Ludwig-Maximilians-Universität München
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Chu Categories

Luis Gambarte

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Advisor: Priv.-Doz. Dr. Iosif Petrakis

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Contents

Abstract

Chu categories were first introduced to generate $*$-autonomous categories from closed symmetric monoidal categories. A special case of the Chu construction, so called Chu spaces were used later on in other areas of mathematics, e.g game theory or quantum logic. In this thesis we present an overview of the theory of Chu categories, as well as a few examples in which the Chu construction actually relates to topics that are not related at first sight. Furthermore we examine the relation of the Chu construction to other categorical notions, especially whether the existence of certain objects, like for example products, in the base category implies the existence of such objects in the Chu category.

Another important tool in category theory are so-called Grothendieck constructions, which have already been examined in relation to Chu categories. We strengthen this relationship by giving a Grothendieck construction that can be associated to any Chu category and is in fact equivalent.

At last we present a few generalizations of Chu categories, e.g. the generalized Chu category which has its origin in the examination of the category of predicates.

Zusammenfassung

Chu-Kategorien wurden eingeführt, um aus einer geschlossenen, symmetrisch monoidalen Kategorie eine $*$-autonome Kategorie zu gewinnen. Später wurden Chu-Räume, ein Spezialfall der Chu-Kategorie, in anderen Bereichen der Mathematik wie Spieltheorie oder Quantenlogik genutzt. In dieser Arbeit wollen wir zum einen einen Überblick über die Chu-Konstruktion sowie einige Beispiele geben, in welchen die Chu-Konstruktion auch in scheinbar nicht verwandten Fachbereichen angewendet werden kann. Ebenso untersuchen wir das Verhältnis der Chu-Konstruktion zu anderen kategoriellen Konstruktionen, insbesondere ob sich die Existenz von Objekten wie z.B. Produkten in der zugrunde liegenden Kategorie auch auf die Chu-Kategorie vererbt.

Ein weiteres wichtiges Hilfsmittel in der Kategorientheorie sind sogenannte Grothendieck-Konstruktionen, welche schon in Zusammenhang mit Chu-Räumen untersucht wurden. Wir vertiefen diesen Zusammenhang und geben eine Grothendieck-Konstruktion, welche zu jeder gegebenen Chu-Kategorie vollführt werden kann und zu dieser äquivalent ist.

Zu guter Letzt geben wir noch einige Verallgemeinerungen der Chu-Kategorie an, z.B. die verallgemeinerte Chu-Kategorie, welche der Untersuchung der Kategorie der Prädikate entsprang.

Acknowledgements

I am deeply indebted towards my advisor, Priv.-Doz. Dr. Iosif Petrakis, who not only aided me in the writing of this thesis, but who also introduced me to the interesting topic of Chu categories and assisted me in understanding their inner workings.

I want to thank Robin Tobias Mader and Clemens Julian Koppelstetter for our insightful discussions regarding category theory, which influenced some of the ways the material in this thesis is presented.

Lastly I want to thank my mother, who supported me during the entire time this thesis was in the making and without whom this work would most likely have never come to completion.

Chapter 1

Introduction

Chu categories were introduced by P.-H. Chu in his master's thesis *Constructing $*$-autonomous categories*, which formed the appendix of the book *$*$-Autonomous Categories* by M. Barr [Bar79]. Later on, with the emergence of linear logic, an invention by J.-Y. Girard, a "logic behind logic", as he calls it in his paper [Gir87], it was shortly after discovered by R. Seely, that $*$-autononmous categories are in a crucial relationship to linear logic. A special kind of Chu categories, the so called *Chu spaces* have found their way into theoretical physics and theoretical computer science, where they are used to model systems with quantum features [Abr12] and game theory.

1.1 A (brief) history of the Chu construction

A (not complete) historical overview of the developments of the theories used to define the Chu construction as well as related topics is summarized in the following table.

1964 F. Lawvere introduces cartesian closed categories in his paper [Law64], in which he establishes the cartesian closed category of sets as an alternative foundation to set theory and therefore lays the foundation for category theory as a base theory for mathematics.

1979 M. Barr introduces $*$-autonomous categories in his paper [Bar79] and H.-P. Chu introduces his Chu categories in his master's thesis which forms the appendix to this paper.

1987 J.-Y. Girard introduces linear logic, a logic in which the usual implication $A \Rightarrow B$ is broken into two operations, $!A$ and $A \multimap B$, where the cut-axiom no longer holds, so $A \& A \multimap B$ is not the same as $A \multimap B$. The classical implication can then be expressed through $!A \multimap B$.

1989 R. A. G. Seely discovers in his paper [See] that Chu categories can be used to model parts of linear logic.

1991 Chu spaces are first used under the term "games" by Y. Lafont and T. Streicher in their paper [LS91].

1.2 Goals & structure of this thesis

The goals of this thesis are twofold.

- We want to present the usefulness of the Chu construction as the correct frameworks for certain questions arising in mathematics, e.g. how to represent the notion of a topological space using a certain Chu space.
- Secondly we want to present the rich theory surrounding the Chu construction itself, as the Chu construction itself exhibits interesting properties depending on the properties of the category it is built upon.

The thesis is structured in the following way.

2 The Chu construction
In this chapter we introduce symmetric monoidal categories as well as $*$-autonomous categories. Furthermore we introduce cartesian closed categories, a special kind of closed symmetric monoidal categories and discuss their relation to λ-calculi, who are special kinds of type theories. We then present the definition of the Chu construction over a cartesian closed category and an object $\gamma \in \mathscr{C}_0$. Lastly we specify Chu spaces, which are objects of the category $\mathsf{Chu}(\mathsf{Set}, X)$ for an arbitrary set X.

3 Chu representations
In this chapter we present a few examples of representations of categories like the category Top of topological spaces or the affine category $\mathsf{Aff}(\mathscr{C}, c)$ over a category $\mathscr{C}$ with an object $c \in \mathscr{C}_0$. All of these representations are distinct from the strict representation we present in section 4.2. By constructing the representation of the category of topological spaces we demonstrate that non-cartesian closed categories can be embedded into Chu constructions.

4 The Chu functors
In this chapter we construct the strict representation of a cartesian closed category $\mathscr{C}$ into its Chu construction with an arbitrary object $\gamma \in \mathscr{C}_0$. We then make the rules of assigning the Chu category to an object, $\mathscr{C}_0 \ni \gamma \mapsto \mathsf{Chu}(\mathscr{C}, \gamma)$ and to a pair $(\mathscr{C}, \gamma) \mapsto \mathsf{Chu}(\mathscr{C}, \gamma)$ rigorous by the internal and the global Chu functor.

5 The Chu construction and categorical constructions
In this chapter we discuss the interaction of the Chu category with the dual category and the product category of two cartesian closed categories, $\mathscr{C}, \mathscr{D}$. To be more precise we want to show that the product category of two Chu categories $\mathsf{Chu}(\mathscr{C}, \gamma) \times \mathsf{Chu}(\mathscr{D}, \delta)$is isomorphic to th $\mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma, \delta))$. As the opposite category of a cartesian closed category need not be cartesian closed, we define the coChu construction.

6 The Chu construction and limits
In this chapter we discuss the interaction of the Chu construction with limits. We start by dissecting the interaction of the Chu construction with products, one of the simplest form of limits. Finally we prove that $\mathsf{Chu}(\mathscr{C}, \gamma)$ is bicomplete if the base category $\mathscr{C}$ is bicomplete.

7 Generalizations of the Chu construction
In this chapter we present various generalizations of the Chu construction. Our first generalization is the generalized Chu category, which allows arrows $a \times x \to \Gamma_0(y)$ for a fixed endofunctor $\Gamma \colon \mathscr{C} \to \mathscr{C}$ instead of just arrows $a \times x \to \gamma$. Next we generalize the Chu category via a Grothendieck construction which allows us to imitate the Chu construction in categories which are not cartesian closed. Then we generalize the Chu construction to products with arbitrary many factors using the previously obtained Grothendieck construction. At last we find a Grothendieck construction equivalent to the generalized Chu category.

8 The Chu construction and topoi
As Chu spaces are objects of a category $\mathsf{Chu}(\mathsf{Set}, K)$ and Set is the prime example of a topos, we want to examine whether the category $\mathsf{Chu}(\mathsf{Set}, K)$ is a topos. To this end we dissect the hypothetical subobject classifier in $\mathsf{Chu}(\mathsf{Set}, K)$ and finish by concluding from this discussion that such a subobject classifier can not exist, ergo $\mathsf{Chu}(\mathsf{Set}, K)$ is not a topos.

1.3 Contributions & Material

The main contribution of this thesis does not lie within the proof of a special theorem, but in presenting the material surrounding Chu categories in a unified fashion.

2 The Chu construction
This chapter consists entirely of presentation of known results. Our approach to symmetric monoidal categories and $*$-autonomous categories is due to [Bar79] and [BW20]. For our definition of cartesian closed categories we use [Awo10], but his definition of a cartesian closed category is the same as in [BW20]. Our definition of a Chu space is due to [Abr18].

3 Chu representations
The material of this chapter is due to [Pet21], we simply gave proofs where they were omitted in this text.

4 The Chu functors
The material in this chapter is due to [Pet21], we simply gave proofs where they were omitted in this text.

5 The Chu construction and categorical constructions
The lemma of the first section is due to [Man17], we only altered the proof slightly. The material of sections 5.2 – 5.6 is due to our own work which was inspired by correspondence with the advisor of this thesis, I. Petrakis.

6 The Chu construction and limits
The material of the first section we came up with after reading [Man17]. The material of the second section is due to [Man17], although we alter his proof using an equivalent characterisation of bicompleteness due to [Awo10].

7 Generalizations of the Chu construction
The material of the first four sections, is either due to [Pet21] or originated from correspondence with the author, I. Petrakis. The last three sections are due to our own work, which was again incited by I. Petrakis.

8 The Chu construction and topoi
The material of this chapter is due to our own work which was incited by I. Petrakis.

All illustrations in this thesis were created by the author using the Ti*k*Z-package.

Chapter 2

The Chu construction

The Chu construction first arose as a way to generate $*$-autonomous categories, which themselves are special forms of closed symmetric monoidal categories. Hence we start by introducing these special kinds of categories.

2.1 Symmetric monoidal categories

Definition 2.1 (Symmetric monoidal categories). Let $\mathscr{C}$ be a closed category. It is *symmetric monoidal* if there exists a bifunctor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and an object $\top$. We often write $A \otimes B$ instead of $\otimes_0(A, B)$ for $A, B \in \mathscr{C}_0$. Furthermore, if $(f, f')\colon (C, D) \to (E, F)$ is an arrow in $\mathscr{C} \times \mathscr{C}$, we write $f \otimes f'$ instead of $\otimes_1(f, f')$. The bifunctor $\otimes$ and $\top$ must fulfil the following axioms.

(SMC$_1$) For all $A, B, C \in \mathscr{C}_0$ there must exist an isomorphism

$$a_{A,B,C}\colon A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C.$$

(SMC$_2$) For all $A \in \mathscr{C}_0$ there must exist an isomorphism $r_A\colon A \otimes \top \xrightarrow{\cong} A$.

(SMC$_3$) For all $A, B \in \mathscr{C}_0$ there must exist an isomorphism $s_{A,B}\colon A \otimes B \xrightarrow{\cong} B \otimes A$.

(SMC$_4$) For all $A, B \in \mathscr{C}_0$ the diagram

$$\begin{tikzcd}
A \otimes (\top \otimes B) \arrow[rr, "a_{A,\top,B}"] \arrow[dr, "(\mathbf{1}_A \otimes r_A) \circ (\mathbf{1}_A \otimes s_{\top,B})"'] & & (A \otimes \top) \otimes B \arrow[dl, "r_A \otimes \mathbf{1}_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes.

(SMC$_5$) For all $A, B, C, D \in \mathscr{C}_0$ the diagram

$$\begin{tikzcd}
A \otimes \big((B \otimes C) \otimes D\big) \arrow[d, "a_{A,B \otimes C,D}"'] & A \otimes \big(B \otimes (C \otimes D)\big) \arrow[l, "\mathbf{1}_A \otimes a_{B,C,D}"'] \arrow[r, "a_{A,B,C \otimes D}"] & (A \otimes B) \otimes (C \otimes D) \arrow[d, "a_{A \otimes B,C,D}"] \\
\big(A \otimes (B \otimes C)\big) \otimes D \arrow[rr, "a_{A,B,C} \otimes \mathbf{1}_D"'] & & \big((A \otimes B) \otimes C\big) \otimes D
\end{tikzcd}$$

commutes.

(SMC$_6$) For all $A, B \in \mathscr{C}_0$ we have $s_{A,B} \circ s_{B,A} = \mathbf{1}_{A \otimes B}$.

(SMC$_7$) For all $A, B, C \in \mathscr{C}_0$ we have that

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\mathbf{1}_A \otimes s_{B,C}"] \arrow[d, "a_{A,B,C}"'] & A \otimes (C \otimes B) \arrow[r, "a_{A,C,B}"] & (A \otimes C) \otimes B \arrow[d, "s_{A,C} \otimes \mathbf{1}_B"'] \\
(A \otimes B) \otimes C \arrow[r, "s_{A \otimes B,C}"'] & C \otimes (A \otimes B) \arrow[r, "a_{C,A,B}"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

commutes.

Remark 2.2. Some authors (for example M. Barr in [BW20]) define a monoidal category without the axiom (SCM$_2$) and instead only require there to be an additional isomorphism $\top \otimes A \to A$ which has to be equal to $r_A \circ s_{\top,A}$ in the case that the category is also symmetric.

Definition 2.3 (Closed symmetric monoidal categories). A symmetric monoidal category is said to be *closed* if for every $A \in \mathscr{C}_0$ the functor $A \otimes \text{-} : \mathscr{C} \to \mathscr{C}$ defined by

$$(A \otimes \text{-})_0(B) = A \otimes B,$$
$$(A \otimes \text{-})_1(f) = \mathbf{1}_A \otimes f$$

has a right adjoint denoted by $A \multimap \text{-}$. We call this right adjoint the *internal hom-functor*.

We shall illuminate these rather abstract notions at the example of the category of sets.

Example 2.4 (The category Set). We first define the category Set. Its objects are sets A and its arrows are functions $f\colon A \to B$ between sets. Now we establish a closed symmetric monoidal structure on Set. For this we need to define $\otimes, \top, \multimap$. Let A, B be arbitrary sets. We then let

$$\begin{aligned} A \otimes B &= \{(a,b) \mid a \in A \,\&\, b \in B\}, && \text{the } \textit{product set}, \\ \top &= \mathbb{1} = \{\emptyset\}, && \text{the } \textit{one-object set}, \\ A \multimap B &= \{f\colon A \to B\}, && \text{the } \textit{function set}. \end{aligned}$$

This establishes the desired structure and gives a first idea why $A \multimap B$ is called the "internal hom"[1]. The elements of $A \multimap B$ are exactly the elements of $\mathrm{Hom}_{\mathsf{Set}}(A,B)$.

This example gives a slightly false impression, as $A \multimap B$ naively "is equal" to the Hom-set $\mathrm{Hom}_{\mathsf{Set}}(A,B)$, but this can not hold in general, as this would mean that the objects of a closed symmetric monoidal category are always sets. A less illustrative, but more accurate approach is to observe that the internal hom has similar properties as the "ordinary" Hom. We have introduced the internal hom $A \multimap \text{-}$ as the right adjoint of $A \otimes \text{-}$. But as the ordinary Hom can be made a functor $\mathrm{Hom}_{\mathscr{C}}(\text{-},\text{-})\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Set}$, it is sensible to ask whether a similar result holds for the internal hom. And unsurprisingly, we have $\text{-} \multimap \text{-} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$. A proof of this can be found in [nLa21b, Proposition 3.1]

Now originally the Chu construction was used to obtain a $*$-autonomous category from a closed symmetric monoidal category. But before we can define $*$-autonomous categories we need to examine $\multimap$ a little further.

Lemma 2.5. *Let $\mathscr{C}$ be a closed symmetric monoidal category and $A, B, C \in \mathscr{C}_0$. Then there exists a unique arrow $c_{A,B,C}\colon (A \multimap B) \otimes (B \multimap C) \to (A \multimap C)$ adjoint to a composition of evaluations.*[2]

Proof: As $A \otimes \text{-}$ is the left adjoint of $A \multimap \text{-}$, we know that for every $B, C \in \mathscr{C}_0$ there exists an isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathscr{C}}(B, A \multimap C).$$

We use this by setting $B = A \otimes C$ and considering the identity $\mathbf{1}_{A \multimap C}$. The identity corresponds to an arrow $e_{A,C}\colon A \otimes (A \multimap C) \to C$, which we shall call the *evaluation* of A and C. Now we can consider the following compositions of arrows.

$$A \otimes (A \multimap B) \otimes (B \multimap C) \xrightarrow{e_{A,B} \otimes \mathbf{1}_{B \multimap C}} B \otimes (B \multimap C) \xrightarrow{e_{B,C}} C.$$

By using the left-adjointness we obtain a unique arrow $c_{A,B,C}\colon (A \multimap B) \otimes (B \multimap C) \to (A \multimap C)$. Q.E.D.

[1] As Set also carries a cartesian closed structure, we will later denote $A \multimap B$ by B^A.

[2] We will give a sufficient definition of evaluations in the proof.

An $*$-autonomous category is a category of the following type.

Definition 2.6 ($*$-autonomous categories). A closed symmetric monoidal category $\mathscr{C}$ is a $*$-*autonomous category*, if there exists a *duality functor* $(\text{-})^*\colon \mathscr{C}^{\,\mathrm{op}} \to \mathscr{C}$, such that for all $A, B \in \mathscr{C}$ there exists an isomorphism $d_{A,B}\colon A \multimap B \to B^* \multimap A^*$.[3] This isomorphism has to adhere to the following condition.

($*$-AC$_1$) Let $A, B, C \in \mathscr{C}_0$ be given. Define $s := s_{B^* \multimap A^*, C^* \multimap B^*}$ to be the isomorphism from property (SMC$_3$) of symmetric monoidal categories. Then the diagram

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap C) \arrow[r, "c_{A,B,C}"] \arrow[d, "d_{A,B} \otimes d_{B,C}"'] & A \multimap C \arrow[d, "d_{A,C}"] \\
(B^* \multimap A^*) \otimes (C^* \multimap B^*) \arrow[r, "c_{C^*,B^*,A^*} \circ s"] & C^* \multimap A^*
\end{tikzcd}$$

has to commute, where the arrows are those arising from the definition of a symmetric monoidal category.

We are not going to use this general approach, but instead focus our attention on cartesian closed categories, an even stronger notion of a category, which we will describe next.

2.2 Cartesian closed categories

Definition 2.7 (Products). Let $\mathscr{C}$ be a category, I be an arbitrary set and c_i be a family of objects of $\mathscr{C}$, indexed by I. A *product of* $(c_i)_{i \in \mathscr{I}}$ is an object P together with arrows $\mathtt{pr}_i\colon P \to c_i$, fulfilling the following universal property.

Let $C \in \mathscr{C}_0$ be an object of $\mathscr{C}$ together with an arrow $f_i\colon C \to c_i$ for every $i \in I$. Then there exists a unique arrow $F\colon C \to P$ such that for every $i \in I$ the diagram

$$\begin{tikzcd}
C \arrow[d, "F"'] \arrow[dr, bend left, "f_i"] & \\
P \arrow[r, "\mathtt{pr}_i"'] & c_i
\end{tikzcd}$$

commutes.

Remark 2.8. It can be shown that products are unique up to unique isomorphism, if they exist, which justifies writing $\prod_{i \in I} c_i$ for the product. We will not denote the arrows $\mathtt{pr}_i$ of the product explicitly, except when it is required to evade confusion. If I is the set with two elements we write $a_1 \times a_2$ instead of $\prod_{i \in I} a_i$.

Definition 2.9 (Exponentials). Let $\mathscr{C}$ be a category such that for all objects $c_1, c_2 \in \mathscr{C}_0$ the product $c_1 \times c_2$ exists. Suppose we are given $a, b \in \mathscr{C}_0$. An *exponential* of b and a is an object $b^a \in \mathscr{C}_0$ together with an arrow $\mathtt{eval}_{b,a}\colon b^a \times a \to b$ fulfilling the following universal property.

Let $d \in \mathscr{C}_0$ be an object with an arrow $f\colon d \times a \to b$. Then there exists a unique $\widehat{f}\colon d \to b^a$ making the diagram

$$\begin{tikzcd}
b^a \times a \arrow[r, "\mathtt{eval}_{b,a}"] & b \\
d \times a \arrow[u, "\widehat{f} \times \mathbf{1}_a"] \arrow[ur, bend right, "f"'] &
\end{tikzcd}$$

[3] We write A^* instead of $(\text{-})^*_0(A)$ for an object A.

commute. The arrow $\widehat{f}$ is called the *transpose of* f whereas $\mathtt{eval}_{b,a}$ is called the *evaluation morphism.*

Definition 2.10 (Terminal objects). Let $\mathscr{C}$ be a category. A *terminal object* of $\mathscr{C}$ is a object $\top$ such that for every $C \in \mathscr{C}_0$ there exists a unique arrow $!_C \colon C \to \top$.

