\forall x_1, x_2 \in X\big(h(x_1) = h(x_2) \longrightarrow f(x_1) = f(x_2)\big).$$

3.4.11 Proposition ($\mathcal{U}$-lifting of openness) *Suppose that X, Y are inhabited sets, $h : X \to Y$ is onto Y, $\Phi \subseteq \mathbb{F}(X, \mathbb{R})$ and $\Theta \subseteq \mathbb{F}(Y, \mathbb{R})$. Then*

$$\forall \phi_0 \in \Phi\, \exists \theta_0 \in \Theta(\phi_0 = \theta_0 \circ h) \longrightarrow \forall \phi \in \mathcal{U}(\Phi)\, \exists \theta \in \mathcal{U}(\Theta)(\phi = \theta \circ h).$$

3.4.12 Proposition ($\bigvee$-lifting of openness) *If $\mathcal{F} = (X, \bigvee \mathbf{F}_0), \mathcal{G} = (Y, \mathbf{G})$ are Bishop spaces and $h \in setEpi_{Bish}(\mathcal{F}, \mathcal{G})$, then*

$$\forall f_0 \in \mathbf{F}_0\, \exists g \in \mathbf{G}(f_0 = g \circ h) \longrightarrow \forall f \in \bigvee \mathbf{F}_0\, \exists g \in \mathbf{G}(f = g \circ h).$$

3.4.13 Proposition *If $\mathcal{F} = (X, \mathbf{F})$ is a Bishop space, and $e : X \to Y$ a bijection, there is a unique Bishop topology $\mathbf{G}_{\mathbf{F}}$ on Y such that e is a Bishop isomorphism between $\mathcal{F}$ and $\mathcal{G}_{\mathcal{F}} = (Y, \mathbf{G}_{\mathbf{F}})$.*

3.5 Relative Bishop spaces

3.5.1 Definition If $\mathcal{F} = (X, \mathbf{F})$ is a Bishop space and $Y \subseteq X$ is inhabited, the *relative topology on Y* is defined by

$$\mathbf{F}|_Y := \bigvee \mathbf{G}_{0,Y},$$

$$\mathbf{G}_{0,Y} := \{f|_Y \mid f \in \mathbf{F}\}.$$

We call the corresponding Bishop space $\mathcal{F}|_Y = (Y, \mathbf{F}|_Y)$ the *relative Bishop space of $\mathcal{F}$* on Y. We denote $\mathbf{r} : (X, \mathbf{F}) \to (Y, \mathbf{F}|_Y)$.

3.5.2 Proposition *If $\mathcal{F} = (X, \bigvee \mathbf{F}_0)$ is a Bishop space, then the relative topology on some inhabited $Y \subseteq X$ is given by*

$$\left(\bigvee \mathbf{F}_0\right)|_Y := \bigvee \mathbf{G}_{00,Y},$$
$$\mathbf{G}_{00,Y} := \{f_0|_Y \mid f_0 \in \mathbf{F}_0\}.$$

3.5.3 Proposition *Suppose that $\mathcal{F} = (X, \mathbf{F}), \mathcal{H} = (Z, \mathbf{H})$ are Bishop spaces, and $Y \subseteq X, B \subseteq Z$ are inhabited.*

i) *$\mathbf{F}|_Y$ is the smallest topology $\mathbf{G}$ on Y satisfying the property $Id_Y \in Mor(\mathcal{G}, \mathcal{F})$.*

ii) *If $e : X \to B$, then $e \in Mor(\mathcal{F}, \mathcal{H}) \longleftrightarrow e \in Mor(\mathcal{F}, \mathcal{H}|_B)$.*

iii) *If $e : X \to Z$, then $e \in Mor(\mathcal{F}, \mathcal{H}) \longrightarrow e|_Y \in Mor(\mathcal{F}|_Y, \mathcal{H})$.*

3.5.4 Proposition *If $\mathcal{F} = (X, \mathbf{F}), \mathcal{G} = (Y, \mathbf{G})$ are Bishop spaces and $A \subseteq X, B \subseteq Y$ are inhabited, then $(\mathbf{F} \times \mathbf{G})|_{A \times B} = \mathbf{F}|_A \times \mathbf{G}|_B$.*

4 Connections between C-spaces and Bishop spaces

In this chapter, the goal is to establish a relationship between the two theories. First, we have a look at maps of type $2^{\mathbb{N}} \to \mathbb{R}$, since we already have explored a C-space with the critical set equal $\mathbb{R}$ and a Bishop space with the set equal $2^{\mathbb{N}}$. After finding a connection for these special maps we will try to expand the results for arbitrary C- and Bishop spaces.

4.1 Continuous functions of type $2^{\mathbb{N}} \to \mathbb{R}$

Here we analyse properties of the two sets

$$C(\mathbb{R}) = Mor\,(\mathcal{C}, \mathcal{R}) = \left\{ f : 2^{\mathbb{N}} \to \mathbb{R} \mid \forall c \in C\big(f \circ c \in C\,(\mathbb{R})\,\big) \right\}$$

and

$$Bic(2)^{\mathbb{N}} = Mor\,(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}}) = \left\{ f : 2^{\mathbb{N}} \to \mathbb{R} \mid \forall r \in Bic\,(\mathbb{R})\left(r \circ f \in Bic\,(2)^{\mathbb{N}} \right) \right\}.$$

$$\begin{tikzcd}
2^{\mathbb{N}} \arrow[r, "f \in Mor(\mathcal{C},\mathcal{R})"] & \mathbb{R} \\
2^{\mathbb{N}} \arrow[u, "C \ni c"] \arrow[ur, "f \circ c \in C(\mathbb{R})"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
2^{\mathbb{N}} \arrow[r, "f \in Mor(\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{R}})"] \arrow[dr, "Bic(2)^{\mathbb{N}} \ni r \circ f"'] & \mathbb{R} \arrow[d, "r \in Bic(\mathbb{R})"] \\
& \mathbb{R}
\end{tikzcd}$$

4.1.1 Remark In corollary 2.4.2 and corollary 3.4.2 we have already demonstrated that $\big(\mathbb{R}, Mor(\mathcal{C}, \mathcal{R})\big)$ is a C-space and $\big(2^{\mathbb{N}}, Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})\big)$ is a Bishop space. Now we want to investigate if the C-continuous functions of type $2^{\mathbb{N}} \to \mathbb{R}$ build a Bishop topology on $2^{\mathbb{N}}$ and the Bishop continuous functions between $2^{\mathbb{N}}$ and $\mathbb{R}$ a C-topology on $\mathbb{R}$.
Before we can prove this, for the second clause of a Bishop topology we need that $C(\mathbb{R})$, the C-continuous functions between $2^{\mathbb{N}}$ and $\mathbb{R}$, is closed under $+$.

4.1.2 Lemma *$C(\mathbb{R})$ is closed under $+$ i.e., $f+g \in C(\mathbb{R})$ for every $f, g \in C(\mathbb{R})$.*

Proof. Let $f, g \in C(\mathbb{R})$ i.e., there are n_f and $n_g \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_f} \beta \longrightarrow f(\alpha) = f(\beta)$$

and

$$\alpha =_{n_g} \beta \longrightarrow g(\alpha) = g(\beta).$$

We choose $n_{f+g} := \max\{n_f, n_g\}$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_{f+g}} \beta$ we get

$$(f+g)(\alpha) = f(\alpha) + g(\alpha) = f(\beta) + g(\beta) = (f+g)(\beta),$$

hence $f + g \in C(\mathbb{R})$. $\square$

4.1.3 Theorem *$\left(2^{\mathbb{N}}, Mor(\mathcal{C}, \mathcal{R})\right)$ is a Bishop space.*

Proof. (BS_1) Let $\overline{a} \in Const\left(2^{\mathbb{N}}, \mathbb{R}\right)$. For every $c \in C$ we get $\overline{a} \circ c \in Const\left(2^{\mathbb{N}}, \mathbb{R}\right)$, hence $\overline{a} \circ c \in C(\mathbb{R})$. Consequently, $\overline{a} \in Mor(\mathcal{C}, \mathcal{R})$.

(BS_2) Let $f, g \in Mor(\mathcal{C}, \mathcal{R})$ i.e., for every $c \in C$ it is

$$f \circ c \in C(\mathbb{R}) \text{ and } g \circ c \in C(\mathbb{R}).$$

Let now $c \in C$ be arbitrary. Then

$$(f+g) \circ c = f \circ c + g \circ c \in C(\mathbb{R})$$

by lemma 4.1.2.

(BS_3) Let $f \in Mor(\mathcal{C}, \mathcal{R})$ and $\Phi \in Bic(\mathbb{R})$ i.e., for every $c \in C$ and for every $\epsilon_{f \circ c} > 0$ there exists a $n_{f \circ c}(\epsilon_{f \circ c}) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_{f \circ c}(\epsilon_{f \circ c})} \beta \longrightarrow |(f \circ c)(\alpha) - (f \circ c)(\beta)| \leq \epsilon_{f \circ c}$$

and for every bounded subset B of $\mathbb{R}$ and for every $\epsilon_\Phi > 0$ there is a $\omega_{\Phi,B}(\epsilon_\Phi)$ such that for all $x, y \in B$ we get

$$|x - y| \leq \omega_{\Phi,B}(\epsilon_\Phi) \longrightarrow |\Phi(x) - \Phi(y)| \leq \epsilon_\Phi.$$

Let now $\epsilon > 0$ be arbitrary and we choose $n_{(\Phi \circ f) \circ c}(\epsilon) := n_{f \circ c}\left(\omega_{\Phi,B}(\epsilon)\right)$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_{(\Phi \circ f) \circ c}(\epsilon)}$ it is

$$|(f \circ c)(\alpha) - (f \circ c)(\beta)| \leq \omega_{\Phi,B}(\epsilon),$$

and thus

$$\left|\left((\Phi \circ f) \circ c\right)(\alpha) - \left((\Phi \circ f) \circ c\right)(\beta)\right| = \left|\left(\Phi \circ (f \circ c)\right)(\alpha) - \left(\Phi \circ (f \circ c)\right)(\beta)\right| =$$

$$= \left|\Phi\big((f \circ c)(\alpha)\big) - \Phi\big((f \circ c)(\beta)\big)\right| \leq \epsilon,$$

hence $\Phi \circ f \in Mor\,(\mathcal{C}, \mathcal{R})$.