Definition 2.11 (Cartesian closed categories). A category $\mathscr{C}$ is cartesian closed, if it fulfills the following requirements.

- The category $\mathscr{C}$ admits all finite products, i.e. for every finite set I and all families $(c_i)_{i\in I}$ of objects of $\mathscr{C}$ the product $\prod_{i\in I} c_i$ exists.
- For all objects $a, b \in \mathscr{C}_0$ the exponential b^a exists.
- There exists a terminal object $\top \in \mathscr{C}_0$.

Remark 2.12. If class of objects of a cartesian closed category is non-empty, the requirement that there exists a terminal object $\top$ can be dropped, as any product over the empty set, $\prod_{i\in\emptyset} c_i$ already is a terminal object. So the last condition can be seen as a way to ensure that $\mathscr{C}$ actually contains objects and the empty set must not be considered when talking about cartesian closed categories.

The way one obtains that a cartesian closed category is in fact a closed symmetric monoidal category is the obvious one. We state this as the following lemma.

Lemma 2.13. *Let $\mathscr{C}$ be a cartesian closed category. The bifunctor $\otimes \colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is given by $(A, B) \mapsto A \times B$, and the object $\top$ is the terminal object of the category $\mathscr{C}$. The right adjoint of $(A \times \text{-}) \colon \mathscr{C} \to \mathscr{C}$ is given by the functor $B \mapsto B^A$.*

Proof (sketch): The conditions (SMC_1) – (SMC_7) with the exception of (SMC_2) are obvious, as all the products involved are naturally isomorphic. The condition (SMC_2) will be proven as lemma 6.15. To see that the functors $(A \times \text{-})$ and $(\text{-})^A$ are adjoint, one needs to prove that

$$\mathrm{Hom}_{\mathscr{C}}(A \times B, D) \cong \mathrm{Hom}_{\mathscr{C}}(B, D^A)$$

for all $B, D \in \mathscr{C}_0$. But this isomorphism is given by sending an arrow $f \colon A \times B \to D$ to its transpose $\widehat{f} \colon B \to D^A$. Q.E.D.

2.3 Typed λ-calculus

J. Lambek and P. J. Scott introduced a special kind of type theory, so called "typed λ-calculus" in their paper [LS84] and related these calculi to cartesian closed categories. In this section, we aim to give a short overview of the topic.

We are going to modify the version of J. Lambek and P. J. Scott, as they introduce cartesian closed categories as positive intuitionistic propositional calculi and require them to have a weak natural numbers object, which we did not do. Therefore we will follow [BW20].

Definition 2.14 (Typed λ-calculi). A *typed λ-calculus* is a formal theory consisting of three classes of *types, terms, and equations.* We will reserve the membership symbol "$\in$" for the metalanguage to symbolize that a term is of a given type, so we would for example write $a \in A$ for the expression "a is of type A". These types, terms and equations have to adhere to the following axioms.

Types:

(λ-Ty_1) There exist a type 1, which we call *basic type.*

(λ-Ty_2) If A and B are types, there exist types $A \times B$ and B^A. We call $A \times B$ the *product type* and B^A the *function type.*

Terms:

(λ-Tm$_1$) For each type A there exist countably many terms x_i^A, $i = 1, 2, \ldots$ of type A, which we call *variables of type* A. We will not always denote the variables by x_i^A, but sometimes by $x \in A$, if the circumstances are understood.

(λ-Tm$_2$) There exists a term $*$ of type 1.

(λ-Tm$_3$) If $a \in A, b \in B$ and $c \in A \times B$, there exists a term $\langle a, b\rangle$ of type $A \times B$ and two terms $\pi_{A,B}(c) \in A, \pi'_{A,B}(c) \in B$.

(λ-Tm$_4$) If $f \in B^A$ and $a \in A$, then there exists a term $\varepsilon_{B,A}(f, a) \in B$.

(λ-Tm$_5$) If $x \in A$ and $\phi(x) \in B$, then $\lambda_{x\in A}\phi(x) \in B^A$.

Before we move on we define what it means for a variable to be bound in a term. A variable x is *bound* in $\phi(x)$, if $\phi(x)$ is of the form $\lambda_{x\in A}\psi(x)$ for a term ψ. A variable x is *free* in $\phi(x)$ if it is not bound. A term a is *substitutable* for x in $\phi(x)$ if all free variables in a are free in $\phi(a)$.

Equations:

(λ-Eq$_1$) Equations are of the form $a =_X a'$, where a and a' are terms of the same type A and X is a finite collection of variables such that all occurences of free variables in a and a' are contained in X.

(λ-Eq$_2$) The equations $a =_X a'$ are symmetric and transitive, which means that if we are given equations $a =_X a', a' =_X a''$ where $a, a', a'' \in A$ and X contains all free variables of a, a', a'', then $a' =_X a, a =_X a''$. Furthermore the equations are reflective, ergo we have $a =_X a$ for all terms of such that X contains all free variables of a.

(λ-Eq$_3$) Let $a =_X a'$ be an equation and $X \subseteq Y$. Then $a =_Y a'$. We will abbreviate this rule by

$$\frac{a =_X a'}{a =_Y a'}.$$

(λ-Eq$_4$) Let $a, b \in A$ and $f \in B^A$. Furthermore let $\phi(x), \phi'(x) \in B$. Suppose we are given equations $a =_X b$ and $\phi(x) =_{X\cup\{x\}} \phi'(x)$. We then obtain equations $\varepsilon_{B,A}(f, a) =_X \varepsilon_{B,A}(f, b)$ and $\lambda_{x\in A}\phi(x) =_X \lambda_{x\in A}\phi'(x)$. We abbreviate these rules by

$$\frac{a =_X b}{\varepsilon_{B,A}(f, a) =_X \varepsilon_{B,A}(f, b)} \quad \text{and} \quad \frac{\phi(x) =_{X\cup\{x\}} \phi'(x)}{\lambda_{x\in A}\phi(x) =_X \lambda_{x\in A}\phi'(x)}.$$

(λ-Eq$_5$) The following list of equations has to hold.

- For all $a \in 1$ we have $a =_X *$.
- For all $a \in A, b \in B$ we have the equations $\pi(\langle a, b\rangle) =_X a$ and $\pi'(\langle a, b\rangle) =_X b$.
- For all $c \in A \times B$ we have $\langle \pi(c), \pi'(c)\rangle =_X c$.
- For all terms $\phi(x) \in B$ such that a is substitutable for x holds the equation

$$\varepsilon_{B,A}\big(\lambda_{x\in A}\phi(x), a\big) =_X \phi(a).$$

- For all $f \in B^A$ with a variable x such that x is not free in f holds the equation

$$\lambda_{x\in A}\varepsilon_{B,A}(f, x) =_X f.$$

- Let $\phi(x) \in B$ and x' be a term substitutable for x in $\phi(x)$ such that x' is not free in $\phi(x)$. Then the equation $\lambda_{x\in A}\phi(x) =_X \lambda_{x'\in A}\phi(x')$ holds.

One way of obtaining a typed λ-calculus is by creating it from a cartesian closed category. This is done in the following way.

Lemma 2.15. *Let $\mathscr{C}$ be a cartesian closed category. We obtain the* internal language $\mathbf{L}(\mathscr{C})$ *of $\mathscr{C}$ in the following manner.*

- *The types are the objects of $\mathscr{C}$, where the type 1 is the terminal object $\top$ and the tyes $A \times B$ and B^A correspond to the product and the exponential respectively.*
- *The terms of $\mathbf{L}(\mathscr{C})$ are the ones required to exist by λ-Tm_1 – λ-Tm_5.*
- *The equations of $\mathbf{L}(\mathscr{C})$ are the ones required to exist by λ-Eq_1 – λ-Eq_5.*

This internal language is a typed λ-calculus.

Proof: See [LS86, Section 11]. One only needs to consider the fact that we do not demand the existence of weak natural number objects in our cartesian closed categories and the existence of the basic type N in a typed λ-calculus as well as the existence of $I(\text{-}\,,\text{-}\,,\text{-})$. Q.E.D.

On the other hand we can also generate a cartesian closed category from a given typed λ calculus $\mathcal{L}$.

Lemma 2.16. *Let $\mathcal{L}$ be a typed λ-calculus. We construct a cartesian closed category $\mathbf{C}(\mathcal{L})$ in the following way.*

- *The objects of $\mathbf{C}(\mathcal{L})$ are the types of $\mathcal{L}$.*
- *The arrows of $\mathbf{C}(\mathcal{L})$ are (equivalence classes of) pairs $(x \in A, \phi(x))$, where x is a variable of type A and $\phi(x)$ is term of type B with only free variable x. An equivalence $(x \in A, \phi(x)) = (x' \in A, \psi(x'))$ is given if and only if $\phi(x) =_{\{x\}} \psi(x)$.*
- *The identity arrows $\mathbf{1}_A$ are obtained by the pairs $(x \in A, x)$. The composition of two arrows $(x \in A, \phi(x))\colon A \to B, (y \in B, \psi(y))\colon B \to C$ is given by $(x \in A, \psi(\phi(x)))\colon A \to C$.*

Now the cartesian closed structure of $\mathbf{C}(\mathcal{L})$ is given as follows.

- *The terminal object is the basic type 1.*
- *The arrow $!_A\colon A \to 1$ is given by $(x \in A, *)$.*
- *The projection $\mathtt{pr}_A\colon A \times B \to A$ is given by $(z \in A \times B, \pi(z))$, and the projection $\mathtt{pr}_B\colon A \times B \to B$ is given by $(z \in A \times B, \pi'(z))$.*
- *The unique arrow $q\colon C \to A \times B$ stemming from two arrows $(z \in C, \phi(z))\colon C \to A, (z \in Z, \psi(z))\colon C \to B$ is the arrow $(z \in C, \langle \phi(z), \psi(z)\rangle)$.*
- *The unique arrow from an object C to the exponential B^A given by an arrow $(z \in C \times A, \phi(z))\colon C \times A \to B$ is the arrow $(x \in C, \lambda_{y \in A}\phi(\langle x, y\rangle))$.*
- *The evalution morphism for an exponential, $\mathtt{eval}_{B,A}\colon B^A \times A \to B$ is given by*

$$\mathtt{eval}_{B,A} = \Big(y \in B^A \times A, \varepsilon_{B,A}\big(\pi(y), \pi'(y)\big)\Big).$$

Proof: See [LS86, Secion 11]. Again one needs to consider the differing definitions of typed λ-calculi and cartesian closed categories. Q.E.D.

Now the main result is to show that typed λ-calculi and cartesian closed categories are "essentially the same". For this one first constructs the category of typed λ-calculi, which requires the notion of a morphism between λ-calculi. To this end we define translations.

Definition 2.17 (Translations). Let $\mathcal{L}, \mathcal{L}'$ be given typed λ-calculi, we define a *translation* $\Phi\colon \mathcal{L} \to \mathcal{L}'$ by the following data:

(Trl$_1$) If A is a type of $\mathcal{L}$, then $\Phi(A)$ is a type of $\mathcal{L}'$. Furthermore if $a \in A$ is a term of type A in $\mathcal{L}$, then $\Phi(a)$ is a term of type $\Phi(A)$ in $\mathcal{L}'$. Additionally, for all variables x_i^A we require $\Phi(x_i^A) = x_i^{\Phi(A)}$ and if a contains no free variables, so does $\Phi(a)$.

(Trl$_2$) The translation preserves the type constructors, i.e. we have $\Phi(1_{\mathcal{L}}) = 1_{\mathcal{L}'}$ and for all types A, B in $\mathcal{L}$ we have $\Phi(A \times B) = \Phi(A) \times \Phi(B)$ where the product type on the right hand side is in $\mathcal{L}$. We also require $\Phi(B^A) = \Phi(B)^{\Phi(A)}$. The term forming operators are preserved up to equality, i.e.

$$\Phi(\pi_{A,B}(c)) = \pi_{\Phi(A),\Phi(B)}(\Phi(c)), \Phi(\lambda_{x \in A}\phi(x)) = \lambda_{\Phi(x)\in\Phi(A)}\Phi(\phi(x)).$$

(Trl$_3$) The translation preserves equalities, that is if we are given an equality $a =_X b$ in $\mathcal{L}$, we obtain an equality $\Phi(a) =_{\Phi(x)} \Phi(b)$. Here the set $\Phi(X)$ is given as

$$\Phi(X) = \{\Phi(x) \mid x \in X\}.$$

With this definition we can define the category of λ-calculi.

Definition 2.18. The category $\lambda\text{-}\mathsf{Calc}$ consist of the following data.

- Its objects are λ-calculi $\mathcal{L}$.
- Its arrows $\Phi\colon \mathcal{L} \to \mathcal{L}'$ are translations.

Now we only need the category of cartesian closed categories, ccCat. We will discuss this category later in section 4.4 with greater detail, but in now suffices to know that its objects are cartesian closed categories $\mathscr{C}$ and its arrows are product preserving functor $F\colon \mathscr{C} \to \mathscr{C}'$, those are functors that for every $a, b \in \mathscr{C}$ there exists an isomorphism $F_{ab}\colon F_0(a\times b) \to F_0(a)\times F_0(b)$. With this at hand we can state the main result.

Theorem 2.19. *Consider the functor* $\mathbf{L}\colon \mathsf{ccCat} \to \lambda\text{-}\mathsf{Calc}$ *assigning to each cartesian closed category* $\mathscr{C}$ *its internal language* $\mathbf{L}(\mathscr{C})$. *Consider also the functor* $\mathbf{C}\colon \lambda\text{-}\mathsf{Calc} \to \mathsf{ccCat}$ *assigning to each typed* λ*-calculus the cartesian closed category* $\mathbf{C}(\mathcal{L})$. *Then* $\mathbf{L}, \mathbf{C}$ *are inverse functors, hence*

$$\mathsf{ccCat} \cong \lambda\text{-}\mathsf{Calc}\,.$$

2.4 The Chu construction over a cartesian closed category

Definition 2.20 (The Chu category over a ccc). Let $\mathscr{C}$ be a cartesian closed category and $\gamma \in \mathscr{C}_0$. Then the *Chu category* $\mathsf{Chu}(\mathscr{C}, \gamma)$ is defined by the following data:

- The *objects* of $\mathsf{Chu}(\mathscr{C}, \gamma)$ are triplets (a, f, x) such that $a, x \in \mathscr{C}_0$ and $f\colon a \times x \to \gamma$ is an arrow in $\mathscr{C}$.
- The *arrows* of $\mathsf{Chu}(\mathscr{C}, \gamma)$ are pairs $(\phi^+, \phi^-)\colon (a, f, x) \to (b, g, y)$ where ϕ^+, ϕ^- are arrows $\phi^+\colon a \to b$, $\phi^-\colon y \to x$ such that the diagram

$$\begin{tikzcd} a \times y \arrow[r, "\mathbf{1}_a \times \phi^-"] \arrow[d, "\phi^+ \times \mathbf{1}_y"'] & a \times x \arrow[d, "f"] \\ b \times y \arrow[r, "g"'] & \gamma \end{tikzcd}$$

  commutes. Sometimes we call arrows in $\mathsf{Chu}(\mathscr{C}, \gamma)$ *Chu morphisms.*
- The *composition* of two arrows $(\phi^+, \phi^-)\colon (a, f, x) \to (b, g, y)$ and $(\psi^+, \psi^-)\colon (b, g, y) \to (c, h, z)$ is given by

$$(\psi^+, \psi^-) \circ (\phi^+, \phi^-) = (\psi^+ \circ \phi^+, \phi^- \circ \psi^-).$$

- The *identities* are given by $\mathbf{1}_{(a,f,x)} = (\mathbf{1}_a, \mathbf{1}_x)$ for $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$.

Theorem 2.21. *Let $\mathscr{C}$ be a cartesian closed category and $\gamma \in \mathscr{C}_0$. Then $\mathsf{Chu}(\mathscr{C}, \gamma)$ is a category.*

Proof: We show that the definition of $\mathsf{Chu}(\mathscr{C}, \gamma)$ gives indeed a category. For this we show the following:

1. *Well-definedness:* Let $(a, f, x), (b, g, y), (c, h, z) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ be given and assume we have two arrows
$$\phi = (\phi^+, \phi^-)\colon (a, f, x) \to (b, g, y), \qquad \psi = (\psi^+, \psi^-)\colon (b, g, y) \to (c, h, z).$$
Then the arrow $\psi \circ \phi\colon (a, f, x) \to (c, h, z)$ is well-defined.
2. *Associativity:* For $(a, f, x), (b, g, y), (c, h, z), (d, j, s) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ and
$$\theta = (\theta^+, \theta^-)\colon (a, f, x) \to (b, g, y), \qquad \psi = (\psi^+, \psi^-)\colon (b, g, y) \to (c, h, z),$$
$$\phi = (\phi^+, \phi^-)\colon (c, h, z) \to (d, j, s)$$
holds the following equality:
$$(\phi \circ \psi) \circ \theta = \phi \circ (\psi \circ \theta).$$
3. *Identity:* For any $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ exists an arrow $\mathbf{1}_{(a,f,x)}\colon (a, f, x) \to (a, f, x)$, such that for any $(b, g, y), (c, h, z) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ and $\theta = (\theta^+, \theta^-)\colon (a, f, x) \to (b, g, y), \psi = (\psi^+, \psi^-)\colon (c, h, z) \to (a, f, x)$ the equalities
$$\mathbf{1}_{(a,f,x)} \circ \psi = \psi \qquad \text{and} \qquad \theta \circ \mathbf{1}_{(a,f,x)} = \theta$$
hold.

Ad 1: Suppose these objects and arrows are given. As $\psi^+ \circ \phi^+\colon a \to c$ and $\phi^- \circ \psi^-\colon z \to x$ it remains to check the commutativity of the diagram

$$\begin{tikzcd} a \times z \arrow[r, "\mathbf{1}_a \times (\phi^- \circ \psi^-)"] \arrow[d, "(\psi^+ \circ \phi^+) \times \mathbf{1}_z"'] & a \times x \arrow[d, "f"] \\ c \times z \arrow[r, "h"] & \gamma. \end{tikzcd}$$

But we can compute

$$\begin{aligned} f \circ \big(\mathbf{1}_a \times (\phi^- \circ \psi^-)\big) &= f \circ (\mathbf{1}_a \times \phi^-) \circ (\mathbf{1}_a \times \psi^-) \\ &= g \circ (\phi^+ \times \mathbf{1}_y) \circ (\mathbf{1}_a \times \psi^-) && \text{(as } (\phi^+, \phi^-) \text{ is a Chu morphism)} \\ &= g \circ (\mathbf{1}_b \times \psi^-) \circ (\phi^+ \times \mathbf{1}_z) && \text{(by [Pet21, p. 2])} \\ &= h \circ (\psi^+ \times \mathbf{1}_z) \circ (\phi^+ \times \mathbf{1}_z) && \text{(as } (\psi^+, \psi^-) \text{ is a Chu morphism)} \\ &= h \circ \big((\psi^+ \circ \phi^+) \times \mathbf{1}_z\big). && \text{(Associativity in } \mathscr{C}) \end{aligned}$$

Hence the composition is well-defined.
Ad 2: Let the objects and arrows required be given. The composition is defined through $(\phi \circ \psi) = (\phi^+ \circ \psi^+, \psi^- \circ \phi^-)$. So we can easily compute

$$\begin{aligned} (\phi \circ \psi) \circ \theta &= (\phi^+ \circ \psi^+, \psi^- \circ \phi^-) \circ (\theta^+, \theta^-) && \text{(by definition)} \\ &= \big((\phi^+ \circ \psi^+) \circ \theta^+, \theta^- \circ (\psi^- \circ \phi^-)\big) && \text{(by definition)} \\ &= \big(\phi^+ \circ (\psi^+ \circ \theta^+), (\theta^- \circ \psi^-) \circ \phi^-\big) && \text{(Associativity in } \mathscr{C}) \end{aligned}$$

$$
\begin{aligned}
&= (\phi^+, \phi^-) \circ (\psi^+ \circ \theta^+, \theta^- \circ \psi^-) && \text{(by definition)}\\
&= \phi \circ (\psi \circ \theta). && \text{(by definition)}
\end{aligned}
$$

Ad 3: Let $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$. We set $\mathbf{1}_{(a,b,x)} := (\mathbf{1}_a, \mathbf{1}_x)$. First we have to show that the diagram

$$
\begin{tikzcd}
a \times x \arrow[r, "\mathbf{1}_a \times \mathbf{1}_x"] \arrow[d, "\mathbf{1}_a \times \mathbf{1}_x"'] & a \times x \arrow[d, "f"] \\
a \times x \arrow[r, "f"] & \gamma
\end{tikzcd}
$$

commutes. But this follows trivially. So $\mathbf{1}_{(a,f,x)}$ defined as above is indeed an arrow in $\mathsf{Chu}(\mathscr{C}, \gamma)_1$. Now let the arrows $\theta = (\theta^+, \theta^-)\colon (a, f, x) \to (b, g, y)$ and $\psi = (\psi^+, \psi^-)\colon (c, h, z) \to (a, f, x)$ be given. We then conclude

$$
\begin{aligned}
\theta \circ \mathbf{1}_{(a,f,x)} &= \theta \circ (\mathbf{1}_x, \mathbf{1}_x) && (\text{definition of } \mathbf{1}_{(a,f,x)})\\
&= (\theta^+ \circ \mathbf{1}_a, \mathbf{1}_x \circ \theta^-)\\
&= (\theta^+, \mathbf{1}_x \circ \theta^-) && (\text{definition of } \mathbf{1}_a)\\
&= (\theta^+, \theta^-) = \theta, && (\text{defintion of } \mathbf{1}_x)\\
\mathbf{1}_{(a,f,x)} \circ \psi &= (\mathbf{1}_a, \mathbf{1}_x) \circ \psi && (\text{definition of } \mathbf{1}_{(a,f,x)})\\
&= (\mathbf{1}_a \circ \psi^+, \psi^- \circ \mathbf{1}_x)\\
&= (\psi^+, \psi^- \circ \mathbf{1}_x) && (\text{definition of } \mathbf{1}_a)\\
&= (\psi^+, \psi^-) = \psi. && (\text{defintion of } \mathbf{1}_x)
\end{aligned}
$$

Q.E.D.