(BS_4') Let $(f_n)_{n \in \mathbb{N}} \subseteq Mor(\mathcal{C}, \mathcal{R})$ and $f \in \mathbb{F}(2^{\mathbb{N}}, \mathbb{R})$ such that $f_n \xrightarrow{\mathrm{u}} f$ i.e., for every $\epsilon_u > 0$ there is a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for every $\alpha \in 2^{\mathbb{N}}$ it is $|f_n(\alpha) - f(\alpha)| \leq \epsilon_u$. Since $(f_n)_{n \in \mathbb{N}} \subseteq Mor(\mathcal{C}, \mathcal{R})$, we get $f_n \circ p \in C(\mathbb{R})$, for every $p \in C$ and for every $n \in \mathbb{N}$ i.e., for every $\epsilon_{f_n \circ p} > 0$ there exists a $n_{f_n \circ p}(\epsilon_{f_n \circ p}) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_{f_n \circ p}(\epsilon_{f_n \circ p})} \beta \longrightarrow |(f_n \circ p)(\alpha) - (f_n \circ p)(\beta)| \leq \epsilon_{f_n \circ p}.$$

We have to verify that $f \in Mor(\mathcal{C}, \mathcal{R})$. For this, let be $\epsilon > 0$. We choose $\epsilon_u := \frac{1}{3}\epsilon$ and $n_f(\epsilon) := n_{f_n \circ Id_{2^{\mathbb{N}}}}(\frac{1}{3}\epsilon)$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_f(\epsilon)} \beta$ we get

$$|f_n(\alpha) - f_n(\beta)| = |(f_n \circ Id_{2^{\mathbb{N}}})(\alpha) - (f_n \circ Id_{2^{\mathbb{N}}})(\beta)| \leq \frac{1}{3}\epsilon,$$

for every $n \in \mathbb{N}$, hence for $n \geq n_0$ it is

$$|f(\alpha) - f(\beta)| = |f(\alpha) + f_n(\alpha) - f_n(\alpha) + f_n(\beta) - f_n(\beta) - f(\beta)| \leq$$

$$\leq \underbrace{|f(\alpha) - f_n(\alpha)|}_{\leq \epsilon_u = \frac{1}{3}\epsilon} + \underbrace{|f_n(\beta) - f(\beta)|}_{\leq \epsilon_u = \frac{1}{3}\epsilon} + \underbrace{|f_n(\alpha) - f_n(\beta)|}_{\leq \frac{1}{3}\epsilon} \leq \epsilon.$$

$\square$

4.1.4 Theorem $\big(\mathbb{R}, Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})\big)$ *is a C-space.*

Proof. (CS_1) Let $\overline{a} \in Const(2^{\mathbb{N}}, \mathbb{R})$ and $r \in Bic(\mathbb{R})$. Then $r \circ \overline{a} \in Const(2^{\mathbb{N}}, \mathbb{R})$, hence $r \circ \overline{a} \in Bic(2)^{\mathbb{N}}$.

(CS_2) Let $f \in Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$ and $t \in C$ i.e., for every $r \in Bic(\mathbb{R})$, for every bounded subset B of $2^{\mathbb{N}}$ and for every $\epsilon_{r \circ f} > 0$ there exists a $n_{r \circ f, B}(\epsilon_{r \circ f}) > 0$ such that for all $\alpha, \beta \in B$ we get

$$\alpha =_{n_{r \circ f, B}(\epsilon_{r \circ f})} \beta \longrightarrow |(r \circ f)(\alpha) - (r \circ f)(\beta)| \leq \epsilon_{r \circ f}$$

and for every $m \in \mathbb{N}$ there is a $n(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n(m)} \beta \longrightarrow t(\alpha) =_m t(\beta).$$

Let B be a bounded subset of $2^{\mathbb{N}}$ and $\epsilon > 0$. We choose $n_{r \circ (f \circ t)}(\epsilon) := n\big(n_{r \circ f, B}(\epsilon)\big)$, then for all $\alpha, \beta \in B$ with $\alpha =_{n_{r \circ (f \circ t)}(\epsilon)} \beta$ we get

$$t(\alpha) =_{n_{r \circ f, B}(\epsilon)} t(\beta)$$

and hence

$$\left|\big(r \circ (f \circ t)\big)(\alpha) - \big(r \circ (f \circ t)\big)(\beta)\right| = \left|\big((r \circ f) \circ t\big)(\alpha) - \big((r \circ f) \circ t\big)(\beta)\right| =$$

$$= \big|(r \circ f)\big(t(\alpha)\big) - (r \circ f)\big(t(\beta)\big)\big| \leq \epsilon.$$

On this account, $f \circ t \in Bic(2)^{\mathbb{N}}$.

(CS_3') Let $f_0, f_1 \in Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$ such that $f(i\alpha) = f_i(\alpha)$, for $i \in \{0,1\}$ i.e., for every $r \in Bic(\mathbb{R})$ it is

$$r \circ f_0 \in Bic(2)^{\mathbb{N}} \text{ and } r \circ f_1 \in Bic(2)^{\mathbb{N}}$$

i.e., for every bounded subset B of $2^{\mathbb{N}}$ and for every $\epsilon > 0$ there are $n_{r\circ f_0}(\epsilon)$ and $n_{r\circ f_1}(\epsilon) \in \mathbb{N}$ such that for all $\alpha, \beta \in B$ it is

$$\alpha =_{n_{r\circ f_0}(\epsilon)} \beta \longrightarrow |(r \circ f_0)(\alpha) - (r \circ f_0)(\beta)| \leq \epsilon$$

and

$$\alpha =_{n_{r\circ f_1}(\epsilon)} \beta \longrightarrow |(r \circ f_1)(\alpha) - (r \circ f_1)(\beta)| \leq \epsilon.$$

Let B be a bounded subset of $2^{\mathbb{N}}$ and $\epsilon > 0$. We choose $\tilde{n}_{r\circ f}(\epsilon) := \max\big\{n_{r\circ f_0}(\epsilon), n_{r\circ f_1}(\epsilon)\big\}$, then for all $\alpha, \beta \in B$ with $\alpha =_{\tilde{n}_{r\circ f}(\epsilon)} \beta$ and for $i \in \{0,1\}$ we get

$$|(r \circ f)(i\alpha) - (r \circ f)(i\beta)| = |(r \circ f_i)(\alpha) - (r \circ f_i)(\beta)| \leq \epsilon.$$

Hence for $n_{r\circ f}(\epsilon) := \max\big\{n_{r\circ f_0}(\epsilon), n_{r\circ f_1}(\epsilon)\big\} + 1$ is $f \in Bic(2)^{\mathbb{N}}$. $\square$

Since both sets $2^{\mathbb{N}}$ and $\mathbb{R}$ produce C- and Bishop spaces with the topologies $Mor(\mathcal{C}, \mathcal{R})$ and $Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$, a connection between the two topologies is likely. In the next theorem, we will demonstrate that the C- and the Bishop continuous functions between $2^{\mathbb{N}}$ and $\mathbb{R}$ are the same.

4.1.5 Theorem The following holds: $Mor(\mathcal{C}, \mathcal{R}) = Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$.

Proof. $(\subseteq)$ Let $f \in Mor(\mathcal{C}, \mathcal{R})$ i.e., for every $t \in C$ we get $f \circ t \in C(\mathbb{R})$ i.e., for every $t \in C$ and for every $\epsilon > 0$ there exists a $n_{f\circ t}(\epsilon) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_{f\circ t}(\epsilon)} \beta \longrightarrow |(f \circ t)(\alpha) - (f \circ t)(\beta)| \leq \epsilon.$$

Let $r \in Bic(\mathbb{R})$ i.e., for every bounded subset B of $\mathbb{R}$ and for every $\epsilon > 0$ there exists a $\omega_{r,B}(\epsilon) > 0$ such that for all $x, y \in B$ we get

$$|x - y| \leq \omega_{r,B}(\epsilon) \longrightarrow |r(x) - r(y)| \leq \epsilon.$$

We have to verify that $r \circ f \in Bic(2)^{\mathbb{N}}$ i.e., for every bounded subset B' of $2^{\mathbb{N}}$ and for every $\tilde{\epsilon} > 0$ we have to find a $n_{r\circ f, B'}(\tilde{\epsilon}) \in \mathbb{N}$ such that for all $\alpha, \beta \in B'$ it is

$$\alpha =_{n_{r\circ f, B'}(\tilde{\epsilon})} \beta \longrightarrow |(r \circ f)(\alpha) - (r \circ f)(\beta)| \leq \tilde{\epsilon}.$$

Let B' be a bounded subset of $2^{\mathbb{N}}$ and $\epsilon > 0$. Let $B := \{f(\alpha) \mid \alpha \in B'\}$. We choose $n_{r\circ f,B'}(\epsilon) := n_{f\circ Id_{2^{\mathbb{N}}}}(\omega_{r,B}(\epsilon)$, then for all $\alpha, \beta \in B'$ with $\alpha =_{n_{r\circ f,B'}(\epsilon)} \beta$ we get

$$|f(\alpha) - f(\beta)| = |(f \circ Id_{2^{\mathbb{N}}})(\alpha) - (f \circ Id_{2^{\mathbb{N}}})(\beta)| \leq \omega_{r,B}(\epsilon),$$

hence

$$|(r \circ f)(\alpha) - (r \circ f)(\beta)| = \big|r\big(f(\alpha)\big) - r\big(f(\beta)\big)\big| \leq \epsilon.$$

Therefore $f \in Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$.

($\supseteq$) Let $f \in Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$ i.e., for every $r \in Bic(\mathbb{R})$ we get $r \circ f \in Bic(2)^{\mathbb{N}}$ i.e., for every $r \in Bic(\mathbb{R})$, for every bounded subset B' of $2^{\mathbb{N}}$ and for every $\epsilon > 0$ there exists a $n_{r\circ f,B'}(\epsilon) \in \mathbb{N}$ such that for all $\alpha, \beta \in B'$ it is

$$\alpha =_{n_{r\circ f,B'}(\epsilon)} \beta \longrightarrow |(r \circ f)(\alpha) - (r \circ f)(\beta)| \leq \epsilon.$$

Let $t \in C$ i.e., for every $m \in \mathbb{N}$ there is a $n_t(m) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_t(m)} \beta \longrightarrow t(\alpha) =_m t(\beta).$$

We have to verify that $f \circ t \in C\,(\mathbb{R})$ i.e., for every $\epsilon > 0$ we need to find a $n_{f\circ t}(\epsilon) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ it is

$$\alpha =_{n_{f\circ t}(\epsilon)} \beta \longrightarrow |(f \circ t)(\alpha) - (f \circ t)(\beta)| \leq \epsilon.$$

Let $\epsilon > 0$. We choose $n_{f\circ t}(\epsilon) := n_t\big(n_{Id_{2^{\mathbb{N}}}\circ f,B'}(\epsilon)\big)$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_{f\circ t}(\epsilon)} \beta$ we get

$$t(\alpha) =_{n_{Id_{2^{\mathbb{N}}}\circ f,B'}(\epsilon)} t(\beta)$$

and accordingly

$$|(f \circ t)(\alpha) - (f \circ t)(\beta)| = \big|f\big(t(\alpha)\big) - f\big(t(\beta)\big)\big| \leq \epsilon,$$

hence $f \in Mor(\mathcal{C}, \mathcal{R})$. $\square$

4.1.6 Definition We use the notation $Mor(C, R) := Mor(\mathcal{C}, \mathcal{R}) = Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$.

4.1.7 Corollary $C(\mathbb{R}) = Bic(2)^{\mathbb{N}}$.

Proof. By corollary 2.4.2, corollary 3.4.2 and theorem 4.1.5 we get

$$C(\mathbb{R}) = Mor(\mathcal{C}, \mathcal{R}) = Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}}) = Bic(2)^{\mathbb{N}}.$$

$\square$

After proving the equality of C- and Bishop continuous functions between $2^{\mathbb{N}}$ and $\mathbb{R}$, we also suggest a dependence between the C- and the Bishop continuous functions between $2^{\mathbb{N}}$ and $2^{\mathbb{N}}$ as well as between $\mathbb{R}$ and $\mathbb{R}$.

4.1.8 Proposition The following hold:

i) $Mor(\mathcal{C},\mathcal{C}) = Mor(\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{C}})$.

ii) $Mor(\boldsymbol{\mathcal{R}},\boldsymbol{\mathcal{R}}) = Mor(\mathcal{R},\mathcal{R})$.

Proof. i) This is exactly corollary 3.15 in [9].

ii) $(\subseteq)$ Let $f \in Mor(\boldsymbol{\mathcal{R}},\boldsymbol{\mathcal{R}})$ i.e., for every $g \in Bic(\mathbb{R})$ we get $g \circ f \in Bic(\mathbb{R})$ i.e., for every $g \in Bic(\mathbb{R})$, for every bounded subset B of $\mathbb{R}$ and for every $\epsilon_{g\circ f,B} > 0$ there exists a $\omega_{g\circ f,B}(\epsilon_{g\circ f,B}) > 0$ such that for all $x, y \in B$ it is

$$|x-y| \leq \omega_{g\circ f,B}(\epsilon_{g\circ f,B}) \longrightarrow |(g\circ f)(x) - (g\circ f)(y)| \leq \epsilon_{g\circ f,B}.$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f\in Mor(\boldsymbol{\mathcal{R}},\boldsymbol{\mathcal{R}})"] \arrow[dr, "Bic(\mathbb{R})\ni g\circ f"'] & \mathbb{R} \arrow[d, "g\in Bic(\mathbb{R})"] \\
& \mathbb{R}
\end{tikzcd}$$

Let $t \in C(\mathbb{R})$ i.e., for every $\epsilon_t > 0$ there exists a $n_t(\epsilon_t) \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ we get

$$\alpha =_{n_t(\epsilon_t)} \beta \longrightarrow |t(\alpha) - t(\beta)| \leq \epsilon_t.$$

We have to show that $f \circ t \in C(\mathbb{R})$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f\in Mor(\mathcal{R},\mathcal{R})"] & \mathbb{R} \\
2^{\mathbb{N}} \arrow[u, "C(\mathbb{R})\ni t"] \arrow[ur, "f\circ t\in C(\mathbb{R})"'] &
\end{tikzcd}$$

Let $\epsilon > 0$. We choose $n_{f\circ t}(\epsilon) := n_t\big(\omega_{Id_{\mathbb{R}}\circ f,B}(\epsilon)\big)$, then for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha =_{n_{f\circ t}(\epsilon)} \beta$ we get

$$|t(\alpha) - t(\beta)| \leq \omega_{Id_{\mathbb{R}}\circ f,B}(\epsilon),$$

hence

$$\begin{aligned}|(f\circ t)(\alpha) - (f\circ t)(\beta)| &= \big|f\big(t(\alpha)\big) - f\big(t(\beta)\big)\big| = \\ &= \big|(Id_{\mathbb{R}}\circ f)\big(t(\alpha)\big) - (Id_{\mathbb{R}}\circ f)\big(t(\beta)\big)\big| \leq \epsilon.\end{aligned}$$

$(\supseteq)$ This direction will be seen by the example 4.2.15 iii).

□

4.1.9 Definition We also write

$Mor(C, C) := Mor(\mathcal{C}, \mathcal{C}) = Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{C}})$.
$Mor(R, R) := Mor(\mathcal{R}, \mathcal{R}) = Mor(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}})$.

4.2 Relationships between general C- and Bishop spaces by means of continuous functions

This section focuses on finding connections between the two theories for arbitrary spaces. For this we define a set $\mathbf{F}_P$ for a given C-topology P and a set $P_{\mathbf{F}}$ for a given Bishop topology $\mathbf{F}$. We will notice that these sets again build a Bishop respectively a C-topology. These relations also satisfy some interesting properties which will be explored in this section.

4.2.1 Definition Let $\mathcal{P} = (X, P)$ be a C-space. We define

$$\mathbf{F}_P := \left\{ f : X \to \mathbb{R} \mid \forall p \in P \big(f \circ p \in Mor(C, R) \big) \right\}.$$

$$\begin{tikzcd}
X \arrow[r, "f\in\mathbf{F}_P"] & \mathbb{R} \\
2^{\mathbb{N}} \arrow[u, "P\ni p"] \arrow[ur, "f\circ p\in Mor(C,R)"'] &
\end{tikzcd}$$

4.2.2 Theorem $\mathcal{F}_{\mathcal{P}} = (X, \mathbf{F}_P)$ *is a Bishop space.*

Proof. (BS_1) Let $\overline{f} \in Const(X, \mathbb{R})$ and $p \in P$. Then $\overline{f} \circ p \in Const(2^{\mathbb{N}}, \mathbb{R}) \subseteq Mor(C, R)$, hence $\overline{f} \in \mathbf{F}_P$.

(BS_2) Let $f, g \in \mathbf{F}_P$ i.e., $f \circ p \in Mor(C, R)$ and $g \circ p \in Mor(C, R)$, for every $p \in P$. Then

$$(f + g) \circ p = f \circ p + g \circ p \in Mor(C, R),$$

since $Mor(C, R)$ is a Bishop topology. On this account, $f + g \in \mathbf{F}_P$.

(BS_3) Let $f \in \mathbf{F}_P$ i.e., $f \circ p \in Mor(C, R)$ for every $p \in P$, and let $\Phi \in Bic(\mathbb{R})$. Then

$$(\Phi \circ f) \circ p = \Phi \circ (f \circ p) \in Mor(C, R),$$

hence $\Phi \circ f \in \mathbf{F}_P$.

(BS_4') Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathbf{F}_P$ and $f \in \mathbb{F}(X, \mathbb{R})$ such that $f_n \xrightarrow{\text{u}} f$ i.e., for every $\epsilon_u > 0$ there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for every $x \in X$ it is $|f_n(x) - f(x)| \leq \epsilon_u$.

Since $(f_n)_{n\in\mathbb{N}} \subseteq \mathbf{F}_P$, we get $f_n \circ p \in Mor(C,R)$, for every $p \in P$ and for every $n \in \mathbb{N}$. We have to verify that $f \in \mathbf{F}_P$. For this, let $p \in P$. It suffices to show that $(f_n \circ p) \xrightarrow{\text{u}} (f \circ p)$, since $Mor(C,R)$ is a Bishop topology.
Let $\epsilon > 0$. We choose $n_{f\circ p}(\epsilon) := n_0$, then for every $n \geq n_{f\circ p}(\epsilon)$ and for every $\alpha \in 2^{\mathbb{N}}$ we get $p(\alpha) \in X$, hence

$$|(f_n \circ p)(\alpha) - (f \circ p)(\alpha)| = \big|f_n\big(p(\alpha)\big) - f\big(p(\alpha)\big)\big| \leq \epsilon.$$

$\square$

4.2.3 Definition We call $\mathcal{F}_{\mathcal{P}} = (X, \mathbf{F}_P)$ the *related Bishop space of* $\mathcal{P} = (X,P)$ and $\mathbf{F}_P$ the *related Bishop topology of* P.

4.2.4 Definition Let $\mathcal{F} = (X, \mathbf{F})$ be a Bishop space. We define

$$P_{\mathbf{F}} := \Big\{p : 2^{\mathbb{N}} \to X \mid \forall f \in \mathbf{F}\big(f \circ p \in Mor(C,R)\big)\Big\}.$$

$$\begin{tikzcd} 2^{\mathbb{N}} \arrow[r, "p\in P_{\mathbf{F}}"] \arrow[dr, "Mor(C,R)\ni f\circ p"'] & X \arrow[d, "f\in\mathbf{F}"] \\ & \mathbb{R} \end{tikzcd}$$

4.2.5 Theorem *$\mathcal{P}_{\mathcal{F}} = (X, P_{\mathbf{F}})$ is a C-space.*

Proof. (CS_1) Let $\overline{p} \in Const(2^{\mathbb{N}}, X)$. For every $f \in \mathbf{F}$ we get $f \circ \overline{p} \in Const(2, \mathbb{R}) \subseteq Mor(C,R)$, hence $\overline{p} \in P_{\mathbf{F}}$.

(CS_2) Let $p \in P_{\mathbf{F}}$ and $t \in C$. For every $f \in \mathbf{F}$ it is

$$f \circ (p \circ t) = \underbrace{(f \circ p)}_{\in Mor(C,R)} \circ\, t \in Mor(C,R),$$

since $Mor(C,R)$ is a C-space.