2.5 Chu spaces

In many other areas of mathematics, for example when examining physical systems, one restricts ones interest to the special case of *Chu spaces*. These are objects of Chu categories $\mathsf{Chu}(\mathsf{Set}, K)$ where $K \in \mathsf{Set}_0$. It is immediate that the objects of $\mathsf{Chu}(\mathsf{Set}, K)$ are sets themselves, so a Chu space is a set itself. It can be described in the following way.

Definition 2.22 (Chu spaces). Let K be a set. A *Chu space over K* is a triple (A, f, X) where A, X are sets and $f\colon A \times X \to K$. The *category of Chu spaces over K* is $\mathsf{Chu}(\mathsf{Set}, K)$.

Now we can apply various set-theoretical notions to Chu spaces. In the following we denote by K^A for sets K, A the function space

$$K^A := \{h\colon A \to K\}.$$

Definition 2.23 (separable/extensional/normal Chu spaces). Let (A, f, X) be a Chu space over B. We then call (A, f, X)

- *separable*, if $\widehat{f}\colon A \to X^B$, defined through the clause
$$\forall a \in A\ \forall b \in B \big(\widehat{f}(a)\big)(b) = f(a, b)$$
is an injection,
- *extensional*, if $\check{f}\colon B \to X^A$, defined through the clause
$$\forall a \in A\ \forall b \in B \big(\check{f}(b)\big)(a) = f(a, b)$$
is an injection,
- *biextensional*, if (A, f, B) is both separable and extensional,
- *normal*, if $B \subseteq X^A$ and $f\colon A \times B \to X$ is given by $f(a, b) = b(a)$.

Chapter 3

Chu representations

In this chapter we give various examples of embeddings of categories arising in other areas of mathematics into Chu categories.

3.1 A Boolean representation of Top

We want to show to model the theory of topological spaces using our Chu construction. For this we present the approach of [Pet21]. First we define the following category.

Definition 3.1 (Category of topological spaces). The *category of topological spaces* Top is defined as follows.

- The objects of Top are pairs (X, T_X) where X is a set and $T_X \in \mathfrak{P}(X)$ is a topology on X.[4]
- The arrows $\phi\colon (X_1, T_1) \to (X_2, T_2)$ are continuous maps.

Remark 3.2. In the following we let $\mathbb{2}$ be the set containing exactly two elements, 0 and 1.

Definition 3.3. Let $E^{\mathsf{Top}}\colon \mathsf{Top} \to \mathsf{Chu}(\mathsf{Set}, \mathbb{2})$ be defined through the following clauses.

- For $(X, T) \in \mathsf{Top}_0$ we set $E_0^{\mathsf{Top}}(X, T) = (X, \in_{X,T}, T)$ where $\in_{X,T}$ is defined as follows.

$$\begin{aligned} \in_{X,T}\colon X \times T &\to \mathbb{2}, \\ (x, G) &\mapsto \begin{cases} 1, & x \in G, \\ 0, & x \notin G. \end{cases} \end{aligned}$$

- If $f\colon (X, T) \to (Y, S)$ is a continuous map, we define $E_1^{\mathsf{Top}}(f) := (f, f^{-1})$, where $f^{-1}\colon S \to T$ is the map assigning to each open set $U \subseteq Y$ its preimage under f, $f^{-1}(U)$.

Proposition 3.4. *The rule* $E^{\mathsf{Top}}\colon \mathsf{Top} \to \mathsf{Chu}(\mathsf{Set}, \mathbb{2})$ *is a functor.*

Proof: The well-definedness can be seen, as $E_0^{\mathsf{Top}}(X, T) \in \mathsf{Chu}(\mathsf{Set}, \mathbb{2})$ and $(f, f^{-1})\colon (X, \in_{X,T}, T) \to (Y, \in_{Y,S}, S)$. The commutativity of the diagram

$$\begin{tikzcd} X \times S \arrow[r, "\mathbf{1}_X \times f^{-1}"] \arrow[d, "f \times \mathbf{1}_S"'] & X \times T \arrow[d, "\in_{X,T}"] \\ Y \times S \arrow[r, "\in_{Y,S}"] & \mathbb{2} \end{tikzcd}$$

is immediate. We show that E^{Top} fulfils the axioms of a functor.

- *Preservation of identities:* Let $(X, T) \in \mathsf{Top}$ be given. Then $\mathbf{1}_X$ is the usual identity on X. We see $\mathbf{1}_X^{-1}(U) = U$, so $E_1^{\mathsf{Top}}(\mathbf{1}_X) = (\mathbf{1}_X, \mathbf{1}_T)$. Furthermore $\mathbf{1}_{E_0^{\mathsf{Top}}(X,T)} = (\mathbf{1}_X, \mathbf{1}_T)$, which proves the hypothesis.

[4] We denote by $\mathfrak{P}(X)$ the *power set* $\mathfrak{P}(X) = \{U \subseteq X\}$.

- *Compatibility with composition:* Assume we are given $(X,T),(Y,S),(Z,V) \in \mathsf{Top}$ and arrows $f\colon (X,T) \to (Y,S), g\colon (Y,S) \to (Z,V)$. Then

$$\begin{aligned} E_1^{\mathsf{Top}}(g \circ f) &= (g \circ f, (g \circ f)^{-1}) = (g \circ f, f^{-1} \circ g^{-1}) = (g, g^{-1}) \circ (f, f^{-1}) \\ &= E_1^{\mathsf{Top}}(g) \circ E_1^{\mathsf{Top}}(f). \end{aligned}$$

Q.E.D.

Now we specify what we mean when we call a functor a representation.

Definition 3.5. Let $\mathscr{A}, \mathscr{B}$ be categories and $F\colon \mathscr{A} \to \mathscr{B}$ be a functor.

- The functor F is *injective (surjective) on objects*, if F_0 is injective (surjective).
- The functor F is *injective (surjective) on arrows*, if F_1 is injective (surjective).
- The functor F is *faithful*, if for every $a, b \in \mathscr{C}_0$ the map

$$\begin{aligned} F_{(a,b)}\colon \operatorname{Hom}_{\mathscr{C}}(a,b) &\to \operatorname{Hom}_{\mathscr{D}}\big(F_0(a), F_0(b)\big), \\ f &\mapsto F_1(f) \end{aligned}$$

  is an injection.
- If $F_{(a,b)}$ is an surjection for every $a, b \in \mathscr{C}_0$, then F is *full*.
- The functor F is an *embedding* if F is injective on objects and faithful.
- The functor F is a *representation* if F is a full embedding.
- The functor F is a *strict representation* if F is injective on arrows and a representation.

Theorem 3.6. *The functor $E^{\mathsf{Top}}\colon \mathsf{Top} \to \mathsf{Chu}(\mathsf{Set}, 2)$ is a representation, i.e. a full embedding.*

Proof: We check the following:

First we have to check that for every $(X,T),(Y,S) \in \mathsf{Top}_0$ the arrow $E^{\mathsf{Top}}_{((X,T),(Y,S))}$ is a bijection. For our convenience we shall write $E^{\mathsf{Top}}_{(X,Y)}$ instead of $E^{\mathsf{Top}}_{((X,T),(Y,S))}$. By definition

$$E^{\mathsf{Top}}_{(X,Y)}\colon \mathsf{Top}_1\big((X,T),(Y,S)\big) \to \mathsf{Chu}(\mathsf{Set},2)_1\big((X,\in_{X,T},T),(Y,\in_{Y,S},S)\big)$$

maps f to $E_1^{\mathsf{Top}}(f)$. We check injectivity and surjectivity separately.
Injectivity: Suppose we have $f\colon (X,T) \to (Y,S), g\colon (X,T) \to (Y,S)$ with $E_1^{\mathsf{Top}}(f) = E_1^{\mathsf{Top}}(g)$. This means $(f, f^{-1}) = (g, g^{-1})$, so $f = g$, so injectivity is proven.
Surjectivity: Suppose we have $\phi \in \mathsf{Chu}(\mathsf{Set},2)_1\big((X,\in_{X,T},T),(Y,\in_{Y,S},S)\big)$. This means we have $\phi = (\phi^+, \phi^-)$ such that

$$\begin{tikzcd} X \times S \arrow[r, "\mathbf{1}_X \times \phi^-"] \arrow[d, "\phi^+ \times \mathbf{1}_S"'] & X \times T \arrow[d, "\in_{X,T}"] \\ Y \times S \arrow[r, "\in_{Y,S}"] & 2 \end{tikzcd}$$

commutes. By the definition of $\mathsf{Chu}(\mathsf{Top}, 2)$ we know that $\phi^+\colon X \to Y$ is an arrow in Top. We seek to show that $\phi^- = (\phi^+)^{-1}$. For this we employ a topological argument. Let $G \in S$ be an open set. Consider $(\phi^+)^{-1}(G)$. Choose $y \in (\phi^+)^{-1}(G)$. Then

$$1 = \in_{Y,S} \circ (\phi^+ \times \mathbf{1}_S)(y, G) = \in_{X,T} \circ (\mathbf{1}_X \times \phi^-)(y, G).$$

This means that $y \in \phi^-(G)$. As this works with all $y \in (\phi^+)^{-1}(G)$, we have the inclusion $(\phi^+)^{-1}(G) \subset \phi^-(G)$. To see $\phi^-(G) \subset (\phi^+)^{-1}(G)$, simply observe that if $x \in \phi^-(G)$ for any x, then

$$1 = \in_{X,T} \circ (\mathbf{1}_X \times \phi^-)(x, G) = \in_{Y,S} \circ (\phi^+ \times \mathbf{1}_S)(x, G),$$

so $x \in (\phi^+)^{-1}(G)$. This shows $\phi^- = (\phi^+)^{-1}$, and we have

$$(\phi^+, \phi^-) = E_1^{\mathsf{Top}}(\phi^+).$$

So we have shown that $E_{(X,Y)}^{\mathsf{Top}}$ is a bijection. Next we show the injecitivity on objects. This can be seen rather simply. If

$$E_0^{\mathsf{Top}}(X,T) = (X, \in_{X,T}, T) = (Y, \in_{Y,S}, S) = E_0^{\mathsf{Top}}(Y,S),$$

then $X = Y$ and $S = T$. This shows that E^{Top} is a full embedding. Q.E.D.

Next we shall examine the remarks of [Pet21, p. 7].

- *The category* Top *is not a cartesian closed category, hence we can not use the Chu representation of a cartesian closed category.*

  Self-explanatory. A treatment of the question what (usable) cartesian closed subcategories of Top exist can be found in [Ste67].

- *The definition of* $\in_{X,T}$ *is classical.*

  This stems from the fact that we use the law of the excluded middle, $(x \in G) \vee \neg(x \in G)$, in the definiton of $\in_{X,T}$, an assumption that does not always hold in intuitionistic logic.

- $(X, \in_{X,T}, T)$ *is separable if and only if* T *is* T_0*.*

  Assume $(X, \in_{X,T}, Y)$ is a separable Chu space. This means that

  $$x \mapsto \left(\begin{array}{ll} \hat{\in}_{X,T}(x)\colon & T \to 2 \\ & U \mapsto \in_{X,T}(x,U) \end{array} \right)$$

  is an injection. Now assume we are given $x, y \in X$ with $x \neq y$. Hence $\hat{\in}_{X,T}(x) \neq \hat{\in}_{X,T}(y)$, so we obtain a $U \in T$ such that $\in_{X,T}(x,U) \neq \in_{X,T}(y,U)$, which means that either $x \in U \wedge y \notin U$ or $x \notin U \wedge y \in U$.

  On the contrary, assume T is T_0. This means that for every $x, y \in X$ with $x \neq y$ there exists $U \in T$ such that either $x \notin U \wedge y \in U$ or $y \notin U \wedge x \in U$. This already means that $\in_{X,T}(x,U) \neq \in_{X,T}(y,U)$, so $\hat{\in}_{X,T}(x) \neq \hat{\in}_{X,T}(y)$.

- $(X, \in_{X,T}, T)$ *is always extensional.*

  To see this we have to show that

  $$U \mapsto \left(\begin{array}{ll} \check{\in}_{X,T}(U)\colon & X \to 2 \\ & x \mapsto \in_{X,T}(x,U) \end{array} \right)$$

  is an injection. To see this, assume we have $U, V \in T$ with $U \neq V$. Without loss of generality there exists $x \in X$ with $x \in U, x \notin V$, otherwise switch U and V. Hence $\in_{X,T}(x,U) \neq \in_{X,T}(x,V)$ and $\check{\in}_{X,T}(U) \neq \check{\in}_{X,T}(V)$.

- *The special properties of a topology* T *on a set* X *play no role in the above, i.e. this representation applies to more general categories.*

  One sees that the only time the property of a category was used, was when the additional data gained from the fact that $f\colon X \to Y$ is continuous was defined to acquire a map $f^{-1}\colon S \to T$. But this additional requirement can be demanded independently from the fact that T is a topology.

3.2 A normal Chu representation of $\mathsf{Aff}(\mathsf{Set}, X)$

We first define the affine category over an arbitrary category.

Definition 3.7 (The affine category). Let $\mathscr{C}$ be an arbitrary category and $c \in \mathscr{C}_0$. The *affine category* $\mathsf{Aff}(\mathscr{C}, c)$ is defined as follows.

- The objects of $\mathsf{Aff}(\mathscr{C}, c)$ are pairs (A, F) where $A \in \mathscr{C}_0$ and $F \subseteq \mathrm{Hom}_{\mathscr{C}}(A, c)$.
- The arrows $h\colon (A, F) \to (B, G)$ are arrows $h \in \mathscr{C}_1$ such that for every $g \in G$ we have $g \circ h \in F$.

We seek to show that the rule $(A, F) \mapsto (A, \mathtt{eval}_{A,F}, F)$ defines a full embedding of $\mathsf{Aff}(\mathsf{Set}, X)$ into $\mathsf{Chu}(\mathsf{Set}, X)$. First we show that this is a functor.

Proposition 3.8. *The rule* $\mathfrak{Aff}\colon (A, F) \mapsto (A, \mathtt{eval}_{A,F}, F)$ *defines a functor* $\mathsf{Aff}(\mathsf{Set}, X) \to \mathsf{Chu}(\mathsf{Set}, X)$.

Proof: One sees that the rule is well defined for objects as $(A, \mathtt{eval}_{A,F}, F) \in \mathsf{Chu}(\mathsf{Set}, X)$, because $A, F \in \mathsf{Set}$, by the definition of $\mathsf{Aff}(\mathsf{Set}, X)$ and $\mathtt{eval}_{A,F}\colon A \times F \to X$ by definition as

$$\mathtt{eval}_{A,F}\colon A \times F \to X, \quad \mathtt{eval}_{A,F}(x, f) = f(x)$$

for $F \subseteq \mathsf{Set}_1(A, X)$. To see that the rule is well-defined for arrows, one takes an arrow $h\colon (A, F) \to (B, G)$ in $\mathsf{Aff}(\mathsf{Set}, X)$. This is a morphism $h\colon A \to B$ such that $g \circ h \in F$ for every $g \in G$. We define $\mathfrak{Aff}_1(h) = (h, \tilde{h})$ where

$$\tilde{h}\colon G \to F, \quad g \mapsto g \circ h.$$

We have to check the commutativity of the diagram

$$\begin{tikzcd} A \times G \arrow[r, "h \times \mathbf{1}_G"] \arrow[d, "\mathbf{1}_A \times \tilde{h}"'] & B \times G \arrow[d, "\mathtt{eval}_{B,G}"] \\ A \times F \arrow[r, "\mathtt{eval}_{A,F}"] & X. \end{tikzcd}$$

As we have

$$\begin{aligned} \big(\mathtt{eval}_{A,F} \circ (\mathbf{1}_A \times \tilde{h})\big)(a, g) &= \mathtt{eval}_{A,F}(a, g \circ h) = (g \circ h)(a) \\ &= g\big(h(a)\big) = \mathtt{eval}_{B,G}(h(a), g) = \\ &= \big(\mathtt{eval}_{B,G} \circ (h \times \mathbf{1}_G)\big)(a, g) \end{aligned}$$

for every $(a, g) \in A \times G$, it follows that $\mathtt{eval}_{A,F} \circ (\mathbf{1}_A \times \tilde{h}) = \mathtt{eval}_{B,G} \circ (h \times \mathbf{1}_G)$. Hence $\mathfrak{Aff}_1(h)$ is well-defined. We check the functoriality.
Preservation of identities: Let $(A, F) \in \mathsf{Aff}(\mathsf{Set}, X)_0$. Then $\mathbf{1}_{(A,X)} = \mathbf{1}_A$. We compute

$$\begin{aligned} \mathfrak{Aff}_1(\mathbf{1}_A) &= (\mathbf{1}_A, \tilde{\mathbf{1}}_A) && (1) \\ &= (\mathbf{1}_A, \mathbf{1}_F), && (2) \end{aligned}$$

where we used in the step from (1) to (2) that $\tilde{\mathbf{1}}_A\colon F \to F, g \mapsto g \circ \mathbf{1}_A = g$ is the identity .
Compatibility with composition: Assume we are given $(A, F), (B, G), (C, H) \in \mathsf{Aff}(\mathsf{Set}, X)_0$ and $\alpha\colon (A, F) \to (B, G), \beta\colon (B, G) \to (C, H)$. As

$$\mathfrak{Aff}_1(\beta \circ \alpha) = (\beta \circ \alpha, \widetilde{(\beta \circ \alpha)}),$$

it remains to show that $\widetilde{(\beta \circ \alpha)} = \tilde{\alpha} \circ \tilde{\beta}$. To see this we observe that for all $h \in H$ we have $\widetilde{(\beta \circ \alpha)}(h) = h \circ \beta \circ \alpha$, similarly $(\tilde{\alpha} \circ \tilde{\beta})(h) = h \circ \beta \circ \alpha$, so we have $\widetilde{(\beta \circ \alpha)} = \tilde{\alpha} \circ \tilde{\beta}$. This finalizes the proof that $\mathfrak{Aff}$ is a functor. Q.E.D.