(CS_3') Let $p_0, p_1 \in P_{\mathbf{F}}$. For $p^* : 2^{\mathbb{N}} \to X$ defined by $p^*(i\alpha) = p_i(\alpha)$, for $i \in \{0,1\}$, we get for every $f \in \mathbf{F}$ and for $i \in \{0,1\}$

$$(f \circ p^*)(i\alpha) = (f \circ p_i)(\alpha) \in Mor(C,R),$$

hence $p^* \in P_{\mathbf{F}}$. $\square$

4.2.6 Definition We call $\mathcal{P}_{\mathcal{F}} = (X, P_{\mathbf{F}})$ the *related C-space of* $\mathcal{F} = (X, \mathbf{F})$ and $P_{\mathbf{F}}$ the *related C-topology of* $\mathbf{F}$.

Since we have found a link between C-spaces and Bishop spaces, we will transform these results into the language of categories. After giving the definition of a functor from [1] we will apply this for the categories **CS** and **BS**.

4.2.7 Definition A *functor*

$$F : \mathbf{C} \to \mathbf{D}$$

between categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping of objects to objects and arrows to arrows, such that the following conditions are satisfied:

i) $F(f : A \to B) = F(f) : F(A) \to F(B)$,

ii) $F(1_A) = 1_{F(A)}$,

iii) $F(g \circ f) = F(g) \circ F(f)$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[d, "F"'] & B \arrow[r, "g"] \arrow[d, "F"'] & C \arrow[d, "F"'] \\
F(A) \arrow[r, "F(f)"'] \arrow[rr, bend right, "F(g\circ f)"'] & F(B) \arrow[r, "F(g)"'] & F(C)
\end{tikzcd}$$

4.2.8 Theorem *Let $\mathcal{P} = (X, P)$ and $\mathcal{Q} = (Y, Q)$ be C-spaces and $r \in Mor(\mathcal{P}, \mathcal{Q})$, hence $\mathcal{P}$ and $\mathcal{Q}$ are objects and $r : X \to Y$ is an arrow of* **CS**. *Then $\beta : \mathbf{CS} \to \mathbf{BS}$ defined by*

$$\beta(X, P) = (X, \mathbf{F}_P),$$
$$\beta(r) = r,$$

is a functor between **CS** *and* **BS**.

Proof. i) For $r \in Mor(\mathcal{P}, \mathcal{Q})$ it is $r \circ p \in Q$ for all $p \in P$.
Let $f \in \mathbf{F}_Q$, then $f \circ q \in Mor(C, R)$ for every $q \in Q$. We have to verify that $f \circ \beta(r) \in \mathbf{F}_P$.
Let $p \in P$, then

$$\big(f \circ \beta(r)\big) \circ p = (f \circ r) \circ p = f \circ \underbrace{(r \circ p)}_{\in Q} \in Mor(C, R).$$

$$\begin{tikzcd}
2^{\mathbb{N}} \arrow[d, "P\ni p"'] \arrow[dr, "r\circ p\in Q"] & \\
(X,P) \arrow[r, "r\in Mor(\mathcal{P},\mathcal{Q})"] \arrow[d, "\beta"'] & (Y,Q) \arrow[d, "\beta"] \\
(X,\mathbf{F}_P) \arrow[r, "\beta(r)=r"] \arrow[dr, "\mathbf{F}_P\ni f\circ\beta(r)"'] & (Y,\mathbf{F}_Q) \arrow[d, "f\in\mathbf{F}_Q"] \\
& \mathbb{R}
\end{tikzcd}$$

ii) By definition, $\beta(Id_{(X,P)}) = Id_{(X,P)} = Id_{(X,\mathbf{F}_P)}$.

iii) Let $s : (Y,Q) \to (Z,S)$ be a C-morphism between $\mathcal{Q}$ and a C-space $\mathcal{S} = (Z,S)$. Then $\beta(s \circ r) = s \circ r = \beta(s) \circ \beta(r)$.
$(X,P) \xrightarrow{r} (Y,Q) \xrightarrow{s} (Z,S)$

□

4.2.9 Theorem *Let $\mathcal{F} = (X,\mathbf{F})$ and $\mathcal{G} = (Y,\mathbf{G})$ be Bishop spaces and $h \in Mor(\mathcal{F},\mathcal{G})$, hence $\mathcal{F}$ and $\mathcal{G}$ are objects and $h : X \to Y$ is an arrow of* **BS**. *Then $\gamma :$* **BS** $\to$ **CS** *defined by*

$$\gamma(X,\mathbf{F}) = (X,P_{\mathbf{F}}),$$
$$\gamma(h) = h,$$

is a functor between **BS** *and* **CS**.

Proof. i) For $h \in Mor(\mathcal{F},\mathcal{G})$ it is $g \circ h \in \mathbf{F}$ for all $g \in \mathbf{G}$.
Let $p \in P_{\mathbf{F}}$, then $f \circ p \in Mor(C,R)$ for every $f \in \mathbf{F}$. We have to verify that $\gamma(h) \circ p \in P_{\mathbf{G}}$.
For this, let $g \in \mathbf{G}$. Then

$$g \circ \big(\gamma(h) \circ p\big) = g \circ (h \circ p) = \underbrace{(g \circ h)}_{\in\mathbf{F}} \circ p \in Mor(C,R).$$

$$\begin{tikzcd}
 & \mathbb{R} \\
(X,\mathbf{F}) \arrow[ur, "\mathbf{F}\ni g\circ h"] \arrow[r, "h"] \arrow[d, "\gamma"'] & (Y,\mathbf{G}) \arrow[u, "g\in\mathbf{G}"'] \arrow[d, "\gamma"] \\
(X,P_{\mathbf{F}}) \arrow[r, "\gamma(h)=h"] & (Y,P_{\mathbf{G}}) \\
2^{\mathbb{N}} \arrow[u, "P_{\mathbf{F}}\ni p"] \arrow[ur, "\gamma(h)\circ p\in P_{\mathbf{G}}"'] &
\end{tikzcd}$$

ii) By definition, $\gamma(Id_{(X,\mathbf{F})}) = Id_{(X,\mathbf{F})} = Id_{(X,P_{\mathbf{F}})}$.

iii) Let $l : (Y,\mathbf{G}) \to (Z,\mathbf{H})$ be a Bishop morphism between $\mathcal{G}$ and a Bishop space $\mathcal{H} = (Z,\mathbf{H})$. Then $\gamma(l \circ h) = l \circ h = \gamma(l) \circ \gamma(h)$.

$$(X,\mathbf{F}) \xrightarrow{h} (Y,\mathbf{G}) \xrightarrow{l} (Z,\mathbf{H})$$

$\square$

4.2.10 Remark The following hold:

i) $\beta(X,P) = \mathcal{F}_{\mathcal{P}}$.

ii) $\gamma(X,F) = \mathcal{P}_{\mathcal{F}}$.

4.2.11 Lemma *Let $\mathcal{P} = (X,P)$ be a C-space and $\mathcal{F} = (Y,\mathbf{F})$ a Bishop space. Then*

i) $\mathbf{F}_P = Mor(\mathcal{P},\mathcal{R})$.

ii) $P_{\mathbf{F}} = Mor(\mathcal{C},\mathcal{F})$.

Proof. i) $\mathbf{F}_P = \{f : X \to \mathbb{R} \mid \forall p \in P\big(f \circ p \in Mor(C,R)\big)\} =$

$$= \Big\{f : X \to \mathbb{R} \mid \forall p \in P\Big(\forall t \in C\big((f \circ p) \circ t \in C(\mathbb{R})\big)\Big)\Big\} =$$

$$= \Big\{f : X \to \mathbb{R} \mid \forall p \in P\Big(\forall t \in C\big(f \circ (p \circ t) \in C(\mathbb{R})\big)\Big)\Big\} =$$

$$= \{f : X \to \mathbb{R} \mid \forall p' \in P\big(f \circ p' \in C(\mathbb{R})\big)\} =$$

$$= Mor(\mathcal{P},\mathcal{R}).$$

ii) $P_{\mathbf{F}} = \Big\{p : 2^{\mathbb{N}} \to Y \mid \forall f \in \mathbf{F}\big(f \circ p \in Mor(C, R)\big)\Big\} =$

$$= \Big\{p : 2^{\mathbb{N}} \to Y \mid \forall f \in \mathbf{F}\Big(\forall g \in Bic(\mathbb{R})\big(g \circ (f \circ p) \in Bic(2)^{\mathbb{N}}\big)\Big)\Big\} =$$

$$= \Big\{p : 2^{\mathbb{N}} \to Y \mid \forall f \in \mathbf{F}\Big(\forall g \in Bic(\mathbb{R})\big((g \circ f) \circ p \in Bic(2)^{\mathbb{N}}\big)\Big)\Big\} =$$

$$= \Big\{p : 2^{\mathbb{N}} \to Y \mid \forall f' \in \mathbf{F}\big(f' \circ p \in Bic(2)^{\mathbb{N}}\big)\Big\} =$$

$$= Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{F}}).$$

□

4.2.12 Proposition *Let $\mathcal{P} = (X, P)$ be a C-space and $\boldsymbol{\mathcal{F}} = (Y, \mathbf{F})$ a Bishop space. Then the following hold:*

i) $P = P_{(\mathbf{F}_P)}$.

ii) $\mathbf{F} = \mathbf{F}_{(P_{\mathbf{F}})}$.

Proof. i) $(\subseteq)$ Let $p \in P$. For $f \in \mathbf{F}_P$ we get that $f \circ p' \in Mor(C, R)$, for every $p' \in P$, especially it is $f \circ p \in Mor(C, R)$, hence $p \in P_{(\mathbf{F}_P)}$.

$(\supseteq)$ By the previous lemma we get $P_{(\mathbf{F}_P)} = Mor\Big(\boldsymbol{\mathcal{C}}, \big(X, Mor(\mathcal{P}, \mathcal{R})\big)\Big)$. Let $p \in P_{(\mathbf{F}_P)}$, then it is

$$\forall g \in Mor(\mathcal{P}, \mathcal{R})\big(g \circ p \in Bic(2)^{\mathbb{N}}\big).$$

Since $g \in Mor(\mathcal{P}, \mathcal{R}) \longleftrightarrow \forall h \in P\big(g \circ h \in C(\mathbb{R})\big)$ we get

$$\forall g\Big(\forall h \in P\big(g \circ h \in C(\mathbb{R})\big)\Big)\big(g \circ p \in Bic(2)^{\mathbb{N}}\big).$$

By corollary 4.1.7 we conclude that $p \in P$.

ii) $(\subseteq)$ Let $f \in \mathbf{F}$. For $p \in P_{\mathbf{F}}$ we get that $f' \circ p \in Mor(C, R)$, for every $f' \in \mathbf{F}$, especially it is $f \circ p \in Mor(C, R)$, hence $f \in \mathbf{F}_{(P_{\mathbf{F}})}$.

$(\supseteq)$ By the previous lemma we get $\mathbf{F}_{(P_{\mathbf{F}})} = Mor\Big(\big(X, Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{F}})\big), \mathcal{R}\Big)$. Let $f \in \mathbf{F}_{(P_{\mathbf{F}})}$, then it is

$$\forall g \in Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{F}})\big(f \circ g \in C(\mathbb{R})\big).$$

Since $g \in Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{F}}) \longleftrightarrow \forall h \in \mathbf{F}\big(h \circ g \in Bic(2)^{\mathbb{N}}\big)$ it is

$$\forall g\Big(\forall h \in \mathbf{F}\big(h \circ g \in Bic(2)^{\mathbb{N}}\big)\Big)\big(f \circ g \in C(\mathbb{R})\big).$$

By corollary 4.1.7 we conclude that $f \in \mathbf{F}$.

$\square$

4.2.13 Remark By the previous proposition we get

$\gamma\big(\beta(X,P)\big) = (X,P)$ and
$\beta\big(\gamma(X,F)\big) = (X,F)$,

hence β and γ are inverse functors.

$$\begin{tikzcd} (X,P) \arrow[r, "\beta"] \arrow[dr, "Id_X"'] & (X,\mathbf{F}_P) \arrow[d, "\gamma"] \\ & (X,P_{\mathbf{F}_P}) \end{tikzcd} \qquad \begin{tikzcd} (X,\mathbf{F}) \arrow[r, "\gamma"] \arrow[dr, "Id_X"'] & (X,P_{\mathbf{F}}) \arrow[d, "\beta"] \\ & (X,\mathbf{F}_{P_{\mathbf{F}}}) \end{tikzcd}$$

4.2.14 Proposition *Let $\mathcal{P}_1 = (X,P_1)$ and $\mathcal{P}_2 = (X,P_2)$ be C-spaces, $\boldsymbol{\mathcal{F}}_1 = (Y,\mathbf{F}_1)$ and $\boldsymbol{\mathcal{F}}_2 = (Y,\mathbf{F}_2)$ Bishop spaces. Then the following hold:*

i) $P_1 \subseteq P_2 \longrightarrow \mathbf{F}_{P_2} \subseteq \mathbf{F}_{P_1}$.

ii) $\mathbf{F}_1 \subseteq \mathbf{F}_2 \longrightarrow P_{\mathbf{F}_2} \subseteq P_{\mathbf{F}_1}$.

Proof. i) By lemma 2.4.3 we get that $Mor(\mathcal{P}_2,\mathcal{Q}) \subseteq Mor(\mathcal{P}_1,\mathcal{Q})$ for any C-space $\mathcal{Q}$, hence by lemma 4.2.11 we get $\mathbf{F}_{P_2} \subseteq \mathbf{F}_{P_1}$.

ii) By lemma 3.4.3 we get that $Mor(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{F}}_2) \subseteq Mor(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{F}}_1)$ for any Bishop space $\boldsymbol{\mathcal{G}}$, hence by lemma 4.2.11 we get $P_{\mathbf{F}_2} \subseteq P_{\mathbf{F}_1}$.

$\square$

This proposition yields that if P_1 is a sub-C-topology of P_2, then the related Bishop topology $\mathbf{F}_{P_1}$ is a superset of $\mathbf{F}_{P_2}$ and respectively if $\mathbf{F}_1$ is a sub-Bishop topology of $\mathbf{F}_2$, then the related C-topology $P_{\mathbf{F}_1}$ is a superset of $P_{\mathbf{F}_2}$.

4.2.15 Examples of related C- and Bishop spaces

i) $\beta(\mathcal{C}) = \beta\big((2^{\mathbb{N}},C)\big) = \big(2^{\mathbb{N}}, Mor(\mathcal{C},\mathcal{R})\big) = \big(2^{\mathbb{N}}, Mor(\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{R}})\big)$ is the Bishop space $\boldsymbol{\mathcal{C}}$.

ii) $\gamma(\boldsymbol{\mathcal{R}}) = \gamma\Big(\big(\mathbb{R}, Bic(\mathbb{R})\big)\Big) = \big(\mathbb{R}, Mor(\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{R}})\big) = \big(\mathbb{R}, Mor(\mathcal{C},\mathcal{R})\big)$ is the C-space $\mathcal{R}$.

iii) $\beta(\mathcal{R}) = \beta\Big(\big(\mathbb{R}, C(\mathbb{R})\big)\Big) = \big(\mathbb{R}, Mor(\mathcal{R}, \mathcal{R})\big)$ and on the other hand
$\beta(\mathcal{R}) = \beta\big(\gamma(\boldsymbol{\mathcal{R}})\big) = \boldsymbol{\mathcal{R}} = \big(\mathbb{R}, Mor(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}})\big)$.

iv) $\beta\Big(\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big)\Big) = \Big(X, Mor\Big(\big(X, Const_{loc}(2^{\mathbb{N}}, X)\big), \mathcal{R}\Big)\Big) = \big(X, \mathbb{F}(X, \mathbb{R})\big)$, by proposition 2.3.6.

v) $\beta\Big(\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big)\Big) = \Big(X, Mor\Big(\big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big), \mathcal{R}\Big)\Big) = \big(X, Const(X, \mathbb{R})\big)$.

vi) $\gamma(\boldsymbol{\mathcal{C}}) = \gamma\Big(\big(2^{\mathbb{N}}, Bic(2)^{\mathbb{N}}\big)\Big) = \big(2^{\mathbb{N}}, Mor(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{C}})\big)$ and on the other hand
$\gamma(\boldsymbol{\mathcal{C}}) = \gamma\big(\beta(\mathcal{C})\big) = \mathcal{C} = \big(2^{\mathbb{N}}, Mor(\mathcal{C}, \mathcal{C})\big)$.

vii) $\gamma\Big(\big(X, Const(X, \mathbb{R})\big)\Big) = \Big(X, Mor\Big(\boldsymbol{\mathcal{C}}, \big(X, Const(X, \mathbb{R})\big)\Big)\Big) = \big(X, \mathbb{F}(2^{\mathbb{N}}, X)\big)$ by remark 3.4.4.

viii) $\gamma\Big(\big(X, \mathbb{F}(X, \mathbb{R})\big)\Big) = \Big(X, Mor\Big(\boldsymbol{\mathcal{C}}, \big(X, \mathbb{F}(X, \mathbb{R})\big)\Big)\Big) = \big(X, Const_{loc}(2^{\mathbb{N}}, X)\big)$.

The following two propositions demonstrate that building related C- and Bishop spaces preserve products.

4.2.16 Proposition *Let $\mathcal{P} = (X, P)$ and $\mathcal{Q} = (Y, Q)$ be C-spaces. Then*

$$\boldsymbol{\mathcal{F}}_{\mathcal{P}\times\mathcal{Q}} = \boldsymbol{\mathcal{F}}_{\mathcal{P}} \times \boldsymbol{\mathcal{F}}_{\mathcal{Q}}.$$

$$\begin{tikzcd}
(X, P) \arrow[dd, "\beta"'] \arrow[dr] & & (Y, Q) \arrow[dl] \arrow[dd, "\beta"] \\
& (X \times Y, P \times Q) \arrow[d, "\beta"] & \\
(X, \mathbf{F}_P) \arrow[dr] & (X \times Y, \mathbf{F}_{P\times Q}) \arrow[d, dotted, leftrightarrow, "Id_{X\times Y}"] & (Y, \mathbf{F}_Q) \arrow[dl] \\
& (X \times Y, \mathbf{F}_P \times \mathbf{F}_Q) &
\end{tikzcd}$$

Proof. By the definitions of products and respective Bishop spaces we get

- $\mathcal{P} \times \mathcal{Q} = (X \times Y, P \times Q)$, where $P \times Q = \bigsqcup \{(p,q) \mid p \in P, q \in Q\}$.
- $\boldsymbol{\mathcal{F}}_{(\mathcal{P}\times\mathcal{Q})} = \Big(X \times Y, \{f : X \times Y \to \mathbb{R} \mid \forall (p,q) \in P \times Q\big(f \circ (p,q) \in Mor(C,R)\big)\}\Big)$.
- $\boldsymbol{\mathcal{F}}_{\mathcal{P}} = (X, \mathbf{F}_P)$, where $\mathbf{F}_P = \big\{f_P : X \to \mathbb{R} \mid \forall p \in P\big(f_P \circ p \in Mor(C,R)\big)\big\}$.
- $\boldsymbol{\mathcal{F}}_{\mathcal{Q}} = (Y, \mathbf{F}_Q)$, where $\mathbf{F}_Q = \big\{f_Q : Y \to \mathbb{R} \mid \forall q \in Q\big(f_Q \circ q \in Mor(C,R)\big)\big\}$.
- $\boldsymbol{\mathcal{F}}_{\mathcal{P}} \times \boldsymbol{\mathcal{F}}_{\mathcal{Q}} = \Big(X \times Y, \bigvee\big(\{f_P \circ \pi_1 \mid f_P \in \mathbf{F}_P\} \cup \{f_Q \circ \pi_2 \mid f_Q \in \mathbf{F}_Q\}\big)\Big)$.