Theorem 3.9. *The functor $\mathfrak{Aff}$ is a full embedding.*

Proof: We first have to show that the rule

$$\begin{aligned}\mathfrak{Aff}_{(A,F),(B,G)}\colon \mathsf{Aff}(\mathsf{Set},X)_1\big((A,F),(B,G)\big)\\ \to \mathsf{Chu}(\mathsf{Set},X)_1\big((A,\mathtt{eval}_{A,F},F),(B,\mathtt{eval}_{B,G},G)\big),\\ f\mapsto \mathfrak{Aff}_1(f)\end{aligned}$$

is a bijection for every $(A,F),(B,G)\in\mathsf{Aff}(\mathsf{Set},X)_0$. We show injectivity and surjectivity separately.
Injectivity: Let $f,g\in\mathsf{Aff}(\mathsf{Set},X)_1((A,F),(B,G))$ such that $f\neq g$. Then

$$\mathfrak{Aff}_1(f)=(f,\tilde f)\neq(g,\tilde g)=\mathfrak{Aff}_1(g),$$

since $f\neq g$.
Surjectivity: Let $h\in\mathsf{Chu}(\mathsf{Set},X)_1((A,\mathtt{eval}_{A,F},F),(B,\mathtt{eval}_{B,G},G))$ be given. Then h is a pair (h_1,h_2) of arrows $h_1\colon A\to B, h_2\colon G\to F$. We seek to show that $h_2=\tilde h_1$. To this end we use the commutativity of the diagram

$$\begin{tikzcd} A\times G \arrow[r, "h_1\times\mathbf{1}_G"] \arrow[d, "\mathbf{1}_A\times h_2"'] & B\times G \arrow[d, "\mathtt{eval}_{B,G}"] \\ A\times F \arrow[r, "\mathtt{eval}_{A,F}"] & X. \end{tikzcd}$$

This means that for any $(a,g)\in A\times G$ we have

$$\begin{aligned}(\mathtt{eval}_{A,F}\circ(\mathbf{1}_A\times h_2))(a,g)&=\mathtt{eval}_{A,F}(a,h_2(g))=h_2(g)(a)\\&=g(h_1(a))=(g\circ h_1)(a)=\mathtt{eval}_{B,G}(h_1(a),g)\\&=(\mathtt{eval}_{B,G}\circ(h_1\times\mathbf{1}_G))(a,g).\end{aligned}$$

So $h_2(g)(a)=g\circ h_1(a)$ for all $a\in A$ and $g\in G$, therefore $h_2(g)=g\circ h_1=\tilde h_1(g)$. At last we show that the rule is injective on objects. But this is immediate, because if

$$(A,\mathtt{eval}_{A,F},F)=(B,\mathtt{eval}_{B,G},G),$$

we already have $A=B, G=F$. Q.E.D.

3.3 A Chu representation of $\mathsf{Sub}(\mathscr{C},\gamma)$

Definition 3.10 (Category of subobjects). Let $\mathscr{C}$ be an arbitrary category and $\gamma\in\mathscr{C}_0$. The *category of subobjects* $\mathsf{Sub}(\mathscr{C},\gamma)$ is defined by the following clauses.

- Its objects are monomorphism $f\in\mathscr{C}_1$ such that the codomain of f is γ.
- Let $i\colon a\hookrightarrow\gamma, j\colon b\hookrightarrow\gamma$ be objects of $\mathsf{Sub}(\mathscr{C},\gamma)$.[5] An arrow $f\colon i\to j$ is an arrow $f\colon a\to b$ such that

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & b \arrow[dl, hook', "j"] \\ & \gamma & \end{tikzcd}$$

  commutes.

[5]We denote monomorphisms by the arrow "$\hookrightarrow$".

Now we present the representation of $\mathsf{Sub}(\mathscr{C},\gamma)$ given in [Pet21, p.59].

Definition 3.11. Let $\mathscr{C}$ be a cartesian closed category. We define $E^{\mathsf{Sub}(\mathscr{C},\gamma)}\colon \mathsf{Sub}(\mathscr{C},\gamma) \to \mathsf{Chu}(\mathscr{C},\gamma)$ by the following rules.

- Let $i\colon a \hookrightarrow \gamma$ be an object of $\mathsf{Sub}(\mathscr{C},\gamma)$. We then set $E_0^{\mathsf{Sub}(\mathscr{C},\gamma)}(i) := (a, i \circ \mathtt{pr}_a, \top)$. Here $\top$ denotes the terminal object of $\mathscr{C}$.
- Let $f\colon i \to j$ be an arrow in $\mathsf{Sub}(\mathscr{C},\gamma)$. We then define $E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(f) := (f, \mathbf{1}_\top)$.

Proposition 3.12. *Let $\mathscr{C}$ be a cartesian closed category. Then $E^{\mathsf{Sub}(\mathscr{C},\gamma)}\colon \mathsf{Sub}(\mathscr{C},\gamma) \to \mathsf{Chu}(\mathscr{C},\gamma)$ is a functor.*

Proof: We first check that $E^{\mathsf{Sub}(\mathscr{C},\gamma)}$ is well-defined. For this let $i\colon a \hookrightarrow \gamma$ be an object of $\mathsf{Sub}(\mathscr{C},\gamma)$. Then $i \circ \mathtt{pr}_a\colon a \times \top \to \gamma$, so $(a, i \circ \mathtt{pr}_a, \top) \in \mathsf{Chu}(\mathscr{C},\gamma)_0$.

Next let $f\colon a \to b$ be an arrow in $\mathsf{Sub}(\mathscr{C},\gamma)$ with $i\colon a \hookrightarrow \gamma, j\colon b \hookrightarrow \gamma$, such that $i = j \circ f$. Then $E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(f) = (f, \mathbf{1}_\top)$. As $f\colon a \to b$ and $\mathbf{1}_\top\colon \top \to \top$, it remains to check the commutativity of the diagram

$$\begin{tikzcd}
a \times \top \arrow[r, "f\times \mathbf{1}_\top"] \arrow[d, "\mathbf{1}_a \times \mathbf{1}_\top"'] & b \times \top \arrow[d, "j\circ \mathtt{pr}_b"] \\
a \times \top \arrow[r, "i\circ \mathtt{pr}_a"'] & \gamma .
\end{tikzcd}$$

To see this we use lemma 6.15. Therefore we know that $a \times \top \cong a, b \times \top \cong b$ and we obtain the diagram

$$\begin{tikzcd}
a \arrow[rrr, "f"] \arrow[dddd, "\mathbf{1}_a"'] \arrow[dr, phantom, "\cong"] & & & b \arrow[dddd, "j"] \arrow[dl, phantom, "\cong"] \\
& a \times \top \arrow[r, "f\times \mathbf{1}_\top"] \arrow[dd, "\mathbf{1}_a \times \mathbf{1}_\top"] & b \times \top \arrow[dd, "j\circ \mathtt{pr}_b"] & \\
& & & \\
& a \times \top \arrow[r, "i\circ \mathtt{pr}_a"'] \arrow[dl, phantom, "\cong"] & \gamma \arrow[dr, phantom, "="] & \\
a \arrow[rrr, "i"'] & & & \gamma ,
\end{tikzcd}$$

where the outer rectangle commutes, as $i = j \circ f$ by definition. But this immediately implies the commutativity of the inner rectangle. Next we check the functoriality.
Preservation of identities: Let $i\colon a \hookrightarrow \gamma$ be an object of $\mathsf{Sub}(\mathscr{C},\gamma)$. Then $\mathbf{1}_i = \mathbf{1}_a$, as $i \circ \mathbf{1}_a = \mathbf{1}_a \circ i = i$. We compute

$$E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(\mathbf{1}_a) = (\mathbf{1}_a, \mathbf{1}_\top) = \mathbf{1}_{(a, i\circ \mathtt{pr}_a, \top)} = \mathbf{1}_{E_0^{\mathsf{Sub}(\mathscr{C},\gamma)}(i)}.$$

Compatibility with composition: Assume we are given $i\colon a \hookrightarrow \gamma, j\colon b \hookrightarrow \gamma, k\colon c \hookrightarrow \gamma$ and $f\colon a \to b, g\colon b \to c$ such that

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, hook, "i"'] & b \arrow[r, "g"] \arrow[d, hook, "j"] & c \arrow[dl, hook', "k"] \\
& \gamma &
\end{tikzcd}$$

commutes. Then

$$E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(g \circ f) = (g \circ f, \mathbf{1}_\top) = (g \circ f, \mathbf{1}_\top \circ \mathbf{1}_\top) = (g, \mathbf{1}_\top) \circ (f, \mathbf{1}_\top) = E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(g) \circ E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(f).$$

So $E^{\mathsf{Sub}(\mathscr{C},\gamma)}$ is indeed a functor. Q.E.D.

Theorem 3.13. *Let $\mathscr{C}$ be a cartesian closed category. Then $E^{\mathsf{Sub}(\mathscr{C},\gamma)}$ is a full embedding.*

Proof: The injectivity on objects is immediate, because if $(a, i \circ \mathtt{pr}_a, \top) = (b, j \circ \mathtt{pr}_a, \top)$, then $i \circ \mathtt{pr}_a = j \circ \mathtt{pr}_b = \mathtt{pr}_a$, as $a = b$, and therefore $i = j$, as $\mathtt{pr}_a$ is an isomorphism.

It remains to show the bijectivity of the rule $f \mapsto E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(f)$ for $f \in \mathsf{Sub}(\mathscr{C},\gamma)_1(a,b)$ for arbitrary $a, b \in \mathsf{Sub}(\mathscr{C},\gamma)_0$.

- *Injectivity:* Assume we have

  $$E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(f) = (f, \mathbf{1}_\top) = (g, \mathbf{1}_\top) = E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}.$$

  Then $f = g$, which proves the injectivity.

- *Surjectivity:* Assume we are given $\phi = (\phi^+, \phi^-) \colon (a, i \circ \mathtt{pr}_a, \top) \to (b, j \circ \mathtt{pr}_b, \top)$. This means that $\phi^+ \colon a \to b$ and $\phi^- \colon \top \to \top$, so $\phi^- = \mathbf{1}_\top$. Therefore $\phi^+ \in \mathsf{Sub}(\mathscr{C},\gamma)_1$ and $\phi = E_1^{\mathsf{Sub}(\mathscr{C},\gamma)}(\phi^+)$.

So $E^{\mathsf{Sub}(\mathscr{C},\gamma)}$ is a full embedding. Q.E.D.

Chapter 4

The Chu functors

In this section we want to examine some rules that can be connected to the Chu construction, i.e. the following:

- Firstly we want to find a functor $\mathscr{C} \to \mathsf{Chu}(\mathscr{C}, \gamma)$ for every cartesian closed category and every $\gamma \in \mathscr{C}_0$ that fully represents the structure of the category $\mathscr{C}$ in $\mathsf{Chu}(\mathscr{C}, \gamma)$. For this we construct the functor $E^{\mathscr{C},\gamma}$, which is a strict representation in the sense discussed in the previous chapter.
- Secondly we want to find a functor $\mathscr{C} \to \mathsf{Cat}$, which assigns to each $\gamma \in \mathscr{C}_0$ the associated Chu category $\mathsf{Chu}(\mathscr{C}, \gamma)$.
- Lastly we want to find a way to assign not only to γ but to each pair $(\mathscr{C}, \gamma)$, where $\mathscr{C}$ is a cartesian closed category and $\gamma \in \mathscr{C}_0$, the associated Chu construction $\mathsf{Chu}(\mathscr{C}, \gamma)$. For this cause we define a version of the Grothendieck construction, whose objects are exactly pairs of this kind.

4.1 Extending functors to Chu categories

In the present section we suppose we are given a functor $F\colon \mathscr{C} \to \mathscr{D}$. We then want to examine under which conditions this functor extends to a functor $\mathsf{Chu}(\mathscr{C}, \gamma) \to \mathsf{Chu}(\mathscr{D}, \delta)$. To this end we define the following kind of functors.

Definition 4.1 (Product preserving functors). Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor. Then F is *product preserving*, if there exists a unique isomorphism $F_{ab}\colon F_0(a) \times F_0(b) \to F_0(a \times b)$ for every $a, b \in \mathscr{C}_0$ such that

$$\begin{tikzcd} & F_0(a) \times F_0(b) \arrow[dl, bend right, "\mathtt{pr}_{F_0(a)}"'] \arrow[dr, bend left, "\mathtt{pr}_{F_0(b)}"] \arrow[d, dashed, "F_{ab}"] & \\ F_0(a) & F_0(a \times b) \arrow[l, "F_1(\mathtt{pr}_a)"'] \arrow[r, "F_1(\mathtt{pr}_b)"] & F_0(b) \end{tikzcd} \tag{3}$$

commutes.

Remark 4.2. Even though we demand the arrow F_{ab} to be unique, it suffices to find an arrow $F_{ab}\colon F_0(a) \times F_0(b) \to F_0(a \times b)$ that makes the diagram (3) commute, as any such arrow is necessarily unique, which can be seen as follows. Assume we have another isomorphism $\tilde{F}_{ab}\colon F_0(a) \times F_0(b) \to F_0(a \times b)$ that makes (3) commute, then we have the commutative diagrams

$$\begin{tikzcd} & F_0(a \times b) \arrow[dl, bend right, "F_1(\mathtt{pr}_a)"'] \arrow[dr, bend left, "F_1(\mathtt{pr}_b)"] \arrow[d, dotted, "\tilde{F}_{ab}^{-1}"] & \\ F_0(a) & F_0(a) \times F_0(b) \arrow[l, "\mathtt{pr}_{F_0(a)}"] \arrow[r, "\mathtt{pr}_{F_0(b)}"'] & F_0(b) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & F_0(a \times b) \arrow[dl, bend right, "F_1(\mathtt{pr}_a)"'] \arrow[dr, bend left, "F_1(\mathtt{pr}_b)"] \arrow[d, dotted, "F_{ab}^{-1}"] & \\ F_0(a) & F_0(a) \times F_0(b) \arrow[l, "\mathtt{pr}_{F_0(a)}"] \arrow[r, "\mathtt{pr}_{F_0(b)}"'] & F_0(b). \end{tikzcd}$$

So by the universal property of the product we have $F_{ab}^{-1} = \tilde{F}_{ab}^{-1}$, and therefore $F_{ab} = \tilde{F}_{ab}$.

Proposition 4.3. *For every $a, a', b, b' \in \mathscr{C}_0$ and every $f\colon a \to a'$, $g\colon b \to b'$ in $\mathscr{C}_1$ the rectangle*

$$\begin{tikzcd}
F_0(a)\times F_0(b) \arrow[r, "F_1(f)\times F_1(g)"] \arrow[d, "F_{ab}"'] & F_0(a')\times F_0(b') \arrow[d, "F_{a'b'}"] \\
F_0(a\times b) \arrow[r, "F_1(f\times g)"] & F_0(a'\times b')
\end{tikzcd} \tag{4}$$

commutes.

Proof: We consider the diagram

$$\begin{tikzcd}
& & F_0(a)\times F_0(b) \arrow[dll, bend right, "\mathtt{pr}_{F_0(a)}"'] \arrow[dd, "F_{ab}"] \arrow[dr, "F_1(f)\times F_1(g)"] \arrow[ddrr, bend left, "\mathtt{pr}_{F_0(b)}"] & & & \\
& & & F_0(a')\times F_0(b') \arrow[ddll, bend right, "\mathtt{pr}_{F_0(a')}"'] \arrow[dd, "F_{a'b'}"] \arrow[ddrr, bend left, "\mathtt{pr}_{F_0(b')}"] & & \\
F_0(a) \arrow[dr, "F_1(f)"'] & & F_0(a\times b) \arrow[ll, "F_1(\mathtt{pr}_a)"'] \arrow[rr, "F_1(\mathtt{pr}_b)"] \arrow[dr, "F_1(f\times g)"] & & F_0(b) \arrow[dr, "F_1(g)"] & \\
& F_0(a') & & F_0(a'\times b') \arrow[ll, "F_1(\mathtt{pr}_{a'})"'] \arrow[rr, "F_1(\mathtt{pr}_{b'})"] & & F_0('b).
\end{tikzcd}$$

To show that the rectangle (4) commutes, it suffices to show that

$$F_1(f)\times F_1(g) = F_{a'b'}^{-1}\circ F_1(f\times g)\circ F_{ab}.$$

This we will prove using the universal propert of the product. Via definition we have the following implication:

$$\left.\begin{array}{l} \mathtt{pr}_{F_0(a')} = F_1(\mathtt{pr}_{a'})\circ F_{a'b'}, \\ \mathtt{pr}_{F_0(b')} = F_1(\mathtt{pr}_{b'})\circ F_{a'b'} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathtt{pr}_{F_0(a')}\circ F_{a'b'}^{-1} = F_1(\mathtt{pr}_{a'}), \\ \mathtt{pr}_{F_0(b')}\circ F_{a'b'}^{-1} = F_1(\mathtt{pr}_{b'}). \end{array}\right. \tag{5}$$

So we compute

$$\begin{aligned}
\mathtt{pr}_{F_0(a')}\circ F_{a'b'}^{-1}\circ F_1(f\times g)\circ F_{ab} &= (\mathtt{pr}_{F_0(a')}\circ F_{a'b'}^{-1})\circ F_1(f\times g)\circ F_{ab} && \text{(by associativity)}\\
&= F_1(\mathtt{pr}_{a'})\circ F_1(f\times g)\circ F_{ab} && \text{(proven in (5))}\\
&= (F_1(\mathtt{pr}_{a'})\circ F_1(f\times g))\circ F_{ab} && \text{(by associativity)}\\
&= \big(F_1(f)\circ F_1(\mathtt{pr}_a)\big)\circ F_{ab} && \text{(definition of } F_1(f\times g))\\
&= F_1(f)\circ (F_1(\mathtt{pr}_a)\circ F_{ab}) && \text{(by associativity)}\\
&= F_1(f)\circ \mathtt{pr}_{F_0(a)}, && \text{(definition of } F_{ab})\\
\mathtt{pr}_{F_0(b')}\circ F_{a'b'}^{-1}\circ F_1(f\times g)\circ F_{ab} &= (\mathtt{pr}_{F_0(b')}\circ F_{a'b'}^{-1})\circ F_1(f\times g)\circ F_{ab} && \text{(by associativity)}\\
&= F_1(\mathtt{pr}_{b'})\circ F_1(f\times g)\circ F_{ab} && \text{(proven in (5))}\\
&= (F_1(\mathtt{pr}_{b'})\circ F_1(f\times g))\circ F_{ab} && \text{(by associativity)}\\
&= \big(F_1(g)\circ F_1(\mathtt{pr}_b)\big)\circ F_{ab} && \text{(definition of } F_1(f\times g))\\
&= F_1(g)\circ (F_1(\mathtt{pr}_b)\circ F_{ab}) && \text{(by associativity)}\\
&= F_1(g)\circ \mathtt{pr}_{F_0(b)}\,. && \text{(definition of } F_{ab})
\end{aligned}$$

So $F_{a'b'}^{-1}\circ F_1(f\times g)\circ F_{ab}$ fulfils the requirement of the product, and by uniqueness we conclude

$$F_{a'b'}^{-1}\circ F_1(f\times g)\circ F_{ab} = F_1(f)\times F_1(g).$$

Q.E.D.

Next we want to prove the following statement:

Proposition 4.4. *If $G\colon \mathscr{D} \to \mathscr{E}$ also preserves products and G_{cd} is the canonical isomorphism $G_{cd}\colon G_0(c) \times G_0(d) \to G_0(c \times d)$ for $c, d \in \mathscr{D}_0$, then $G \circ F$ also preserves products and for every $a, b \in \mathscr{C}_0$ we have that*

$$(G \circ F)_{ab} = G_1(F_{ab}) \circ G_{F_0(a)F_0(b)}.$$

Proof: For $a, b \in \mathscr{C}_0$ we obtain the following diagrams, where F_{ab} and $(G \circ F)_{ab}$ are unique:

$$\begin{tikzcd}
& F_0(a) \times F_0(b) \arrow[dl, bend right, "\mathtt{pr}_{F_0(a)}"'] \arrow[dr, bend left, "\mathtt{pr}_{F_0(b)}"] \arrow[d, dotted, "F_{ab}"] & \\
F_0(a) & F_0(a \times b) \arrow[l, "F_1(\mathtt{pr}_a)"'] \arrow[r, "F_1(\mathtt{pr}_b)"] & F_0(b)
\end{tikzcd}$$

and

$$\begin{tikzcd}
& G_0(F_0(a)) \times G_0(F_0(b)) \arrow[dl, bend right, "\mathtt{pr}_{G_0(F_0(a))}"'] \arrow[dr, bend left, "\mathtt{pr}_{G_0(F_0(b))}"] \arrow[d, dotted, "(G\circ F)_{ab}"] & \\
G_0(F_0(a)) & G_0\big(F_0(a \times b)\big) \arrow[l, "(G\circ F)_1(\mathtt{pr}_a)"'] \arrow[r, "(G\circ F)_1(\mathtt{pr}_b)"] & G_0(F_0(b)).
\end{tikzcd}$$

We apply the functor G on the first diagram to obtain the diagram

$$\begin{tikzcd}
& G_0\big(F_0(a) \times F_0(b)\big) \arrow[dl, bend right, "G_1(\mathtt{pr}_{F_0(a)})"'] \arrow[dr, bend left, "G_1(\mathtt{pr}_{F_0(b)})"] \arrow[d, dotted, "G_1(F_{ab})"] & \\
G_0\big(F_0(a)\big) & G_0\big(F_0(a \times b)\big) \arrow[l, "G_1(F_1(\mathtt{pr}_a))"] \arrow[r, "G_1(F_1(\mathtt{pr}_b))"'] & G_0\big(F_0(b)\big).
\end{tikzcd}$$

As $G\colon \mathscr{D} \to \mathscr{E}$ is a product preserving functor and $F_0(a), F_0(b) \in \mathscr{D}_0$, we also have a commutative diagram

$$\begin{tikzcd}
& G_0\big(F_0(a)\big) \times G_0\big(F_0(b)\big) \arrow[dl, bend right, "\mathtt{pr}_{G_0(F_0(a))}"'] \arrow[dr, bend left, "\mathtt{pr}_{G_0(F_0(b))}"] \arrow[d, dotted, "G_{F_0(a)F_0(b)}"] & \\
G_0\big(F_0(a)\big) & G_0\big(F_0(a) \times F_0(b)\big) \arrow[l, "G_1(\mathtt{pr}_{F_0(a)})"'] \arrow[r, "G_1(\mathtt{pr}_{F_0(b)})"] & G_0\big(F_0(b)\big).
\end{tikzcd}$$

This gives rise to a bigger commutative diagram

$$\begin{tikzcd}
& G_0\big(F_0(a)\big) \times G_0\big(F_0(b)\big) \arrow[ddl, bend right, "\mathtt{pr}_{G_0(F_0(a))}"'] \arrow[ddr, bend left, "\mathtt{pr}_{G_0(F_0(b))}"] \arrow[d, dotted, "G_{F_0(a)F_0(b)}"] & \\
& G_0\big(F_0(a) \times F_0(b)\big) \arrow[dl, "G_1(\mathtt{pr}_{F_0(a)})"'] \arrow[dr, "G_1(\mathtt{pr}_{F_0(b)})"] \arrow[d, dotted, "G_1(F_{ab})"] & \\
G_0\big(F_0(a)\big) & G_0\big(F_0(a \times b)\big) \arrow[l, "G_1(F_1(\mathtt{pr}_a))"] \arrow[r, "G_1(F_1(\mathtt{pr}_b))"'] & G_0\big(F_0(b)\big).
\end{tikzcd}$$

This allows us to conclude that

$$\begin{aligned}
\mathtt{pr}_{G_0(F_0(b))} &= G_1(\mathtt{pr}_{F_0(b)}) \circ G_{F_0(a)F_0(b)} \\
&= (G \circ F)_1(\mathtt{pr}_b) \circ G_1(F_{ab}) \circ G_{F_0(a)F_0(b)} \\
\mathtt{pr}_{G_0(F_0(a))} &= G_1(\mathtt{pr}_{F_0(a)}) \circ G_{F_0(a)F_0(b)} \\
&= (G \circ F)_1(\mathtt{pr}_a) \circ G_1(F_{ab}) \circ G_{F_0(a)F_0(b)}.
\end{aligned}$$

So $G_1(F_{ab}) \circ G_{F_0(a)F_0(b)}$ fulfils the requirements and $G \circ F$ preserves products. Q.E.D.

This allows us to define a functor $\mathsf{Chu}(\mathscr{C},\gamma) \to \mathsf{Chu}(\mathscr{D},\delta)$, given a functor $F\colon \mathscr{C} \to \mathscr{D}$.

Definition 4.5. Let $\mathscr{C}, \mathscr{D}$ be cartesian closed categories, $F\colon \mathscr{C} \to \mathscr{D}$ be a product preserving functor and $\gamma \in \mathscr{C}_0, \delta \in \mathscr{D}_0$ with an arrow $\psi\colon F_0(\gamma) \to \delta$. We define $F_*\colon \mathsf{Chu}(\mathscr{C},\gamma) \to \mathsf{Chu}(\mathscr{D},\delta)$ in the following way:

$$\begin{aligned}(F_*)_0(a,f,b) &= \big(F_0(a), \psi \circ F_1(f) \circ F_{ab}, F_0(b)\big),\\ (F_*)_1(\phi^+,\phi^-) &= (F_1(\phi^+), F_1(\phi^-)).\end{aligned}$$

Theorem 4.6. *$F_*\colon \mathsf{Chu}(\mathscr{C},\gamma) \to \mathsf{Chu}(\mathscr{D},\delta)$ is a functor.*

Proof: Let $\mathscr{C}, \mathscr{D}$ be cartesian closed categories, $\gamma \in \mathscr{C}_0, \delta \in \mathscr{D}_0$ and $F\colon \mathscr{C} \to \mathscr{D}$ be a product preserving functor with canonical isomorphisms $F_{ab}\colon F_0(a) \times F_0(b) \to F_0(a \times b)$ for $a, b \in \mathscr{C}_0$. We seek to show that F_* is a functor.
Well-definedness: We show that $(F_*)_0(a,f,x) \in \mathsf{Chu}(\mathscr{C},\delta)_0$ for all $(a,f,x) \in \mathsf{Chu}(\mathscr{C},\gamma)_0$. It immediately follows that $F_0(a), F_0(x) \in \mathscr{D}_0$. Now $\psi \circ F_1(f) \circ F_{ab}\colon F_0(a) \times F_0(b) \to \delta$, as desired.

Now we seek to show that the image of arrows is as desired. Let $(\phi^+,\phi^-)\colon (a,f,x) \to (b,g,y)$ be given. We first show that $F_1(\phi^+)\colon F_0(a) \to F_0(b)$ and $F_1(\phi^-)\colon F_0(y) \to F_0(x)$.This is immediate from the definition of F and ϕ^+, ϕ^-. It remains to show that the diagram

$$\begin{tikzcd} F_0(a) \times F_0(y) \arrow[r, "\mathbf{1}_{F_0(a)} \times F_1(\phi^-)"] \arrow[d, "F_1(\phi^+) \times \mathbf{1}_{F_0(y)}"'] & F_0(a) \times F_0(x) \arrow[d, "\psi \circ F_1(f) \circ F_{ax}"] \\ F_0(b) \times F_0(y) \arrow[r, "\psi \circ F_1(g) \circ F_{by}"] & \delta \end{tikzcd}$$

commutes. To this end it suffices to show that

$$F_1(g) \circ F_{by} \circ (F_1(\phi^+) \times \mathbf{1}_{F_0(a)}) = F_1(f) \circ F_{ax} \circ (\mathbf{1}_{F_0(a)} \times F_1(\phi^-)).$$

As we already know that $g \circ (\phi^+ \times \mathbf{1}_y) = f \circ (\mathbf{1}_a \times \phi^-)$, we have

$$\begin{aligned} F_1(g) \circ F_{by} \circ &(F_1(\phi^+) \times \mathbf{1}_{F_0(y)}) \\ &= F_1(g) \circ F_{by} \circ (F_1(\phi^+) \times F_1(\mathbf{1}_y)) \\ &= F_1(g) \circ F_1(\phi^+ \times \mathbf{1}_y) \circ F_{ay} && \text{(by proposition 4.3)} \\ &= F_1(g \circ (\phi^+ \times \mathbf{1}_y)) \circ F_{ay} \\ &= F_1(f \circ (\mathbf{1}_a \times \phi^-)) \circ F_{ay} \\ &= F_1(f) \circ F_1(\mathbf{1}_a \times \phi^-) \circ F_{ay} \\ &= F_1(f) \circ F_{ax} \circ (F_1(\mathbf{1}_a) \times F_1(\phi^-)) && \text{(by proposition 4.3)} \\ &= F_1(f) \circ F_{ax} \circ \mathbf{1}_{F_0(a)} \times F_1(\phi^-), \end{aligned}$$

as desired.
Preservation of composition: Suppose we are given $\phi = (\phi^+,\phi^-)\colon (a,f,x) \to (b,g,y)$ and $\theta = (\theta^+,\theta^-)\colon (b,g,y) \to (c,h,z)$. We seek to show that $(F_*)_1(\theta \circ \phi) = (F_*)_1(\theta) \circ F_1(\phi)$. One quickly sees

$$\begin{aligned}(F_*)_1(\theta \circ \phi) &= \big(F_1((\theta \circ \phi)^+), F_1((\theta \circ \phi)^-)\big)\\ &= \big(F_1(\theta^+ \circ \phi^+), F_1(\phi^- \circ \theta^-)\big)\\ &= \big(F_1(\theta^+) \circ F_1(\phi^+), F_1(\phi^-) \circ F_1(\theta^-)\big)\\ &= \big(F_1(\theta^+), F_1(\theta^-)\big) \circ \big(F_1(\phi^+), F_1(\phi^-)\big)\end{aligned}$$

$$= (F_*)_1(\theta) \circ (F_*)_1(\phi),$$

as desired.
Preservation of unity: Let $(a,f,x) \in \mathsf{Chu}(\mathscr{C},\gamma)$ be arbitray. Then

$$\begin{aligned}
(F_*)_1(\mathbf{1}_{(a,f,x)}) &= \big(F_1(\mathbf{1}_a), F_1(\mathbf{1}_x)\big) && \text{(definition of } \mathbf{1}_{(a,f,x)}\text{)}\\
&= \big(\mathbf{1}_{F_0(a)}, \mathbf{1}_{F_0(x)}\big) && \text{(functoriality of } F\text{)}\\
&= \mathbf{1}_{(F_0(a), \psi\circ F_1(f)\circ F_{ax}, F_0(x))}\\
&= \mathbf{1}_{(F_*)_0(a,f,x)}.
\end{aligned}$$

Q.E.D.

4.2 Strict representations of a cartesian closed category $\mathscr{C}$ into $\mathsf{Chu}(\mathscr{C},\gamma)$

We want to define a a functor $E^{\mathscr{C},\gamma} \colon \mathscr{C} \to \mathsf{Chu}(\mathscr{C},\gamma)$ for a cartesian closed category $\mathscr{C}$ and $\gamma \in \mathscr{C}_0$ that transfers all the information of $\mathscr{C}$ into $\mathsf{Chu}(\mathscr{C},\gamma)$, i.e. that is a strict representation. We do this in the following way:

Definition 4.7. Let $\mathscr{C}$ be a cartesian closed category and $\gamma \in \mathscr{C}_0$. We define $E^{\mathscr{C},\gamma} \colon \mathscr{C} \to \mathsf{Chu}(\mathscr{C},\gamma)$ by the following clauses:
- For all $a \in \mathscr{C}_0$ we set $E_0^{\mathscr{C},\gamma}(a) = (a, \mathtt{eval}_{\gamma,a}, \gamma^a)$.
- For all $f \colon a \to b$ in $\mathscr{C}$ we set $E_1^{\mathscr{C},\gamma}(f) = (f, \widehat{h}) \colon (a, \mathtt{eval}_{\gamma,a}, \gamma^a) \to (b, \mathtt{eval}_{\gamma,b}, \gamma^b)$ where $h = \mathtt{eval}_{\gamma,b} \circ (\mathbf{1}_{\gamma^b} \times f)$. We will denote $\widehat{h}$ by f^- to aid the simplicity of the notation.

Proposition 4.8. $E^{\mathscr{C},\gamma} \colon \mathscr{C} \to \mathsf{Chu}(\mathscr{C},\gamma)$ *defined as above for a cartesian closed category* $\mathscr{C}$ *and an arbitrary* $\gamma \in C_0$ *is a covariant functor and a strict representation of* $\mathscr{C}$ *in* $\mathsf{Chu}(\mathscr{C},\gamma)$.

Proof: We shall proceed along the following steps:
1. Show that $E^{\mathscr{C},\gamma}$ is well-defined.
2. Show that $E_0^{\mathscr{C},\gamma}$ is an injection.
3. Show that $E_1^{\mathscr{C},\gamma}$ is an injection.
4. Show that for every $a, b \in C_0$ the map

$$E_{(a,b)}^{\mathscr{C},\gamma} \colon \mathrm{Hom}_{\mathscr{C}}(a,b) \to \mathrm{Hom}_{\mathsf{Chu}(\mathscr{C},\gamma)}\big(E_0^{\mathscr{C},\gamma}(a), E_0^{\mathscr{C},\gamma}(b)\big)$$

is a bijection.

These three steps are sufficient as a proof for all the statements made in proposition 4.8.
Ad step 1: We can see, that $E_0^{\mathscr{C},\gamma}(a) = (a, \mathtt{eval}_{\gamma,a}, \gamma^a) \in \mathsf{Chu}(\mathscr{C},\gamma)$, as $a, \gamma^a \in \mathscr{C}$ and $\mathtt{eval}_{\gamma,a} \colon a \times \gamma^a \to \gamma$ (up to an isomorphism). Now suppose we are given $a, b \in C_0$ with an arrow $f \colon a \to b$). We have to show that

$$(f, f^-) \colon (a, \mathtt{eval}_{\gamma,a}, \gamma^a) \to (b, \mathtt{eval}_{\gamma,b}, \gamma^b).$$

That is, we want that $f \colon a \to b$ and $f^- \colon \gamma^b \to \gamma^a$ make the diagram

$$\begin{tikzcd}
a \times \gamma^b \arrow[r, "\mathbf{1}_a \times f^-"] \arrow[d, "f^+ \times \mathbf{1}_{\gamma^b}"'] & a \times \gamma^a \arrow[d, "\mathtt{eval}_{\gamma,a}"] \\
b \times \gamma^b \arrow[r, "\mathtt{eval}_{\gamma,b}"] & \gamma
\end{tikzcd}$$

commute. For this we first have to examine the definition of f^-. The arrow f^- is defined as the transpose $f = \hat{h}$ of the arrow $h = \mathtt{eval}_{\gamma,b} \circ (\mathbf{1}_{\gamma^b} \times f)$. This immediately yields, that $f^- \colon \gamma^b \to \gamma^a$. The commutativity follows immediately from the definition of the transpose.

Ad step 2: If we assume that we are given $a, a' \in C_0$ with $E_0^{\mathscr{C},\gamma}(a) = E_0^{\mathscr{C},\gamma}(a')$, it follows that $(a, \mathtt{eval}_{\gamma,a}, \gamma^a) = (a', \gamma_{\gamma,a'}, \gamma^{a'})$ and therefore $a = a'$.

Ad step 3: Suppose we are given two arrow $f\colon a \to b, g\colon a \to b$ in $\mathscr{C}$ such that $f \neq g$. We need to show that $E_1^{\mathscr{C},\gamma}(f) \neq E_1^{\mathscr{C},\gamma}(g)$. Assume the contrary. Then $(f, \widehat{h}_1) = (g, \widehat{h}_2)$, where $\widehat{h}_1 = \mathtt{eval}_{\gamma,b} \circ (f \times \mathbf{1}_{\gamma^b})$ and $\widehat{h}_2 = \mathtt{eval}_{\gamma,b} \circ (g \times \mathbf{1}_{\gamma^b})$. But this immediately implies $f = g$, contrary to the assumption $f \neq g$. So $E_1^{\mathscr{C},\gamma}(f) \neq E_1^{\mathscr{C},\gamma}(g)$.

Ad step 4: Suppose we are given $a, b \in C_0$ and two arrows $f, g\colon a \to b$ such that $E_1^{\mathscr{C},\gamma}(f) = E_1^{\mathscr{C},\gamma}(g)$. Then $(f, f^-) = (g, g^-)$ and by extension $f = g$. This shows the injectivity.

To see the surjectivity, we assume we have an arrow $g \in \mathrm{Hom}_{\mathsf{Chu}(\mathscr{C},\gamma)}\big(E_0^{\mathscr{C},\gamma}(a), E_0^{\mathscr{C},\gamma}(b)\big)$. This arrow consists of g^+, g^- such that $g^+\colon a \to b, g^-\colon \gamma^b \to \gamma^a$ and the rectangle

$$\begin{tikzcd}
a \times \gamma^b \arrow[r, "\mathbf{1}_a \times g^-"] \arrow[d, "g^+ \times \mathbf{1}_{\gamma^b}"'] & a \times \gamma^a \arrow[d, "\mathtt{eval}_{\gamma,a}"] \\
b \times \gamma^b \arrow[r, "\mathtt{eval}_{\gamma,b}"] & \gamma
\end{tikzcd}$$

commutes. We seek to show that we automatically have $g^- = \hat{h}$ for $h = \mathtt{eval}_{\gamma,b} \circ (g^+ \times \mathbf{1}_{\gamma^b})$. Since we have that $\mathtt{eval}_{\gamma,a} \circ (\mathbf{1}_a \times g^-) = h$, we automatically have from the uniqueness of the transpose that $g^- = \hat{h}$. Q.E.D.

4.3 The internal Chu functor

Now we want to define the rule that assigns to each $\gamma \in \mathscr{C}_0$ for a given cartesian closed category $\mathscr{C}$ its Chu category $\mathsf{Chu}(\mathscr{C}, \gamma)$. Before we can do this we need to define the category of locally small categories.

Definition 4.9. The category Cat is defined by the following clauses:
- Its objects are locally small categories.
- The arrows of Cat are functors $F\colon \mathscr{C} \to \mathscr{D}$ where $\mathscr{C}, \mathscr{D} \in \mathsf{Cat}_0$.
- The composition of arrows is the composition of functors.

Definition 4.10 (The internal (local) Chu functor). Let a cartesian closed category $\mathscr{C}$ be given. We define the rule $\mathsf{Chu}^{\mathscr{C}}\colon \mathscr{C} \to \mathsf{Cat}$ in the following way.
- Let $\gamma \in \mathscr{C}_0$. Then
$$\mathsf{Chu}_0^{\mathscr{C}}(\gamma) = \mathsf{Chu}(\mathscr{C}, \gamma).$$
- Let $u\colon \gamma \to \delta$ be an arrow in $\mathscr{C}$. We define the functor
$$\mathsf{Chu}_1^{\mathscr{C}}(u) =: u_*\colon\ \mathsf{Chu}(\mathscr{C}, \gamma) \to \mathsf{Chu}(\mathscr{C}, \delta)$$
by the following clauses:
$$(u_*)_0(a, f, b) = (a, u \circ f, b), \qquad (u_*)_1(\phi^+, \phi^-) = (\phi^+, \phi^-)$$
for all $(a, f, b) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ and all $(\phi^+, \phi^-) \in \mathsf{Chu}(\mathscr{C}, \gamma)_1$.

Proposition 4.11. *Let $\mathscr{C}$ be a cartesian closed category. Then $\mathsf{Chu}^{\mathscr{C}}\colon \mathscr{C} \to \mathsf{Cat}$ is a functor. If $u\colon \delta \hookrightarrow \gamma$ is a monomorphism in $\mathscr{C}$, then u_* is a full embedding.*

Proof: We first show that $\mathsf{Chu}^{\mathscr{C}}\colon \mathscr{C} \to \mathsf{Cat}$ is a functor. We have to show the following three properties:
1. The functor $\mathsf{Chu}^{\mathscr{C}}$ is well-defined.

2. For $\delta, \gamma, \chi \in \mathscr{C}_0$ and $u\colon \delta \to \gamma, w\colon \gamma \to \chi$ we have

$$\mathsf{Chu}_1^{\mathscr{C}}(w \circ u) = \mathsf{Chu}_1^{\mathscr{C}}(w) \circ \mathsf{Chu}^{\mathscr{C}}(u).$$

3. For any $\gamma \in \mathscr{C}_0$ we have $\mathsf{Chu}^{\mathscr{C}}(\mathbf{1}_\gamma) = \mathbf{1}_{\mathsf{Chu}_0^{\mathscr{C}}(\gamma)}$.

Ad 1: As $\mathscr{C}$ is a locally small category, we immediately can conclude that $\mathsf{Chu}(\mathscr{C}, \gamma)$ is a locally small category, because

$$\mathrm{Hom}_{\mathsf{Chu}(\mathscr{C},\gamma)}\big((a, f, x), (b, g, y)\big) \subseteq \mathrm{Hom}_{\mathscr{C}}(a, b) \times \mathrm{Hom}_{\mathscr{C}}(y, x).$$

It remains to show that for $u\colon \delta \to \gamma$ the functor $u_*\colon\ \mathsf{Chu}(\mathscr{C}, \delta) \to \mathsf{Chu}(\mathscr{C}, \gamma)$ is well-defined. So let u be given. By definition we have

$$\mathsf{Chu}_1^{\mathscr{C}}(u) = u_*\colon\ \mathsf{Chu}(\mathscr{C}, \delta) \to \mathsf{Chu}(\mathscr{C}, \gamma),$$

where $(u_*)_0(a, f, b) = (a, u \circ f, b)$ and $(u_*)_1(\phi^+, \phi^-) = (\phi^+, \phi^-)$. We see that this is well defined, as $u \circ f\colon a \times x \to \gamma$, so $(a, u \circ f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$. Assume we are given $(a, f, x), (b, g, y) \in \mathsf{Chu}(\mathscr{C}, \delta)_0$ and $\phi = (\phi^+, \phi^-)\colon (a, f, y) \to (b, g, y)$. Then $(\phi^+, \phi^-)\colon (a, u \circ f, x) \to (b, u \circ g, y)$, as $\phi^+\colon a \to b$ and $\phi^-\colon y \to x$. Furthermore, the diagram

$$\begin{tikzcd}
a \times y \arrow[r, "\mathbf{1}_a \times \phi^-"] \arrow[d, "\phi^+ \times \mathbf{1}_y"'] & a \times x \arrow[d, "u \circ f"] \\
b \times y \arrow[r, "u \circ g"] & \gamma
\end{tikzcd}$$

still commutes, so the condition on Chu morphisms is satisfied. So u_* is indeed an arrow in Cat_1.