It suffices to investigate the Bishop topologies.

$$\begin{aligned}
&\bigvee\big(\{f_P \circ \pi_1 \mid f_P \in \mathbf{F}_P\} \cup \{f_Q \circ \pi_2 \mid f_Q \in \mathbf{F}_Q\}\big) = \\
&\quad = \bigvee\big(\{(f_P, f_Q) : X \times Y \to \mathbb{R} \mid f_P \in \mathbf{F}_P, f_Q \in \mathbf{F}_Q\}\big) = \\
&\quad = \bigvee\Big(\big\{(f_P, f_Q) : X \times Y \to \mathbb{R} \mid \forall p \in P, q \in Q\big((f_P, f_Q) \circ (p,q) \in Mor(C,R)\big)\big\}\Big) = \\
&\quad = \bigvee\Big(\big\{(f_P, f_Q) : X \times Y \to \mathbb{R} \mid \forall (p,q) \in P \times Q\big((f_P, f_Q) \circ (p,q) \in Mor(C,R)\big)\big\}\Big) = \\
&\quad = \bigvee \mathbf{F}_{P\times Q} = \\
&\quad = \mathbf{F}_{P\times Q}.
\end{aligned}$$
∎

4.2.17 Proposition *Let $\boldsymbol{\mathcal{F}} = (X, \mathbf{F})$ and $\boldsymbol{\mathcal{G}} = (Y, \mathbf{G})$ be Bishop spaces. Then*

$$\mathcal{P}_{(\boldsymbol{\mathcal{F}}\times\boldsymbol{\mathcal{G}})} = \mathcal{P}_{\boldsymbol{\mathcal{F}}} \times \mathcal{P}_{\boldsymbol{\mathcal{G}}}.$$

$$\begin{tikzcd}
(X, \mathbf{F}) \arrow[dr] \arrow[dd, "\gamma"'] & & (Y, \mathbf{G}) \arrow[dl] \arrow[dd, "\gamma"] \\
& (X \times Y, \mathbf{F} \times \mathbf{G}) \arrow[d, "\gamma"] & \\
(X, P_{\mathbf{F}}) \arrow[dr] & (X \times Y, P_{\mathbf{F}\times\mathbf{G}}) \arrow[d, dotted, leftrightarrow, "Id_{X\times Y}"] & (Y, P_{\mathbf{G}}) \arrow[dl] \\
& (X \times Y, P_{\mathbf{F}} \times P_{\mathbf{G}}) &
\end{tikzcd}$$

Proof. By the definitions of products and respective C-spaces it is

- $\mathcal{F} \times \mathcal{G} = (X \times Y, \mathbf{F} \times \mathbf{G})$, where
  $\mathbf{F} \times \mathbf{G} = \bigvee\big(\{f \circ \pi_1 \mid f \in \mathbf{F}\} \cup \{g \circ \pi_2 \mid g \in \mathbf{G}\}\big)$.
- $\mathcal{P}_{(\mathcal{F}\times\mathcal{G})} = (X \times Y, P_{\mathbf{F}\times\mathbf{G}})$, where
  $P_{\mathbf{F}\times\mathbf{G}} = \big\{p : 2^{\mathbb{N}} \to X \times Y \mid \forall (f,g) \in \mathbf{F} \times \mathbf{G}\big((f,g) \circ p \in Mor(C,R)\big)\big\}$.
- $\mathcal{P}_{\mathcal{F}} = (X, P_{\mathbf{F}})$, where $P_{\mathbf{F}} = \big\{p_{\mathbf{F}} : 2^{\mathbb{N}} \to X \mid \forall f \in \mathbf{F}\big(f \circ p_{\mathbf{F}} \in Mor(C,R)\big)\big\}$.
- $\mathcal{P}_{\mathcal{G}} = (Y, P_{\mathbf{G}})$, where $P_{\mathbf{G}} = \big\{p_{\mathbf{G}} : 2^{\mathbb{N}} \to Y \mid \forall g \in \mathbf{G}\big(g \circ p_{\mathbf{G}} \in Mor(C,R)\big)\big\}$.
- $\mathcal{P}_{\mathcal{F}} \times \mathcal{P}_{\mathcal{G}} = \big(X \times Y, \bigsqcup\{(p_{\mathbf{F}}, p_{\mathbf{G}}) \mid p_{\mathbf{F}} \in P_{\mathbf{F}}, p_{\mathbf{G}} \in P_{\mathbf{G}}\}\big)$.

It suffices to analyze the C-topologies.

$$\begin{aligned}
&\bigsqcup\{(p_{\mathbf{F}}, p_{\mathbf{G}}) \mid p_{\mathbf{F}} \in P_{\mathbf{F}}, p_{\mathbf{G}} \in P_{\mathbf{G}}\} = \\
&\quad = \bigsqcup\big\{(p_{\mathbf{F}}, p_{\mathbf{G}}) \mid \forall f \in \mathbf{F}\big(f \circ p_{\mathbf{F}} \in Mor(C,R)\big), \forall g \in \mathbf{G}\big(g \circ p_{\mathbf{G}} \in Mor(C,R)\big)\big\} = \\
&\quad = \bigsqcup\big\{(p_{\mathbf{F}}, p_{\mathbf{G}}) \mid \forall (f,g) \in \mathbf{F} \times \mathbf{G}\big((f,g) \circ (p_{\mathbf{F}}, p_{\mathbf{G}}) \in Mor(C,R)\big)\big\} = \\
&\quad = \bigsqcup P_{\mathbf{F}\times\mathbf{G}} = \\
&\quad = P_{\mathbf{F}\times\mathbf{G}}.
\end{aligned}$$

□

4.2.18 Lemma *Let $f \in \mathbb{F}(X, \mathbb{R})$. Then*

$$\forall p \in \bigsqcup P_0\big(f \circ p \in Mor(C,R)\big) \longleftrightarrow \forall p \in P_0\big(f \circ p \in Mor(C,R)\big).$$

Proof. $(\longrightarrow)$ is clear, since $P_0 \subseteq \bigsqcup P_0$.
$(\longleftarrow)$ We use the induction principle $Ind_{\sqcup}$.

i) For $p \in P_0$ we just use our premise.

ii) For $p \in Const(2^{\mathbb{N}}, X)$ it is $f \circ p \in Const(2^{\mathbb{N}}, \mathbb{R}) \subseteq Mor(C,R)$.

iii) For $p = p_0 \circ t$, where $p_0 \in \bigsqcup P_0$ such that $f \circ p_0 \in Mor(C,R)$ and $t \in C$, it is
$$f \circ p = f \circ (p_0 \circ t) = (f \circ p_0) \circ t \in Mor(C,R).$$

iv) For $p^* : 2^{\mathbb{N}} \to X$ defined by $p^*(i\alpha) = p_i(\alpha)$, where $p_i \in \bigsqcup P_0$ such that $f \circ p_i \in Mor(C,R)$, for $i \in \{0,1\}$ we get for $\alpha = i\tilde{\alpha} \in 2^{\mathbb{N}}$
$$(f \circ p^*)(\alpha) = (f \circ p^*)(i\tilde{\alpha}) = f\big(p^*(i\tilde{\alpha})\big) = f\big(p_i(\tilde{\alpha})\big) = (f \circ p_i)(\tilde{\alpha}),$$
hence $f \circ p^* \in Mor(C,R)$.

□

4.2.19 Proposition *Let $\mathcal{P} = (X, \bigsqcup P_0)$ be a C-space, then $\mathbf{F}_{\bigsqcup P_0} = \mathbf{F}_{P_0}$.*

Proof. By the previous lemma we conclude

$$\begin{aligned}\mathbf{F}_{\bigsqcup P_0} &= \Big\{ f : X \to \mathbb{R} \mid \forall p \in \bigsqcup P_0 \big(f \circ p \in Mor(C, R) \big) \Big\} = \\ &= \big\{ f : X \to \mathbb{R} \mid \forall p \in P_0 \big(f \circ p \in Mor(C, R) \big) \big\} = \\ &= \mathbf{F}_{P_0}.\end{aligned}$$

□

This proposition yields the $\bigsqcup$-lifting of building related Bishop spaces.

4.2.20 Lemma *Let $p \in \mathbb{F}(2^{\mathbb{N}}, X)$. Then*

$$\forall f \in \bigvee \mathbf{F}_0 \big(f \circ p \in Mor(C, R) \big) \longleftrightarrow \forall f \in \mathbf{F}_0 \big(f \circ p \in Mor(C, R) \big).$$

Proof. ($\longrightarrow$) is clear, since $\mathbf{F}_0 \subseteq \bigvee \mathbf{F}_0$.
($\longleftarrow$) We use the induction principle $Ind_{\vee}$.

i) For $f \in \mathbf{F}_0$ we just use our premise.

ii) For $f \in Const(X, \mathbb{R})$ obviously $f \circ p \in Const(2^{\mathbb{N}}, \mathbb{R}) \subseteq Mor(C, R)$.

iii) For $f = f_0 + g_0$ with $f_0, g_0 \in \bigvee \mathbf{F}_0$ such that $f_0 \circ p \in Mor(C, R)$ and $g_0 \circ p \in Mor(C, R)$ it is

$$f \circ p = (f_0 + g_0) \circ p = (f_0 \circ p) + (g_0 \circ p) \in Mor(C, R).$$

iv) For $f = \phi \circ f_0$, where $\phi \in Bic(\mathbb{R})$ and $f_0 \in \bigvee \mathbf{F}_0$ such that $f_0 \circ p \in Mor(C, R)$ it is

$$f \circ p = (\phi \circ f_0) \circ p = \phi \circ (f_0 \circ p) \in Mor(C, R).$$

v) Suppose that $\epsilon > 0$ and $f_0 \in \bigvee \mathbf{F}_0$ such that $f_0 \circ p \in Mor(C, R)$, and for every $x \in X$ it is $|f_0(x) - f(x)| \leq \epsilon$. Then

$$f \circ p = (f \circ p) - (f_0 \circ p) + (f_0 \circ p) = \underbrace{\big((f - f_0) \circ p\big)}_{\leq \bar{\epsilon}} + (f_0 \circ p) \in Mor(C, R).$$

□

4.2.21 Proposition *Let $\mathcal{F} = (X, \bigvee \mathbf{F}_0)$ be a Bishop space, then $P_{\bigvee \mathbf{F}_0} = P_{\mathbf{F}_0}$.*

Proof. By the previous lemma we conclude

$$\begin{aligned} P_{\bigvee \mathbf{F}_0} &= \left\{ p : 2^{\mathbb{N}} \to X \mid \forall f \in \bigvee \mathbf{F}_0 \big(f \circ p \in Mor(C, R) \big) \right\} = \\ &= \left\{ p : 2^{\mathbb{N}} \to X \mid \forall f \in \mathbf{F}_0 \big(f \circ p \in Mor(C, R) \big) \right\} = \\ &= P_{\mathbf{F}_0}. \end{aligned}$$

□

The result of the previous proposition is the $\bigvee$-lifting of building related C-spaces. By the following proposition we verify that for inductively generated C- respectively Bishop spaces we get inductively generated related Bishop respectively C-spaces.