Ad 2: Now consider $\mathsf{Chu}_1^{\mathscr{C}}(w \circ u) = (w \circ u)_*$. We have

$$\begin{aligned}
\big((w \circ u)_*\big)_0(a, f, x) &= (a, w \circ u \circ f, x),\\
\big((w \circ u)_*\big)_1(\phi) &= \phi
\end{aligned}$$

for all $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \delta)_0$ and $\phi \in C_1$. Analogously, we have

$$\begin{aligned}
(w_*)_0\big((u_*)_0(a, f, x)\big) &= (w_*)_0(a, u \circ f, x) = (a, w \circ u \circ f, x),\\
(w_*)_1\big((u_*)_1(\phi)\big) &= (w_*)_1(\phi) = \phi,
\end{aligned}$$

so $\mathsf{Chu}_1^{\mathscr{C}}(w \circ u) = \mathsf{Chu}_1^{\mathscr{C}}(w) \circ \mathsf{Chu}_1^{\mathscr{C}}(u)$ holds.

Ad 3: We have $\mathsf{Chu}_1^{\mathscr{C}}(\mathrm{id}_\gamma) = (\mathbf{1}_\gamma)_*$, where

$$\begin{aligned}
\big((\mathbf{1}_\gamma)_*\big)_0(a, f, b) &= (a, \mathbf{1}_\gamma \circ f, b) = (a, f, b),\\
\big((\mathbf{1}_\gamma)_*\big)_1(\phi) &= \phi.
\end{aligned}$$

From this we see that $\mathsf{Chu}_1^{\mathscr{C}}(\mathbf{1}_\gamma) = \mathbf{1}_{\mathsf{Chu}_0^{\mathscr{C}}(\gamma)}$.

At last we want to show that u_* is a full embedding if $u\colon \delta \hookrightarrow \gamma$ is a monomorphism. One can immediately see, that for any $(a, f, x), (b, g, y) \in \mathsf{Chu}(\mathscr{C}, \delta)_0$ we have

$$\mathrm{Hom}_{\mathsf{Chu}(\mathscr{C},\gamma)}\big((a, f, x), (b, g, y)\big) = \mathrm{Hom}_{\mathsf{Chu}(\mathscr{C},\delta)}\big((a, u \circ f, x)(b, u \circ g, y)\big),$$

since $(u_*)_1(\phi) = \phi$ for all $\phi \in \mathsf{Chu}(\mathscr{C}, \gamma)_1$. So it remains to show that u_* is injective on the objects. Assume $(a, u \circ f, x) = (b, u \circ g, y)$. Then $a = b, x = y$ and $u \circ f = u \circ g$ implies $f = g$, as u is a monomorphism. So $(a, f, x) = (b, g, y)$. Q.E.D.

4.4 The Grothendieck category

Before we can define the global Chu functor, we need a specific construction. For this cause we define the following variant of the Grothendieck category.

We begin by making the entity of all cartesian closed categories a category.

Definition 4.12 (The category ccCat). The category ccCat is defined by the following clauses.

- The *objects* of ccCat are cartesian closed categories $\mathscr{C}$.
- An *arrow* $F\colon \mathscr{C} \to \mathscr{D}$ in ccCat is a product preserving functor.
- The *identity* of a cartesian closed category is $\mathrm{id}^{\mathscr{C}}$, as in Cat.

Definition 4.13 (The Grothendieck category). Let ccCat be the category of cartesian closed categories and $\mathrm{id}^{\mathsf{ccCat}}\colon \mathsf{ccCat} \to \mathsf{Cat}$ be the full embedding as a subcategory. Then the category $\mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})$ is defined by the following data:

- The *objects* of $\mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})$ are given by pairs $(\mathscr{C}, \gamma)$, where $\mathscr{C}$ is a cartesian closed category and
$$\gamma \in [\mathrm{id}^{\mathsf{ccCat}}]_0(\mathscr{C}) = \mathscr{C}_0.$$
- The *arrows* of $\mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})$ are given by pairs $(F, \phi)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$, where $F\colon \mathscr{C} \to \mathscr{D}$ is a product preserving functor and $\phi\colon F_0(\gamma) \to \delta$ is an arrow in $\mathscr{D}$.
- The *identity* of $(\mathscr{C}, \gamma)$ is given by $(\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma)$.

Lemma 4.14. *The category* $\mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})$ *defined above is a category.*

Proof: *Associativity:* Let

$$(F, \phi)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta), \qquad (G, \psi)\colon (\mathscr{D}, \delta) \to (\mathscr{E}, \epsilon),$$
$$(H, \chi)\colon (\mathscr{E}, \epsilon) \to (\mathscr{J}, \kappa)$$

be given. We compute

$$\begin{aligned}
\big((H, \chi) \circ (G, \psi)\big) \circ (F, \phi) &= \big(H \circ G, \chi \circ H_1(\psi)\big) \circ (F, \phi) \\
&= \big((H \circ G) \circ F, (\chi \circ H_1(\psi)) \circ (H \circ G)_1(\phi)\big) \\
&= \big(H \circ (G \circ F), \chi \circ (H_1(\psi)) \circ (H \circ G)_1(\phi))\big) \\
&= \big(H \circ (G \circ F), \chi \circ H_1(\psi \circ (G)_1(\phi))\big) \\
&= (H, \chi) \circ (G \circ F, \psi \circ G_1(\phi)) \\
&= (H, \chi) \circ \big((G, \psi) \circ (F, \phi)\big)
\end{aligned}$$

Identity: Let $(\mathscr{C}, \gamma) \in \mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})_0$. Let $(\mathscr{D}, \delta), (\mathscr{E}, \epsilon) \in \mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})_0$ with arrows $(F, \phi)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$ and $(G, \psi)\colon (\mathscr{E}, \epsilon) \to (\mathscr{C}, \gamma)$ be given. Then

$$\begin{aligned}
(F, \phi) \circ \mathbf{1}_{(\mathscr{C}, \gamma)} &= (F, \phi) \circ (\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma) && \text{(by definition)} \\
&= (F \circ \mathrm{id}^{\mathscr{C}}, \phi \circ F_1(\mathbf{1}_\gamma)) \\
&= (F, \phi \circ \mathbf{1}_{F_0(\gamma)}) && \text{(definition of } \mathrm{id}^{\mathscr{C}}) \\
&= (F, \phi), \\
\mathbf{1}_{(\mathscr{C}, \gamma)} \circ (G, \psi) &= (\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma) \circ (G, \psi) && \text{(by definition)} \\
&= (\mathrm{id}^{\mathscr{C}} \circ G, \mathbf{1}_\gamma \circ \mathrm{id}_0^{\mathscr{C}}(\psi)) \\
&= (G, \mathbf{1}_\gamma \circ \psi) \\
&= (G, \psi).
\end{aligned}$$

Q.E.D.

4.5 The global Chu functor

We seek to define a rule that assigns to a pair $(\mathscr{C}, \gamma)$ of a cartesian closed category $\mathscr{C}$ and an object $\gamma \in \mathscr{C}$ its Chu category $\mathsf{Chu}(\mathscr{C}, \gamma)$. For this we use the approach of [Pet21].

Definition 4.15. We define the rule $\mathsf{Chu}\colon \mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}}) \to \mathsf{Cat}$ in the following way.

- For all $(\mathscr{C}, \gamma) \in \mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})_0$ we set $\mathsf{Chu}_0(\mathscr{C}, \gamma) = \mathsf{Chu}(\mathscr{C}, \gamma)$.
- For all arrows $(F, \phi)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$ in $\mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})$ we set $\mathsf{Chu}_1(F, \phi) = F_*$, where F_* is defined in definition 4.5.

Lemma 4.16. *The rule* $\mathsf{Chu}\colon \mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})$ *defined as above gives rise to a functor.*

Proof: We prove this by checking the preservation of composition and identities.
Preservation of Composition: Assume we are given arrows $(F, \phi)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$ and $(G, \psi)\colon (\mathscr{D}, \gamma) \to (\mathscr{E}, \epsilon)$. Then $\mathsf{Chu}_1((G, \psi) \circ (F, \phi)) = (G \circ F)_*$ and $\mathsf{Chu}_1(G, \psi) \circ \mathsf{Chu}_1(F, \phi) = G_* \circ F_*$. So we need to show

$$(G \circ F)_* = G_* \circ F_*.$$

It remains to check the following two conditions:

- For all $(a, f, b) \in \mathsf{Chu}(\mathscr{C}, \gamma)$ holds
$$((G \circ F)_*)_0(a, f, b) = (G_*)_0\big((F_*)_0(a, f, b)\big).$$
- For all $(\phi^+, \phi^-)\colon (a, f, b) \to (c, g, d)$ holds
$$((G \circ F)_*)_1(\phi^+, \phi^-) = (G_*)_1\big((F_*)_1(\phi^+, \phi^-)\big).$$

So assume $(a, f, b) \in \mathsf{Chu}(\mathscr{C}, \gamma)$ is given. Then

$$\begin{aligned}
((G \circ F)_*)_0(a, f, b) &= ((G \circ F)_0(a), (\psi \circ G_1(\phi)) \circ (G \circ F)_1(f) \circ (\mathcal{G} \circ \mathcal{F})_{ab}, (G \circ F)_0(b)) \\
&= (G_0(F_0(a)), \psi \circ G_1(\phi) \circ G_1(F_1(f)) \circ (G \circ F)_{ab}, (G \circ F)_0(b)) \\
&= (G_0(F_0(a)), \psi \circ G_1(\phi) \circ G_1(F_1(f)) \circ (G)_1(F_{ab}) \circ G_{F_0(a)F_0(b)}, (G \circ F)_0(b)) \\
&= (G_0(F_0(a)), \psi \circ G_1(\phi \circ F_1(f) \circ F_{ab}) \circ G_{F_0(a)F_0(b)}, G_0(F_0(b))) \\
&= (G_*)_0(F_0(a), \phi \circ F_1(f) \circ F_{ab}, F_0(b)) \\
&= (G_*)_0((F_*)_0(a, f, b)).
\end{aligned}$$

Now assume we are given $(\phi^+, \phi^-)\colon (a, f, b) \to (c, g, d)$. Then

$$\begin{aligned}
((G \circ F)_*)_1(\phi^+, \phi^-) &= \big((G \circ F)_1(\phi^+), (G \circ F)_1(\phi^-)\big) \\
&= \big(G_1(F_1(\phi^+)), G_1(F_1(\phi^-))\big) \\
&= (G_*)_1\big(F_1(\phi^+), F_1(\phi^-)\big) \\
&= (G_*)_1\big((F_*)_1(\phi^+, \phi^-)\big).
\end{aligned}$$

Preservation of identity: Assume we are given $(\mathscr{C}, \gamma) \in \mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}})$. We need to prove that $\mathsf{Chu}_1(\mathbf{1}_{(\mathscr{C},\gamma)}) = \mathbf{1}_{\mathsf{Chu}(\mathscr{C},\gamma)}$. By definition we have $\mathsf{Chu}_1(\mathbf{1}_{(\mathscr{C},\gamma)}) = (\mathrm{id}^{\mathscr{C}})_*$. To see that $(\mathrm{id}^{\mathscr{C}})_* = \mathbf{1}_{\mathsf{Chu}(\mathscr{C},\gamma)}$, it suffices to check the equalities

$$\big((\mathrm{id}^{\mathscr{C}})_*\big)_0(a, f, x) = (a, f, x) \quad \text{and} \quad \big((\mathrm{id}^{\mathscr{C}})_*\big)_1(\phi^+, \phi^-) = (\phi^+, \phi^-)$$

for all objects $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ and all arrows $\phi = (\phi^+, \phi^-)\colon (a, f, x,) \to (b, g, y)$ in $\mathsf{Chu}(\mathscr{C}, \gamma)$. Using definition 4.5 we can compute

$$\big((\mathrm{id}^{\mathscr{C}})_*\big)_0(a, f, x) = \big((\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma)_*\big)_1(a, f, x) = (\mathrm{id}^{\mathscr{C}}_0(a), \mathbf{1}_\gamma \circ \mathrm{id}^{\mathscr{C}}_1(f) \circ \mathrm{id}^{\mathscr{C}}_{ab}, x) = (a, f, x)$$

and

$$\big((\mathrm{id}^{\mathscr{C}})_*\big)_1(\phi^+, \phi^-) = (\phi^+, \phi^-).$$

So Chu is indeed a functor. Q.E.D.

4.6 A covariant Grothendieck functor

Let $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1)\colon \mathscr{C} \to \mathsf{Cat}$. We imitate the definition given in [Pet20a, Definition 7]. So we define a category $\int(\mathscr{C}, \mathcal{P})$ in the following way:

Definition 4.17 (The Grothendieck functor). The *covariant Grothendieck functor* is defined by the following clauses.

- *Objects:* The *objects* of $\int(\mathscr{C}, \mathcal{P})$ are given as pairs of objects (a, x), where $a \in \mathscr{C}_0$ and $x \in \big(\mathcal{P}_0(a)\big)_0$.
- *Morphisms:* Assume we are given $(a, x), (b, y) \in \int(\mathscr{C}, \mathcal{P})_0$. A *morphism* is a pair $(f, \phi)\colon (a, x) \to (b, y)$, where $f\colon a \to b$ is an arrow in $\mathscr{C}$ and $\phi\colon (\mathcal{P}_1(f))_0(x) \to y$ is an arrow in $\mathcal{P}_0(b)$. One can visualize this as in figure 4.1.

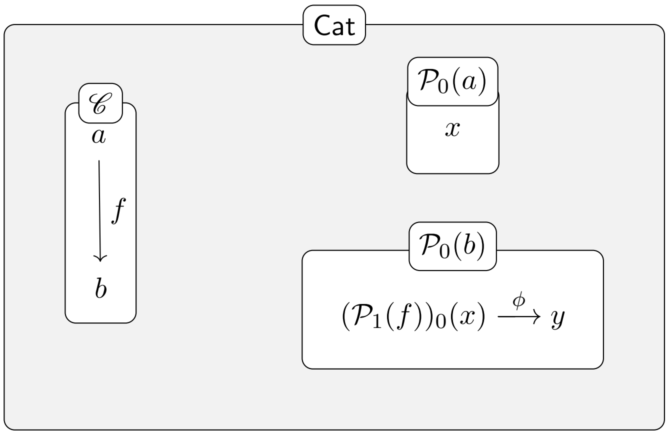


Figure 4.1: Illustration regarding the arrows in $\int(\mathscr{C}, \mathcal{P})$

- *Composition:* Assume we are given $(a, x), (b, y), (c, z) \in \int(\mathscr{C}, \mathcal{P})_0$ and $(f, \phi)\colon (a, x) \to (b, y), (g, \psi)\colon (b, y) \to (c, z)$. Then we set

$$(g, \psi) \circ (f, \phi) = (g \circ f, \psi \circ (\mathcal{P}_1(g))_1(\phi)),$$

  where $\psi \circ (\mathcal{P}_1(g))_1(\phi)\colon (\mathcal{P}_1(g \circ f))_0(x) \to z$. We could visualize this as in figure 4.2.
- *Identity:* We set $\mathbf{1}_{(a,x)} = (\mathbf{1}_a, \mathbf{1}_x)$ and see that this fulfils the required conditions.

Remark 4.18. Finally we want to compare the Grothendieck construction with the Grothendieck functor.

Objects: The objects of the Grothendieck category are given as pairs $(\mathscr{C}, \gamma)$, where $\mathscr{C}$ is a cartesian closed category and $\gamma \in \mathscr{C}_0$. The objects of the Grothendieck functor on the other hand are given as pairs (a, x) where $a \in \mathscr{C}_0$ for an arbitrary category $\mathscr{C}$ and $x \in \mathcal{P}_0(a)$.

Arrows: The arrows of the Grothendieck category are pairs $(F, f)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$, where $F\colon \mathscr{C} \to \mathscr{D}$ is a product preserving functor and $f\colon F_0(\gamma) \to \delta$ is an arrow in $\mathscr{D}$. The arrows of the covariant Grothendieck functor on the other hand are given by pairs $(F, f)\colon (a, x) \to (b, y)$, where $F\colon a \to b$ is an arrow in $\mathscr{C}$ and $f\colon (\mathcal{P}_1(F))_0(x) \to y$ is an arrow in $\mathcal{P}_0(b)$.

Hence we can observe the following.

> If we take $\mathscr{C} = \mathsf{ccCat}$, the category of cartesian closed categories and let $\mathcal{P}$ be the inclusion functor $\iota\colon \mathsf{ccCat} \hookrightarrow \mathsf{Cat}$, then we can recover the Grothendieck category from the Grothendieck functor as $\int(\mathsf{ccCat}, \iota)$.

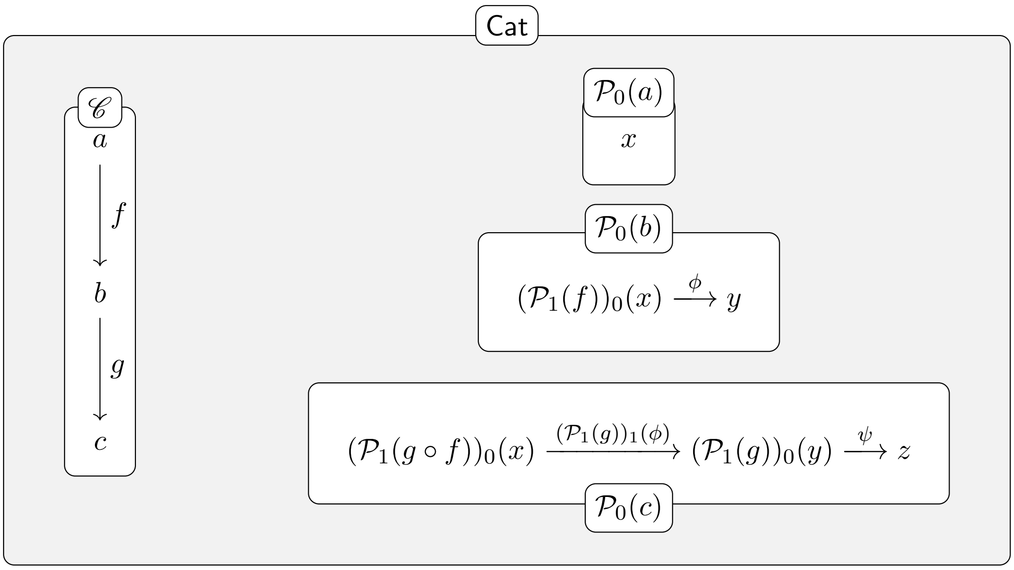


Figure 4.2: Illustration regarding the composition in $\int(\mathscr{C}, \mathcal{P})$

Chapter 5

The Chu construction and categorical constructions

In this chapter we want to dissect the interplay of the Chu construction with various constructions using categories, i.e. the opposite category or product categories.

5.1 Chu and duality

When working with the opposite category there are two ways to approach this problem. On one hand we can examine the opposite of the Chu category, $\mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}$, and on the other hand we can examine the Chu construction over the opposite category, $\mathsf{Chu}(\mathscr{C}^{\mathrm{op}},\gamma)$. As we have only defined the Chu construction over cartesian closed categories $\mathscr{C}$ and the opposite category $\mathscr{C}^{\mathrm{op}}$ need not be closed for a cartesian closed category $\mathscr{C}$, we need a greater framework for the latter approach.

For this reason we start by discussing the opposite category $\mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}$ and work on the latter approach in sections 5.2 to 5.5.

Lemma 5.1. *Let $\mathscr{C}$ be a cartesian closed category and $\gamma \in \mathscr{C}_0$. Then the Chu category is isomorphic to its dual,*

$$\mathsf{Chu}(\mathscr{C},\gamma) \cong \mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}.$$

Proof: We have to find a covariant functor $F\colon \mathsf{Chu}(\mathscr{C},\gamma) \to \mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}$ and a covariant functor $G\colon \mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}} \to \mathsf{Chu}(\mathscr{C},\gamma)$ such that $F \circ G = \mathrm{id}_{\mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}}$ and $G \circ F = \mathrm{id}_{\mathsf{Chu}(\mathscr{C},\gamma)}$. We define these functors in the following way.

- The functor F is defined by the following clauses: Let $(a,f,x) \in \mathsf{Chu}(\mathscr{C},\gamma)$. Then $F_0(a,f,x) = (x,f,a)$. If $(\phi^+,\phi^-)\colon (a,f,x) \to (b,g,y)$ is an arrow in $\mathsf{Chu}(\mathscr{C},\gamma)$, we set $F_1(\phi^+,\phi^-) = (\phi^-,\phi^+)$.
- The functor G is defined as follows: Let $(a,f,x) \in \mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}$. Then $G_0(a,f,x) = (x,f,a)$. If $(\psi^+,\psi^-)\colon (a,f,x) \to (b,g,y)$ is an arrow in $\mathsf{Chu}(\mathscr{C},\gamma)$, we set $G_1(\psi^+,\psi^-) = (\psi^-,\psi^+)$.