4.2.22 Proposition *Let $\mathcal{P} = (X, \bigsqcup P_0)$ be a C-space and $\mathcal{F} = (Y, \bigvee \mathbf{F}_0)$ a Bishop space. Then there exist $\mathbf{G}(P_0) \in \mathbb{F}(X, \mathbb{R})$ and $Q(\mathbf{F}_0) \in \mathbb{F}(2^{\mathbb{N}}, X)$ such that*

i) $\mathbf{F}_{\bigsqcup P_0} = \bigvee \mathbf{G}(P_0)$,

ii) $P_{\bigvee \mathbf{F}_0} = \bigsqcup Q(\mathbf{F}_0)$.

Proof. For $\mathbf{G}(P_0) := \mathbf{F}_{P_0}$ and $Q(\mathbf{F}_0) := P_{\mathbf{F}_0}$ the claim follows directly. □

4.2.23 Corollary *Let $\mathcal{P} = (X, \bigsqcup P_0)$ be a C-space and $\mathcal{F} = (Y, \bigvee \mathbf{F}_0)$ a Bishop space. Then*

i) $\bigsqcup P_0 = \bigsqcup Q\big(\mathbf{G}(P_0)\big)$,

ii) $\bigvee \mathbf{F}_0 = \bigvee \mathbf{G}\big(Q(\mathbf{F}_0)\big)$.

Proof. By proposition 4.2.12 and proposition 4.2.22 it is

i) $\bigsqcup P_0 = P_{(\mathbf{F}_{\bigsqcup P_0})} = P_{\left(\bigvee \mathbf{G}(P_0)\right)} = \bigsqcup Q\big(\mathbf{G}(P_0)\big)$.

ii) $\bigvee \mathbf{F}_0 = \mathbf{F}_{(P_{\bigvee \mathbf{F}_0})} = \mathbf{F}_{\left(\bigsqcup Q(\mathbf{F}_0)\right)} = \bigvee \mathbf{G}\big(Q(\mathbf{F}_0)\big)$.

□

Of course finer sets than $\mathbf{G}(P_0)$ and $Q(\mathbf{F}_0)$ can exist. Some have already been analyzed.

4.2.24 Examples

- $\mathbf{F}_{\bigsqcup Id_{2^{\mathbb{N}}}} = \mathbf{F}_C = Mor(C,R) = Bic(2)^{\mathbb{N}} = \bigvee_{n\in\mathbb{N}} \pi_n$.
- $\mathbf{F}_{\bigsqcup i} = \mathbf{F}_{C(\mathbb{R})} = Mor(R,R) = Bic(\mathbb{R}) = \bigvee Id_{\mathbb{R}}$.
- $\mathbf{F}_{\bigsqcup \emptyset} = \mathbf{F}_{Const_{loc}(2^{\mathbb{N}},X)} = \mathbb{F}(X,\mathbb{R}) = \bigvee \mathbb{F}(X,\mathbb{R})$.
- $\mathbf{F}_{\bigsqcup \mathbb{F}(2^{\mathbb{N}},X)} = \mathbf{F}_{\mathbb{F}(2^{\mathbb{N}},X)} = Const(X,\mathbb{R}) = \bigvee \emptyset$.
- $P_{\bigvee Id_{\mathbb{R}}} = P_{Bic(\mathbb{R})} = Mor(C,R) = C(\mathbb{R}) = \bigsqcup i$.
- $P_{\bigvee_{n\in\mathbb{N}} \pi_n} = P_{Bic(2)^{\mathbb{N}}} = Mor(C,C) = C = \bigsqcup Id_{2^{\mathbb{N}}}$.
- $P_{\bigvee \emptyset} = P_{Const(X,\mathbb{R})} = \mathbb{F}(2^{\mathbb{N}},X) = \bigsqcup \mathbb{F}(2^{\mathbb{N}},X)$.
- $P_{\bigvee (X,\mathbb{R})} = P_{\mathbb{F}(X,\mathbb{R})} = Const_{loc}(2^{\mathbb{N}},X) = \bigsqcup \emptyset$.

The following two propositions lead to the result that building related Bishop respectively C-spaces also preserve relativation.

4.2.25 Proposition *Let $\mathcal{P} = (X,P)$ be a C-space and $Y \subseteq X$. Then $\mathbf{F}_{(P|_Y)} = (\mathbf{F}_P)|_Y$.*

$$\begin{tikzcd}
(X,P) \arrow[r, "r"] \arrow[d, "\beta"'] & (Y,P|_Y) \arrow[d, "\beta"] \\
(X,\mathbf{F}_P) \arrow[r, "\mathbf{r}"] & \big(Y,\mathbf{F}_{(P|_Y)}\big) = \big(Y,(\mathbf{F}_P)|_Y\big)
\end{tikzcd}$$

Proof. By the definitions of related spaces and respective Bishop spaces it is

- $P|_Y = \bigsqcup \big\{p \in P \mid \forall \alpha \in 2^{\mathbb{N}}\big(p(\alpha) \in Y\big)\big\}$.
- $\mathbf{F}_{(P|_Y)} = \big\{f : Y \to \mathbb{R} \mid \forall p|_Y \in P|_Y\big(f \circ p|_Y \in Mor(C,R)\big)\big\}$.
- $\mathbf{F}_P = \big\{f : X \to \mathbb{R} \mid \forall p \in P\big(f \circ p \in Mor(C,R)\big)\big\}$.
- $(\mathbf{F}_P)|_Y = \bigvee \big(\,\{f_P|_Y \mid f_P \in \mathbf{F}_P\}\,\big) =$

$$= \bigvee \Big(\,\big\{f : Y \to \mathbb{R} \mid \forall p \in P\big(f \circ p \in Mor(C,R)\big)\big\}\,\Big) =$$
$$= \big\{f : Y \to \mathbb{R} \mid \forall p \in P|_Y\big(f \circ p \in Mor(C,R)\big)\big\} =$$
$$= \mathbf{F}_{(P|_Y)}.$$

$\square$

4.2.26 Proposition *Let $\mathcal{F} = (X, \mathbf{F})$ be a Bishop space and $Y \subseteq X$. Then $P_{(\mathbf{F}|_Y)} = (P_{\mathbf{F}})|_Y$.*

$$\begin{tikzcd}
(X, \mathbf{F}) \arrow[r, "\mathbf{r}"] \arrow[d, "\gamma"'] & (Y, \mathbf{F}|_Y) \arrow[d, "\gamma"] \\
(X, P_{\mathbf{F}}) \arrow[r, "r"] & \big(Y, P_{(\mathbf{F}|_Y)}\big) = \big(Y, (P_{\mathbf{F}})|_Y\big)
\end{tikzcd}$$

Proof. By the definitions of related spaces and respective C-spaces we get

- $\mathbf{F}|_Y = \bigvee\big(\{f|_Y \mid f \in \mathbf{F}\}\big)$.
- $P_{(\mathbf{F}|_Y)} = \big\{p : 2^{\mathbb{N}} \to Y \mid \forall f|_Y \in \mathbf{F}|_Y\big(f|_Y \circ p \in Mor(C, R)\big)\big\}$.
- $P_{\mathbf{F}} = \big\{p : 2^{\mathbb{N}} \to X \mid \forall f \in \mathbf{F}\big(f \circ p \in Mor(C, R)\big)\big\}$.
- $$\begin{aligned}(P_{\mathbf{F}})|_Y &= \bigsqcup\Big\{p \in P_{\mathbf{F}} \mid \forall \alpha \in 2^{\mathbb{N}}\big(p(\alpha) \in Y\big)\Big\} = \\ &= \bigsqcup\Big\{p : 2^{\mathbb{N}} \to Y \mid \forall f \in \mathbf{F}\big(f \circ p \in Mor(C, R)\big)\Big\} = \\ &= \bigsqcup\Big\{p : 2^{\mathbb{N}} \to Y \mid \forall f \in \mathbf{F}|_Y\big(f \circ p \in Mor(C, R)\big)\Big\} = \\ &= P_{(\mathbf{F}|_Y)}.\end{aligned}$$

□

5 Open questions

The following questions and facts need further elaboration:

i) The Tychonoff embedding theorem

The general Tychonoff embedding theorem for Bishop spaces says that a Bishop space $\mathcal{F} = (X, \bigvee \mathbf{F}_0)$ is completely regular if and only if $\mathcal{F}$ is topologically embedded into the Euclidean Bishop space $\mathcal{R}^{\mathbf{F}_0}$. This theorem is proven in [10, page 133]. To find an analogous theorem for C-spaces, definitions for *apartness* and for *topological embedding* are needed. The difficulty here is that the set X we are interested in is the codomain of a probe in a C-space $\mathcal{P} = (X, P)$. It may is possible to solve this problem by defining by means of the respective Bishop space.

A C-space $\mathcal{P} = (X, P)$ could be called *completely regular* if the Bishop space $\mathcal{F}_{\mathcal{P}} = (X, \mathbf{F}_P)$ is completely regular i.e.,

$$\forall f \in \mathbf{F}_P \big(f(x) = f(y) \longrightarrow x = y \big).$$

If $\mathcal{P} = (X, P)$, $\mathcal{Q} = (Y, Q)$ are C-spaces, a function $e : X \to Y$ could be a *topological embedding* of $\mathcal{P}$ into $\mathcal{Q}$ if e is a C-isomorphism between $\mathcal{P}$ and $\mathcal{Q}|_{e(X)}$.

It remains unclear, how the respective space, now called $\mathcal{X}(\mathcal{P}_0) = \big(X(\mathcal{P}_0), P(\mathcal{P}_0)\big)$, for a C-space $\mathcal{P}_0 = (X, \bigsqcup P_0)$, of the Euclidean Bishop space $\mathcal{R}^{\mathbf{F}_0}$ should look like for C-spaces.
The respective general Tychonoff embedding theorem would read

Suppose that $\mathcal{P}_0 = (X, \bigsqcup P_0)$ is a C-space. Then, $\mathcal{P}$ is completely regular if and only if $\mathcal{P}$ is topologically embedded into the C-space $\mathcal{X}(\mathcal{P}_0)$.