One can see that these definitions are indeed allowed: If $(a,f,x) \in \mathsf{Chu}(\mathscr{C},\gamma)$, then $(x,f,a) \in \mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}$, as $f\colon a \times x \to \gamma$ and therefore $(x,f,a) \in \mathsf{Chu}(\mathscr{C},\gamma)_0$ and $\mathsf{Chu}(\mathscr{C},\gamma)_0{}^{\mathrm{op}} = \mathsf{Chu}(\mathscr{C},\gamma)_0$. So let $(\phi^+,\phi^-)\colon (a,f,x) \to (b,g,y)$ be an arrow in $\mathsf{Chu}(\mathscr{C},\gamma)$. This means that $\phi^+\colon a \to b$ and $\phi^-\colon y \to x$ such that the diagram

$$\begin{tikzcd}
a \times y \arrow[r, "\mathbf{1}_a \times \phi^-"] \arrow[d, "\phi^+ \times \mathbf{1}_y"'] & a \times x \arrow[d, "f"] \\
b \times y \arrow[r, "g"'] & \gamma
\end{tikzcd}$$

commutes. We have to check that $F_1(\phi^+,\phi^-)\colon (x,f,a) \to (y,g,b)$ is an arrow in $\mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}$. So we have to show that $(\phi^-,\phi^+)\colon (y,g,b) \to (x,f,a)$ is an arrow in $\mathsf{Chu}(\mathscr{C},\gamma)$. By definition we have $\phi^-\colon y \to x$ and $\phi^+\colon a \to b$, and the diagram

$$\begin{tikzcd} y \times a \arrow[r, "\mathbf{1}_y \times \phi^+"] \arrow[d, "\phi^- \times \mathbf{1}_a"'] & y \times b \arrow[d, "g"] \\ x \times a \arrow[r, "f"'] & \gamma \end{tikzcd}$$

commutes as the diagram of (ϕ^+, ϕ^-) commutes. Similarly, the well-definedness of G can be shown in the following way: If $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0{}^{\mathrm{op}}$, then $(x, f, a) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$. Now let $(\psi^+, \psi^-)\colon (a, f, x) \to (b, g, y)$ be an arrow in $\mathsf{Chu}(\mathscr{C}, \gamma)^{\mathrm{op}}$. This means that $\psi^+\colon b \to a$ and $\psi^-\colon x \to y$, where

$$\begin{tikzcd} b \times x \arrow[r, "\mathbf{1}_b \times \psi^-"] \arrow[d, "\psi^+ \times \mathbf{1}_x"'] & b \times y \arrow[d, "g"] \\ a \times x \arrow[r, "f"] & \gamma \end{tikzcd}$$

commutes. But this is also the diagram required for $(\psi^-, \psi^+)\colon (x, f, a) \to (y, g, b)$ to be an arrow in $\mathsf{Chu}(\mathscr{C}, \gamma)$. So F, G are well-defined. It remains to check their functoriality.

- *Compatibility with identities:* Suppose we are given $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$. Then $\mathbf{1}_{(a,f,x)} = (\mathbf{1}_a, \mathbf{1}_x)$ and $F_1(\mathbf{1}_{(a,f,x)}) = (\mathbf{1}_x, \mathbf{1}_a) = \mathbf{1}_{(x,f,a)}$. Conversely, let $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0{}^{\mathrm{op}}$ be given. Then $\mathbf{1}_{(a,f,x)}) = (\mathbf{1}_a, \mathbf{1}_x)$ and $G_1(\mathbf{1}_{(a,f,x)}) = (\mathbf{1}_x, \mathbf{1}_a) = \mathbf{1}_{(x,f,a)}$.
- *Compatibility with composition:* Suppose we are given arrows

  $$(\phi^+, \phi^-)\colon (a, f, x) \to (b, g, y), \qquad (\psi^+, \psi^-)\colon (b, g, y) \to (c, h, z)$$

  in $\mathsf{Chu}(\mathscr{C}, \gamma)$. To reduce confusion we will denote the composition in $\mathsf{Chu}(\mathscr{C}, \gamma)^{\mathrm{op}}$ by $\circ^{\mathrm{op}}$. We compute

  $$\begin{aligned} F_1\big((\psi^+, \psi^-) \circ (\phi^+, \phi^-)\big) &= F_1(\psi^+ \circ \phi^+, \phi^- \circ \psi^-) = (\phi^- \circ \psi^-, \psi^+ \circ \phi^+) \\ &= (\phi^-, \phi^+) \circ (\psi^-, \psi^+) = (\psi^-, \psi^+) \circ^{\mathrm{op}} (\phi^-, \phi^+) \\ &= F_1(\psi^+, \psi^-) \circ^{\mathrm{op}} F_1(\phi^+, \phi^-). \end{aligned}$$

  Conversely, suppose we are given arrows

  $$(\phi^+, \phi^-)\colon (a, f, x) \to (b, g, y), \qquad (\psi^+, \psi^-)\colon (b, g, y) \to (c, h, z)$$

  in $\mathsf{Chu}(\mathscr{C}, \gamma)^{\mathrm{op}}$. Then we can compute

  $$\begin{aligned} G_1\big((\psi^+, \psi^-) \circ^{\mathrm{op}} (\phi^+, \phi^-)\big) &= G_1\big((\phi^+, \phi^-) \circ (\psi^+, \psi^-)\big) = G_1(\phi^+ \circ \psi^+, \psi^- \circ \phi^-) \\ &= (\psi^- \circ \phi^-, \phi^+ \circ \psi^+) = (\psi^-, \psi^+) \circ (\phi^-, \phi^+) \\ &= G_1(\psi^+, \psi^-) \circ G_1(\phi^+, \phi^-). \end{aligned}$$

Lastly we show that $G \circ F = \mathrm{id}_{\mathsf{Chu}(\mathscr{C},\gamma)}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}}$. Let $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$. Then

$$(G \circ F)_0(a, f, x) = G_0\big(F_0(a, f, x)\big) = G_0(x, f, a) = (a, f, x).$$

Now suppose we are given an arrow $(\phi^+, \phi^-)\colon (a, f, x) \to (b, g, y)$ in $\mathsf{Chu}(\mathscr{C}, \gamma)$. Then

$$(G \circ F)_1(\phi^+, \phi^-) = G_1\big(F_1(\phi^+, \phi^-)\big) = G_1(\phi^-, \phi^+) = (\phi^+, \phi^-).$$

The equality for $F \circ G = \mathrm{id}_{\mathsf{Chu}(\mathscr{C},\gamma)^{\mathrm{op}}}$ follows identically. Q.E.D.

5.2 The dual Chu construction

Now we want to consider the category $\mathscr{C}^{\mathrm{op}}$ for a given cartesian closed category $\mathscr{C}$. But if we wanted to construct $\mathsf{Chu}(\mathscr{C}, \gamma)$ for $\gamma \in \mathscr{C}_0$, we would run into the problem that $\mathscr{C}^{\mathrm{op}}$ is ad hoc not cartesian closed. So we need a different construction. We do this by analysing the key ingredients of the Chu construction and dualizing them. Those ingredients are

- A cartesian closed category $\mathscr{C}$.
- A specific object $\gamma \in \mathscr{C}_0$.
- Triplets (a, f, x) with $f\colon a \times x \to \gamma$.

So we first dualize the notion of a product. We then arrive at the notion of a coproduct, which we recall to be the following.

Definition 5.2 (Coproducts). Let $\mathscr{C}$ be an arbitrary category and $(c_n)_{n\in\mathscr{I}}$ be a family of objects in $\mathscr{C}$ indexed by the set I. A *coproduct* of $(c_n)_{n\in I}$ is an object $C \in \mathscr{C}_0$ together with arrows $i_n\colon c_n \to C$ satisfying the following universal property.

Let $a \in \mathscr{C}_0$ with an arrow $g_n\colon c_n \to a$ for each $n \in I$ be given. Then there exists a unique arrow $g\colon C \to a$ such that for each $n \in I$ the diagram

$$\begin{tikzcd} & a \\ c_n \arrow[ur, bend left, "g_n"] \arrow[r, "i_n"'] & C \arrow[u, "g"'] \end{tikzcd}$$

commutes.

Definition 5.3 (Initial objects). Let $\mathscr{C}$ be an arbitrary category. An *initial object* of $\mathscr{C}$ is an object $\perp \in \mathscr{C}_0$ such that for every $c \in \mathscr{C}_0$ there exists exactly one arrow, $¡_c\colon \perp \to c$.

Remark 5.4. The coproduct of $(c_n)_{n\in I}$ is determined up to unique isomorphism, if it exists, therefore we shall write $\coprod_{n\in I} c_i$ instead of C. If I is the set containing only two elements, we write $c_1 + c_2$ for the coproduct.

The dualization of the Chu construction is given as such:

Definition 5.5 (coChu categories). Let $\mathscr{C}$ be a category with binary coproducts, denoted by $+$. Then the *dual Chu category* $\mathsf{coChu}(\mathscr{C}, \xi)$ for $\xi \in \mathscr{C}_0$ is the category defined as such:

- *Objects:* The objects of $\mathsf{coChu}(\mathscr{C}, \xi)$ are pairs (c, f, x) with $c, x \in \mathscr{C}_0$ and $f\colon \xi \to c + x$.
- *Arrows:* The arrows of $\mathsf{coChu}(\mathscr{C}, \xi)$ are pairs $(\phi_+, \phi_-)\colon (c, f, x) \to (d, g, y)$ of arrows $\phi_+\colon d \to c, \phi_-\colon x \to y$. These arrows have to make the diagram

$$\begin{tikzcd} \xi \arrow[r, "f"] \arrow[d, "g"] & c + x \arrow[d, "\mathbf{1}_c + \phi_-"] \\ d + y \arrow[r, "\phi_+ + \mathbf{1}_y"] & c + y \end{tikzcd}$$

  commute. The identity arrow is simply $(\mathbf{1}_c, \mathbf{1}_x)$.
- *Composition:* Let arrows

$$(\phi_+, \phi_-)\colon (c, f, x) \to (d, g, y),$$
$$(\psi_+, \psi_-)\colon (d, g, y) \to (e, h, z)$$

  be given. We then set $(\psi_+, \psi_-) \circ (\phi_+, \phi_-) := (\phi_+ \circ \psi_+, \psi_- \circ \phi_-)$.

We start by showing the following lemma.

Lemma 5.6. *Let $\mathscr{C}$ be a category with coproducts. Then for $a, b, c, d, e, f \in \mathscr{C}_0$ and $g+g' \colon a+b \to c+d, h+h' \colon c+d \to e+f$ we have*

$$(h+h') \circ (g+g') = (h \circ g) + (h' \circ g').$$

Proof: We know that we have the commutative diagrams

$$\begin{tikzcd} a \arrow[r, "i_a"] \arrow[d, "g"] & a+b \arrow[d, dotted, "g+g'"] & b \arrow[l, "i_b"'] \arrow[d, "g'"] \\ c \arrow[r, "i_c"] & c+d & d \arrow[l, "i_d"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} c \arrow[r, "i_c"] \arrow[d, "h"] & c+d \arrow[d, dotted, "h+h'"] & d \arrow[l, "i_d"'] \arrow[d, "h'"] \\ e \arrow[r, "i_e"] & e+f & f. \arrow[l, "i_f"'] \end{tikzcd}$$

As both $(h+h') \circ (g+g')$ and $(h \circ g) + (h' \circ g')$ make the diagram

$$\begin{tikzcd} a \arrow[r, "i_a"] \arrow[d, "h \circ g"] & a+b \arrow[d, dotted] & b \arrow[l, "i_b"'] \arrow[d, "h' \circ g'"] \\ e \arrow[r, "i_e"] & e+f & f. \arrow[l, "i_f"'] \end{tikzcd}$$

commute, we have the desired equality. Q.E.D.

Proposition 5.7. *Let $\mathscr{C}$ be a category with binary coproducts and $\gamma \in \mathscr{C}_0$. Then $\mathsf{coChu}(\mathscr{C}, \gamma)$ is a well-defined category.*

Proof: We shall show that the composition is well-defined. For this let the arrows (ϕ_+, ϕ_-), (ψ_+, ψ_-) as described in definition 5.5 be given. We have to check the commutativity of the diagram

$$\begin{tikzcd} \xi \arrow[r, "f"] \arrow[d, "h"] & c+x \arrow[d, "\mathbf{1}_c+(\psi_-\circ\phi_-)"] \\ e+z \arrow[r, "(\phi_+\circ\psi_+)+\mathbf{1}_z"] & c+z. \end{tikzcd}$$

To this end we use the commutative diagrams of each arrow and compose them into a big diagram

$$\begin{tikzcd} c+x \arrow[r, "\mathbf{1}_c+\phi_-"] & c+y \\ \xi \arrow[u, "f"] \arrow[r, "g"] \arrow[d, "h"] & d+y \arrow[u, "\phi_++\mathbf{1}_y"'] \arrow[d, "\mathbf{1}_d+\psi_-"] \\ e+z \arrow[r, "\psi_++\mathbf{1}_z"] & d+z. \end{tikzcd}$$

This can be expanded into the commutative diagram

$$\begin{tikzcd} c+x \arrow[r, "\mathbf{1}_c+\phi_-"] \arrow[rrd, bend left, "\mathbf{1}_c+(\psi_+\circ\phi_-)"] & c+y \arrow[rd, "\mathbf{1}_c+\psi_-"] & \\ \xi \arrow[u, "f"] \arrow[r, "g"] \arrow[d, "h"] & d+y \arrow[u, "\phi_++\mathbf{1}_y"'] \arrow[d, "\mathbf{1}_d+\psi_-"'] & c+z \\ e+z \arrow[r, "\psi_++\mathbf{1}_z"'] \arrow[rru, bend right, "(\phi_+\circ\psi_+)+\mathbf{1}_z"'] & d+z. \arrow[ru, "\phi_++\mathbf{1}_z"'] & \end{tikzcd}$$

This yields the desired equality. Q.E.D.

Next we will quickly mimic the standard functors associated to the Chu construction for the coChu construction.

5.3 The dual Chu representation

To define a dual Chu representation, we first examine which prerequisites a category had to fulfil to employ the Chu representation. Such a category needed to be cartesian closed, but we only needed the existence of binary products and exponential. As the coproduct is the notion dual to the product, we need a notion dual to the exponential.

Definition 5.8 (Coexponentials). Let $\mathscr{C}$ be a category with coproducts and $a, x \in \mathscr{C}_0$. A *coexponential* of a, x is an object ${}^x a$ with an arrow $\mathsf{coev}_{a,x} \colon a \to {}^x a + x$ such that for every $b \in \mathscr{C}_0$ and $f \colon a \to b + x$ there exists an unique arrow $\check{f} \colon {}^x a \to b$ that makes

$$\begin{tikzcd} a \arrow[r, "\mathsf{coev}_{a,x}"] \arrow[dr, "f"'] & {}^x a + x \arrow[d, "\check{f} + \mathbf{1}_x"] \\ & b + x \end{tikzcd}$$

commute.

This allows us to define the dual to the Chu representation.

Theorem 5.9. *Let $\mathscr{C}$ be a category with binary coproducts and coexponentials and $\gamma \in \mathscr{C}_0$. We define a contravariant functor $C^{\mathscr{C},\gamma} \colon \mathscr{C} \to \mathsf{coChu}(\mathscr{C}, \gamma)$ via*

$$C_0^{\mathscr{C},\gamma}(a) = (a, \mathsf{coev}_{\gamma,a}, {}^a\gamma),$$
$$C_1^{\mathscr{C},\gamma}(f \colon a \to b) = (f, \check{h}),$$

where $h = (\mathbf{1}_{{}^a\gamma} + f) \circ \mathsf{coev}_{\gamma,a}$ and $\check{h}$ arises from the commutative diagram

$$\begin{tikzcd} \gamma \arrow[rr, "\mathsf{coev}_{\gamma,b}"] \arrow[dr, "\mathsf{coev}_{\gamma,a}"'] & & {}^b\gamma + b \arrow[dd, "\check{h} + \mathbf{1}_b"] \\ & {}^a\gamma + a \arrow[dr, "\mathbf{1}_{{}^a\gamma} + f"'] & \\ & & {}^a\gamma + b. \end{tikzcd} \tag{6}$$

Proof: We first show that these definitions are well-defined. By definition we have that $\mathsf{coev}_{\gamma,a} \colon \gamma \to {}^a\gamma + a$, so it remains to check $C_1^{\mathscr{C},\gamma}(f)$. Let $f \colon a \to b$ be given. We have to check that $C_1^{\mathscr{C},\gamma}(f) = (f, \check{h}) \colon (b, \mathsf{coev}_{\gamma,b}, {}^b\gamma) \to (a, \mathsf{coev}_{\gamma,a}, {}^a\gamma)$ is an arrow in $\mathsf{coChu}(\mathscr{C}, \gamma)$. This means to check that

$$f \colon a \to b \quad \text{and} \quad \check{h} \colon {}^b\gamma \to {}^a\gamma$$

as well as the arrow condition. The first condition is immediate by the definition of the arrow f, so it remains to show the second condition. As $h \colon \gamma \to {}^a\gamma + b$, we have by definition of the coexponential that $\check{h} \colon {}^b\gamma \to {}^a\gamma$. By definition of h the diagram (6) commutes, so $(f, \check{h})$ is an arrow in $\mathsf{coChu}(\mathscr{C}, \gamma)$.

We proceed by showing that $C^{\mathscr{C},\gamma}$ is indeed a functor $\mathscr{C}^{\mathrm{op}} \to \mathsf{coChu}(\mathscr{C}, \gamma)$. For his we check the following:

- *Compatibility with composition:* Let two arrows in $\mathscr{C}$, $f\colon a \to b, g\colon b \to c$ be given. We have to check that $C_1^{\mathscr{C},\gamma}(g \circ f) = C_1^{\mathscr{C},\gamma}(f) \circ C_1^{\mathscr{C},\gamma}(g)$. As we have

$$\begin{aligned} C_1^{\mathscr{C},\gamma}(g \circ f) &= (g \circ f, \check{h}_1), \\ C_1^{\mathscr{C},\gamma}(f) \circ C_1^{\mathscr{C},\gamma}(g) &= (f, \check{h}_2) \circ (g, \check{h}_3) = (g \circ f, \check{h}_2 \circ \check{h}_3) \end{aligned}$$

where

$$\begin{aligned} h_1 &= (\mathbf{1}_{{}^a\gamma} + g \circ f) \circ \mathtt{coev}_{\gamma,a}, & h_2 &= (\mathbf{1}_{{}^a\gamma} + f) \circ \mathtt{coev}_{\gamma,a}, \\ h_3 &= (\mathbf{1}_{{}^b\gamma} + g) \circ \mathtt{coev}_{\gamma,b}, \end{aligned}$$

it remains to show that $\check{h}_1 = \check{h}_2 \circ \check{h}_3$. To this end we show that both $h = \check{h}_1 + \mathbf{1}_c$ and $h = (\check{h}_2 \circ \check{h}_3) + \mathbf{1}_c$ make the diagram

$$\begin{tikzcd} \gamma \arrow[rr, "\mathtt{coev}_{\gamma,c}"] \arrow[dr, "\mathtt{coev}_{\gamma,a}"'] & & {}^c\gamma + c \arrow[dd, "h"] \\ & {}^a\gamma + a \arrow[dr, "\mathbf{1}_{{}^a\gamma} + (g \circ f)"'] & \\ & & {}^a\gamma + c \end{tikzcd} \tag{7}$$

commute. For the arrow $\check{h}_1 + \mathbf{1}_c$ this is immediate by definition, so we divert our attention to $(\check{h}_2 \circ \check{h}_3) + \mathbf{1}_c$. It is immediate that

$$(\check{h}_2 \circ \check{h}_3) + \mathbf{1}_c = (\check{h}_2 + \mathbf{1}_c) \circ (\check{h}_1 + \mathbf{1}_c),$$

and we have a diagram

$$\begin{tikzcd} {}^c\gamma + c \arrow[r, "\check{h}_3 + \mathbf{1}_c"] & {}^b\gamma + c \arrow[dr, "\check{h}_2 + \mathbf{1}_c"] & \\ \gamma \arrow[u, "\mathtt{coev}_{\gamma,c}"] \arrow[r, "\mathtt{coev}_{\gamma,b}"] \arrow[d, "\mathtt{coev}_{\gamma,a}"'] & {}^b\gamma + b \arrow[u, "\mathbf{1}_{{}^b\gamma} + g"'] \arrow[d, "\check{h}_2 + \mathbf{1}_b"'] & {}^a\gamma + c \\ {}^a\gamma + a \arrow[r, "\mathbf{1}_{{}^a\gamma} + f"'] & {}^a\gamma + b \arrow[ur, "\mathbf{1}_{{}^a\gamma} + g"'] & \end{tikzcd}$$

where the two squares on the left commute, so we can establish that

$$\begin{aligned} (\check{h}_2 + \mathbf{1}_c) \circ (\check{h}_3 + \mathbf{1}_c) \circ \mathtt{coev}_{\gamma,c} &= (\check{h}_2 + \mathbf{1}_c) \circ (\mathbf{1}_{{}^b\gamma} + g) \circ \mathtt{coev}_{\gamma,b} \\ &= (\check{h}_2 + g) \circ \mathtt{coev}_{\gamma,b} \\ &= (\mathbf{1}_{{}^a\gamma} + g) \circ (\check{h}_2 + \mathbf{1}_b) \circ \mathtt{coev}_{\gamma,b} \\ &= (\mathbf{1}_{{}^a\gamma} + g) \circ (\mathbf{1}_{{}^a\gamma} + f) \circ \mathtt{coev}_{\gamma,a} \\ &= (\mathbf{1}_{{}^a\gamma} + (g \circ f)) \circ \mathtt{coev}_{\gamma,a}, \end{aligned}$$

which gives us the commutativity of (7) in the case $h = (\check{h}_2 \circ \check{h}_3) + \mathbf{1}_c$, so by the uniqueness of h we have

$$(\check{h}_2 \circ \check{h}_3) + \mathbf{1}_c = \check{h}_1 + \mathbf{1}_c,$$

as desired.