If one could prove that the existence of a Bishop-isomorphism $e : X \to \mathbb{R}^{\mathbf{G}(P_0)}$ between $\mathcal{F}_{\mathcal{P}_0}$ and $\mathcal{R}^{\mathbf{G}(P_0)}|_{e(X)}$ is equivalent to the existence of a C-isomorphism $\tilde{e} : X(\mathcal{P}_0) \to X$ between $\mathcal{X}(\mathcal{P}_0)|_{\tilde{e}(X)}$ and $\mathcal{P}_0$, then by means of the general Tychonoff embedding theorem for Bishop spaces the respective general Tychonoff embedding theorem for C-spaces could be verified by the following way:

$\mathcal{P}_0$ is completely regular

$\longleftrightarrow \boldsymbol{\mathcal{F}}_{\mathcal{P}_0} = (X, \mathbf{F}_{\bigsqcup P_0}) = \big(X, \bigvee \mathbf{G}(P_0)\big)$ is completely regular

$\longleftrightarrow \boldsymbol{\mathcal{F}}_{\mathcal{P}_0}$ is topologically embedded into the Euclidian Bishop space $\boldsymbol{\mathcal{R}}^{\mathbf{G}(P_0)}$

$\longleftrightarrow \exists e : X \to \mathbb{R}^{\mathbf{G}(P_0)}$ Bishop-isomorphism between $\boldsymbol{\mathcal{F}}_{\mathcal{P}_0}$ and $\boldsymbol{\mathcal{R}}^{\mathbf{G}(P_0)}|_{e(X)}$

$\longleftrightarrow \exists \tilde{e} : X(\mathcal{P}_0) \to X$ C-isomorphism between $\mathcal{X}(\mathcal{P}_0)|_{\tilde{e}(X)}$ and $\mathcal{P}_0$

$\longleftrightarrow \mathcal{P}_0$ is topologically embedded into $\mathcal{X}(\mathcal{P}_0)$.

ii) Restrictions on X for C-spaces

While in this Thesis the definition of a C-space is just repeated, in [13] also a derivation of how to get this specific definition is explained. The beginning is at the definition of *quasi-topological spaces* introduced by Spanier in [12]:

A *quasi-topological space* is a set X endowed with a *quasi-topology*.
A *quasi-topology* on X assigns to each compact Hausdorff space K a set $Q(K, X)$ of functions of type $K \to X$ such that:

a) $Const(K, X) \subseteq Q(K, X)$.

b) If $t' : K' \to K$ is continuous and $q \in Q(K, X)$, then $q \circ t' \in Q(K', X)$.

c) If $\{t_i : K_i \to K\}_{i \in I}$ is a finite, jointly surjective family and $q : K \to X$ is a map with $q \circ t_i \in Q(K_i, X)$, for every $i \in I$, then $q \in Q(K, X)$.

A function $f : X \to Y$ between quasi-topological spaces X and Y is called *continuus* if $f \circ q \in Q(K, Y)$, for every $q \in Q(K, X)$.

$$\begin{tikzcd} X \arrow[r, "f \text{ continuous}"] & Y \\ K \arrow[u, "Q(K,X)\ni q"] \arrow[ur, "f\circ q\in Q(K,Y)"'] & \end{tikzcd}$$

By considering just one compact Hausdorff space, the Cantor space $2^{\mathbb{N}}$, and by restricting the jointly surjective finite families of continuous maps to the covering families $\{cons_s\}_{s \in 2^n}$, where $cons_s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by $cons_s(\alpha) = s\alpha$, we get the definition of a C-space. It also follows that the category **CS** is cartesian closed.

In comparison to Bishop spaces we note that for every functions in a Bishop topology the critical set X performs as the domain, hence we have no restriction on it.

In C-topologies we always start with the compact Hausdorff space $2^{\mathbb{N}}$ as the domain and have C-continuous maps as probes. Since for continuous functions the image of a compact set is again compact in classical mathematics, we should study the relation between the images of $2^{\mathbb{N}}$ under the probes and the compact subsets of X.

iii) Fan functional

In this Thesis mainly results of [10] are studied in C-spaces. By means of the bridge we have found that it is possible to transform the results of [13] into the theory of Bishop spaces.

For example in chapter 3.5 the fan fuctional

$$\text{fan} \ : \mathbb{N}^{2^{\mathbb{N}}} \to \mathbb{N},$$

which C-continuously calculates least moduli of uniform continuity, is defined. Maybe this could help to study the FAN functional

$$\text{FAN} \ : Mor\Big(2^{\mathbb{N}}, \big(\mathbb{N}, \mathbb{F}(\mathbb{N}, \mathbb{R})\big)\Big) \to \mathbb{N}$$

$$\Phi \mapsto \text{FAN}\ (\Phi)$$

$$\forall \alpha, \beta \Big(\overline{\alpha}\big(\text{FAN}\ (\Phi)\big) = \overline{\beta}\big(\text{FAN}\ (\Phi)\big) \longrightarrow \Phi(\alpha) = \Phi(\beta) \Big)$$

mentioned in [10], chapter 8.

One could also try to find a **fan** functional

$$\textbf{fan} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R},$$

which Bishop continuously calculates least moduli of uniform continuity in **BS**.

While the existence of exponentials has already been proven in the theory of C-spaces, this is still an open question in the theory of Bishop spaces. It is expected that **BS** is not cartesian closed, as it is the case in the category of topological spaces **Top**. It would be interesting to determine cartesian closed subcategories of **BS**. We denote the set of them by $\mathbf{BS}_c$. Now one could also investigate the following points:

iv) Limit spaces and Kleene-Kreisel-spaces

The definition of limit spaces is given in [13], chapter 4 by:

A *limit space* is a set X together with a family of functions $x : \mathbb{N}_\infty \to X$, written as $(x_i) \to x_\infty$ and called *convergent sequences* in X, satisfying the following conditions:

a) The constant sequence (x) converges to x.

b) If (x_i) converges to x_∞, then so does every subsequence of (x_i).

c) If (x_i) is a sequence such that every subsequence of (x_i) contains a subsequence converging to x_∞, then (x_i) converges to x_∞.

The collection of convergent sequences in X is called the *limit structure* on X. A function $f : X \to Y$ of limit spaces is said to be *continuous* if it preserves convergent sequences i.e.,

$$\big((x_i) \to x_\infty\big) \longrightarrow \big((fx_i) \to fx_\infty\big).$$

The category of limit spaces and continuous maps is denoted by **Lim**.

By the lemmas 4.2.3 and 4.2.4 we get functors between **Lim** and **CS**:

- The functor $G : \mathbf{Lim} \to \mathbf{CS}$ is given by
  a) For any limit space X, the limit probes form a C-topology on X.
  b) For any two limit spaces X and Y, a function $X \to Y$ is limit-continuous if and only if it is continuous w.r.t. the limit probes.
- The functor $F : \mathbf{CS} \to \mathbf{Lim}$ is given by
  a) For any C-space X, the probe-continuous maps $\mathbb{N}_\infty \to X$ form a limit structure on X.
  b) For any two C-spaces X and Y, if a function $X \to Y$ is probe-continuous then it is limit-continuous w.r.t. the above limit structures.

Now one could try to compose this functors with the functors $\beta : \mathbf{CS} \to \mathbf{BS}$ and $\gamma : \mathbf{BS} \to \mathbf{CS}$. Here we have to restrict the functors to $\beta_c : \mathbf{CS} \to \mathbf{BS}_c$ and $\gamma_c : \mathbf{BS}_c \to \mathbf{CS}$. Then there is to verify if the compositions

$\beta_c \circ G : \mathbf{Lim} \to \mathbf{BS}_c$ and
$F \circ \gamma_c : \mathbf{BS}_c \to \mathbf{Lim}$

are functors between the category **Lim** and $\mathbf{BS}_c$.

$$\begin{tikzcd}
\mathbf{BS}_c \arrow[d, shift left, "\gamma_c"] \arrow[dr, shift left, "F\circ\gamma_c"] & \\
\mathbf{CS} \arrow[u, shift left, "\beta_c"] \arrow[r, shift left, "F"] & \mathbf{Lim} \arrow[l, shift left, "G"] \arrow[ul, shift left, "\beta_c\circ G"]
\end{tikzcd}$$

After this, one could also investigate like in [13], chapter 4, if limit spaces may form a reflective subcategory or an exponential ideal of Bishop spaces, if $F \circ \gamma_c : \mathbf{BS}_c \to \mathbf{Lim}$ preserve finite products and if $\beta_c \circ G : \mathbf{Lim} \to \mathbf{BS}_c$ is cartesian closed.

In remark 4.2.13 we have demonstrated that $\beta\big(\gamma(X, \mathbf{F})\big) = (X, \mathbf{F})$ for a Bishop space $\mathcal{F} = (X, \mathbf{F})$. Does this equation also hold for β_c and γ_c? If this would be the case, then by lemma 4.2.10 of [13] we notice:

If $\mathcal{F} = (X, \mathbf{F})$ is a Bishop space such that $\gamma_c(X, \mathbf{F})$ is a discrete C-space, then

$$\big((\beta_c \circ G) \circ (F \circ \gamma_c)\big)(X) = X.$$

If all the above questions can be answered by yes an analogous assertion for the Kleene-Kreisel spaces defined in [13] can be made:

The Kleene-Kreisel spaces can be calculated within $\mathbf{BS}_c$ by starting from the natural numbers object and iterating products and exponentials.

v) In [10] the relationship between the category **Top** and the category of quasi-topological spaces is explored. Now it would be interesting to see if a similar relationship could exist between **BS** and another category. An approach could be the investigation of the Bishop spaces generated by C-spaces. Do these spaces again build a category? If this holds, then the suggestion is that this category is cartesian closed and exponentials exist, since it is the case in the category of C-spaces. One could also compare the cartesian closed subcategories of **BS** we mentioned before with the Bishop spaces generated by C-spaces.

Of course this is just a small list of questions that can be worked out in the future.

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