- *Preservation of identities:* Let $a \in \mathscr{C}_0$ be given. We consider $C_1^{\mathscr{C},\gamma}(\mathbf{1}_a)$. By definition we have
$$C_1^{\mathscr{C},\gamma}(\mathbf{1}_a) = (\mathbf{1}_a, \check{h})$$
for $h = (\mathbf{1}_{^a\gamma} + \mathbf{1}_a) \circ \mathtt{coev}_{\gamma,a}$. We can immediately verify that $h = \mathtt{coev}_{\gamma,a}$, so we have to check that
$$\begin{tikzcd} \gamma \arrow[r, "\mathtt{coev}_{\gamma,a}"] \arrow[dr, "\mathtt{coev}_{\gamma,a}"'] & {^a\gamma + a} \arrow[d, "\mathbf{1}_{^a\gamma}+\mathbf{1}_a"] \\ & {^a\gamma + a} \end{tikzcd}$$
commutes. But this is immediate, so we have $\check{h} = \mathbf{1}_{^a\gamma}$.

So $C^{\mathscr{C},\gamma}$ is a functor $\mathscr{C}^{\mathrm{op}} \to \mathsf{coChu}(\mathscr{C},\gamma)$. Q.E.D.

As $E^{\mathscr{C},\gamma}$ was a full embedding of $\mathscr{C}$ into $\mathsf{Chu}(\mathscr{C},\gamma)$, we would like to have a dual statement to this. Fortunately, the following holds.

Proposition 5.10. *$C^{\mathscr{C},\gamma}$ is a full embedding of $\mathscr{C}^{\mathrm{op}}$ into $\mathsf{coChu}(\mathscr{C},\gamma)$.*

Proof: We first show that $C^{\mathscr{C},\gamma}$ is injective on objects. This is immediate, as if
$$(a, \mathtt{coev}_{\gamma,a}, {^a\gamma}) = (b, \mathtt{coev}_{\gamma,b}, {^b\gamma}),$$
we already have $a = b$.

Next we show that $C^{\mathscr{C},\gamma}_{(a,b)}$ is an bijection for all $a, b \in \mathscr{C}$. For this we have to be a little careful, as we have
$$C^{\mathscr{C},\gamma}_{(a,b)} \colon \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(a,b) = \mathrm{Hom}_{\mathscr{C}}(b,a) \to \mathrm{Hom}_{\mathsf{coChu}(\mathscr{C},\gamma)}\big((a, \mathtt{coev}_{\gamma,a}, {^a\gamma}), (b, \mathtt{coev}_{\gamma,b}, {^b\gamma})\big)$$
with $f \mapsto C_1^{\mathscr{C},\gamma}(f)$.

- *Injectivity:* Let $f, g \colon b \to a$ be given. We then have
$$\begin{aligned} C_1^{\mathscr{C},\gamma}(f) &= (f, \check{h}) = (g, \check{j}) \\ &= C_1^{\mathscr{C},\gamma}(g) \end{aligned}$$
for unique arrows $h, j \colon {^a\gamma} \to {^b\gamma}$, which are not necessary for the further argument, as we already have that $f = g$.
- *Surjectivity:* Let $(\phi_+, \phi_-) \colon (a, \mathtt{coev}_{\gamma,a}, {^a\gamma}) \to (b, \mathtt{coev}_{\gamma,b}, {^b\gamma}))$ be given. We want to find $f \colon b \to a$ such that $C_1^{\mathscr{C},\gamma}(f) = (\phi_+, \phi_-)$. To this end it suffices to show that $\check{h} = \phi_-$ for $h = (\mathbf{1}_{^b\gamma} + \phi_+) \circ \mathtt{coev}_{\gamma,b}$. We simply have to show that $(\phi_- + \mathbf{1}_a) \circ \mathtt{coev}_{\gamma,a} = h$. But this is immediate from the commutativity of the diagram
$$\begin{tikzcd} \gamma \arrow[r, "\mathtt{coev}_{\gamma,a}"] \arrow[d, "\mathtt{coev}_{\gamma,b}"'] & {^a\gamma + a} \arrow[d, "\phi_- + \mathbf{1}_a"] \\ {^b\gamma} \arrow[r, "\mathbf{1}_{^b\gamma}+\phi_+"'] & {^b\gamma + a,} \end{tikzcd}$$
which is given by the arrow (ϕ_+, ϕ_-).

So $C^{\mathscr{C},\gamma}$ is a full embedding of $\mathscr{C}^{\mathrm{op}}$ into $\mathsf{coChu}(\mathscr{C},\gamma)$. Q.E.D.

5.4 A dual of the internal Chu functor

As the internal functor $\mathsf{Chu}^{\mathscr{C}}$ is a functor $\mathscr{C} \to \mathsf{Cat}$ we want to define a functor $\mathsf{coChu}^{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Cat}$.

Theorem 5.11. *Let $\mathscr{C}$ be a category with coproducts. The rule $\mathsf{coChu}^{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Cat}$ defined by $\mathsf{coChu}_0^{\mathscr{C}}(\gamma) = \mathsf{coChu}(\mathscr{C}, \gamma)$ and*

$$\begin{aligned}\mathsf{coChu}_1^{\mathscr{C}}(u\colon \delta \to \gamma) = u^*\colon\ &\mathsf{coChu}(\mathscr{C}, \gamma) \to \mathsf{coChu}(\mathscr{C}, \delta),\\ &(u^*)_0(a, f, x) = (a, f \circ u, x),\\ &(u^*)_1(\phi_+, \phi_-) = (\phi_+, \phi_-),\end{aligned}$$

is a covariant functor.

Proof: Let $\mathscr{C}$ be given. To show the well-definedness, it remains to check $\mathsf{coChu}_1^{\mathscr{C}}$ is a functor, as $\mathsf{coChu}(\mathscr{C}, \gamma) \in \mathsf{Cat}_0$ for all $\gamma \in \mathscr{C}$.

So let $u\colon \delta \to \gamma$ be given. We have to prove that u^* is a functor. To this end we observe that $(a, f \circ u, b) \in \mathsf{coChu}(\mathscr{C}, \delta)$ for $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$, as $f \circ u\colon \delta \to a + x$. To see that any arrow $(\phi_+, \phi_-)\colon (a, f, x) \to (b, g, y)$ is also an arrow $(a, f \circ u, x) \to (b, g \circ u, y)$, one only has to see that

$$(\mathbf{1}_a + \phi_-) \circ f \circ u = (\phi_+ + \mathbf{1}_b) \circ g \circ u,$$

as already $(\mathbf{1}_a + \phi_-) \circ f = (\phi_+ + \mathbf{1}_b) \circ g$. The compatibility with composition and the preservation of identities are now immediate, so $\mathsf{coChu}^{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Cat}$ is indeed a functor Q.E.D.

Proposition 5.12. *If $u\colon \delta \to \gamma$ is an epimorphism, then u^* is a full embedding.*

Proof: For any $(a, f, x), (b, g, y) \in \mathsf{coChu}(\mathscr{C}, \gamma)_0$ we have

$$\mathrm{Hom}_{\mathsf{Chu}(\mathscr{C},\gamma)}\big((a, f, x), (b, g, y)\big) = \mathrm{Hom}_{\mathsf{Chu}(\mathscr{C},\delta)}\big((a, f \circ u, x)(b, g \circ u, y)\big),$$

since $(u_*)_1(\phi) = \phi$ for all $\phi \in \mathsf{coChu}(\mathscr{C}, \gamma)_1$. So it remains to show that u^* is injective on the objects. Assume $(a, f \circ u, x) = (b, g \circ u, y)$. Then $a = b, x = y$ and $f \circ u = g \circ u$ implies $f = g$, as u is an epimorphism. So $(a, f, x) = (b, g, y)$. Q.E.D.

5.5 The global coChu functor

Before we can define a global Chu functor, we first examine the global Chu functor. This was a functor

$$\mathsf{Chu}\colon \mathsf{Groth}(\mathsf{ccCat}, \mathrm{id}^{\mathsf{ccCat}}) \to \mathsf{Cat},$$

where
- Groth is the covariant Grothendieck construction,
- ccCat is the category of cartesian closed categories,
- $\mathrm{id}^{\mathsf{ccCat}}$ is the full embedding $\mathsf{ccCat} \hookrightarrow \mathsf{ccCat}$.

We would like to keep the covariant Grothendieck construction in our definition of the global coChu functor, so it remains to modify ccCat and $\mathrm{id}^{\mathsf{ccCat}}$.

Definition 5.13. We denote by cocCat the category defined as such:
- *Objects:* Categories $\mathscr{C}$ with finite coproducts and coexponentials and initial objects.
- *Arrows:* Covariant coproduct preserving functors $F\colon \mathscr{C} \to \mathscr{D}$.

The composition is defined in the usual way and the identities are simply the identity functors. We call objects $\mathscr{C} \in \mathsf{cocCat}_0$ *cocartesian closed categories.*

Next we have to deal with coproduct preserving functors.

Definition 5.14. Let $\mathscr{C}, \mathscr{D}$ be cocartesian closed categories. A functor $F\colon \mathscr{C} \to \mathscr{D}$ is *coproduct preserving*, if for every $a, b \in \mathscr{C}_0$ there exists a unique isomorphism F^{ab} that makes the diagram

$$\begin{tikzcd}
F_0(a) \arrow[r, "F_1(i_a)"] \arrow[dr, "i_{F_0(a)}"'] & F_0(a+b) \arrow[d, dotted, "F^{ab}"] & F_0(b) \arrow[l, "F_1(i_b)"'] \arrow[dl, "i_{F_0(b)}"] \\
 & F_0(a)+F_0(b) &
\end{tikzcd} \tag{8}$$

commute.

Remark 5.15. Analogously to the case of product preserving functors one only needs to find an isomorphism F^{ab} making the diagram (8) commute, as such an isomorphism is unique by the universal property of the coproduct.

Lemma 5.16. *Let $\mathscr{C}, \mathscr{D}$ be cocartesian closed categories and $a, b, c, d \in \mathscr{C}_0$ such that $\phi\colon a \to b$ and $\psi\colon c \to d$. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a coproduct preserving functor. Then the diagram*

$$\begin{tikzcd}
F_0(a+c) \arrow[r, "F^{ac}"] \arrow[d, "F_1(\phi+\psi)"] & F_0(a)+F_0(c) \arrow[d, "F_1(\phi)+F_1(\psi)"] \\
F_0(b+d) \arrow[r, "F^{bd}"] & F_0(b)+F_0(d)
\end{tikzcd}$$

commutes.

Proof: It suffices to show that

$$\begin{tikzcd}
F_0(a) \arrow[r, "F_1(i_a)"] \arrow[d, "F_1(\phi)"'] & F_0(a+c) \arrow[d, dashed, "\Xi"] & F_0(c) \arrow[l, "F_1(i_c)"'] \arrow[d, "F_1(\psi)"] \\
F_0(b) \arrow[r, "i_{F_0(b)}"] & F_0(b)+F_0(d) & F_0(d) \arrow[l, "i_{F_0(d)}"']
\end{tikzcd} \tag{9}$$

commutes for $\Xi = (F_1(\phi) + F_1(\psi)) \circ F^{ac}$, as $\Xi = F^{bd} \circ F_1(\phi + \psi)$ is the unique arrow making the diagram commute. But this can be checked directly. We know that

$$\begin{tikzcd}
F_0(a) \arrow[r, "i_{F_0(a)}"] \arrow[d, "F_1(\phi)"'] & F_0(a+c) \arrow[d, dashed, "F_1(\phi)+F_1(\psi)"] & F_0(c) \arrow[l, "i_{F_0(c)}"'] \arrow[d, "F_1(\psi)"] \\
F_0(b) \arrow[r, "i_{F_0(b)}"] & F_0(b)+F_0(d) & F_0(d) \arrow[l, "i_{F_0(d)}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F_0(a) \arrow[r, "F_1(i_a)"] \arrow[dr, "i_{F_0(a)}"'] & F_0(a+c) \arrow[d, dotted, "F^{ac}"] & F_0(c) \arrow[l, "F_1(i_c)"'] \arrow[dl, "i_{F_0(c)}"] \\
 & F_0(a)+F_0(c) &
\end{tikzcd}$$

commute, so we can combine the two diagrams to form a bigger, still commutative diagram

$$\begin{tikzcd}
 & F_0(a+c) \arrow[d, dotted, "F^{ac}"] & \\
F_0(a) \arrow[ur, bend left, "F_1(i_a)"] \arrow[r, "i_{F_0(a)}"] \arrow[d, "F_1(\phi)"'] & F_0(a)+F_0(c) \arrow[d, dashed, "F_1(\phi)+F_1(\psi)"] & F_0(c) \arrow[ul, bend right, "F_1(i_c)"'] \arrow[l, "i_{F_0(c)}"'] \arrow[d, "F_1(\psi)"] \\
F_0(b) \arrow[r, "i_{F_0(b)}"] & F_0(b)+F_0(d) & F_0(d). \arrow[l, "i_{F_0(d)}"']
\end{tikzcd}$$

But the outermost vertices of the diagram with the arrows gives the desired commutative diagram (9) Q.E.D.

Remark 5.17. Just as in the case of product preserving functors, if two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{E}$ are coproduct preserving, then $G \circ F$ is coproduct preserving and we have

$$(G \circ F)^{ab} = G^{F_0(a)F_0(b)} \circ G_1(F^{ab}).$$

Furthermore we have the following lemma.

Lemma 5.18. *Let $\mathscr{C}, \mathscr{D}$ be cocartesian closed categories and $\gamma \in \mathscr{C}_0, \delta \in \mathscr{D}_0$. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a coproduct preserving functor and F^{ab} be the canonical isomorphisms of F for $a, b \in \mathscr{C}_0$ and $\psi\colon \delta \to F_0(\gamma)$ be an arrow. The rule*

$$\begin{aligned} F^*\colon\ & \mathsf{coChu}(\mathscr{C}, \gamma) \to \mathsf{coChu}(\mathscr{D}, \delta), \\ (F^*)_0(a, f, b) &= (F_0(a), F^{ab} \circ F_1(f) \circ \psi, F_0(b)), \\ (F^*)_1(\phi_+, \phi_-) &= (F_1(\phi_+), F_1(\phi_-)) \end{aligned}$$

is a functor.

Proof: We first show the well-definedness. As $F^{ab} \circ F_1(f) \circ \psi\colon \delta \to F_0(a) + F_0(b)$, we have $\big(F_0(a), F^{ab} \circ F_1(f) \circ \psi, F_0(b)\big) \in \mathsf{coChu}(\mathscr{D}, \delta)_0$. Now let an arrow $(\phi_+, \phi_-)\colon (a, f, x) \to (b, g, y)$ be given. We have to check that $(F_1(\phi_+), F_1(\phi_-))$ is well defined, i.e. that

$$\begin{tikzcd} \delta \arrow[r, "F^{ax} \circ F_1(f) \circ \psi"] \arrow[d, "F^{by} \circ F_1(g) \circ \psi"'] & F_0(a) + F_0(x) \arrow[d, "\mathbf{1}_{F_0(a)} + F_1(\phi_-)"] \\ F_0(b) + F_0(y) \arrow[r, "F_1(\phi_+) + F_1(\mathbf{1}_y)"] & F_0(a) + F_0(y) \end{tikzcd}$$

commutes. This can be computed as follows: First we observe that we have a commutative diagram

$$\begin{tikzcd} F_0(\gamma) \arrow[r, "F_1(f)"] \arrow[d, "F_1(g)"'] & F_0(a + x) \arrow[d, "F_1(\mathbf{1}_a + \phi_+)"] \\ F_0(b + y) \arrow[r, "F_1(\phi_+ + \mathbf{1}_y)"'] & F_0(a + y). \end{tikzcd} \tag{10}$$

By lemma 5.16 we have commutative diagrams

$$\begin{tikzcd} F_0(a + x) \arrow[r, "F^{ax}"] \arrow[d, "F_1(\mathbf{1}_a + \phi_-)"] & F_0(a) + F_0(x) \arrow[d, "F_1(\mathbf{1}_a) + F_1(\phi_-)"] \\ F_0(a + y) \arrow[r, "F^{ay}"] & F_0(a) + F_0(y). \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_0(b + y) \arrow[r, "F^{by}"] \arrow[d, "F_1(\phi_+ + \mathbf{1}_y)"] & F_0(b) + F_0(y) \arrow[d, "F_1(\phi_+) + F_1(\mathbf{1}_y)"] \\ F_0(a + y) \arrow[r, "F^{ay}"] & F_0(a) + F_0(y). \end{tikzcd} \tag{11}$$

So we can compute

$$\begin{aligned} \big(\mathbf{1}_{F_0(a)} + F_1(\phi_-)\big) \circ F^{ax} \circ F_1(f) \circ \psi &= \big(F_1(\mathbf{1}_a) + F_1(\phi_-)\big) \circ F^{ax} \circ F_1(f) \circ \psi && (12) \\ &= F^{ay} \circ F_1(\mathbf{1}_a + \phi_-) \circ F_1(f) \circ \psi && (13) \\ &= F^{ay} \circ F_1(\phi_+ + \mathbf{1}_y) \circ F_1(g) \circ \psi && (14) \\ &= \big(F_1(\phi_+) + F_1(\mathbf{1}_y\big) \circ F^{by} \circ F_1(g) \circ \psi, && (15) \end{aligned}$$

where we used the commutativity of the left diagram of (11) in (12) = (13), the commutativity of (10) in (13) = (14) and finally he commutativity of the right diagram of (11) in (14) = (15). It remains to check the axioms of a functor.

- *Compatibility with composition:* Let

$$(\phi_+, \phi_-)\colon (a, f, x) \to (b, g, y) \text{ and } (\psi_+, \psi_-)\colon (b, g, y) \to (c, h, z)$$

be given. We have to check that $(F^*)_1(\phi_+ \circ \psi_+, \psi_- \circ \phi_-) = (F^*)_1(\psi_+, \psi_-) \circ F_1(\phi_+, \phi_-)$. But this can be computed by

$$\begin{aligned}(F^*)_1(\phi_+ \circ \psi_+, \psi_- \circ \phi_-) &= (F_1(\phi_+ \circ \psi_+), F_1(\psi_- \circ \phi_-)) \\ &= \big(F_1(\phi_+) \circ F_1(\psi_+), F_1(\psi_-) \circ F_1(\phi_-)\big) \\ &= \big(F_1(\psi_+), F_1(\psi_-)\big) \circ \big(F_1(\phi_+), F_1(\phi_-)\big) \\ &= F_1(\psi_+, \psi_-) \circ F_1(\phi_+, \phi^-).\end{aligned}$$

- *Preservation of identities:* Let $(a, f, x) \in \mathsf{coChu}(\mathscr{C}, \gamma)$. Then

$$F_1(\mathbf{1}_a, \mathbf{1}_x) = (F_1(\mathbf{1}_a), F_1(\mathbf{1}_x)) = (\mathbf{1}_{F_0(a)}, \mathbf{1}_{F_0(x)})$$

by definition.

So $F^*\colon \mathsf{coChu}(\mathscr{C}, \gamma) \to \mathsf{coChu}(\mathscr{D}, \delta)$ is indeed a functor. Q.E.D.

It remains to find the adequate Grothendieck construction. Unfortunately we can not use the standard embedding $\mathsf{cocCat} \hookrightarrow \mathsf{Cat}$, but we have to use the modified embedding $\iota^{\,\mathrm{op}}\colon \mathsf{cocCat} \hookrightarrow \mathsf{Cat}$, which sends a cocartesian closed category $\mathscr{C}$ to its dual category $\mathscr{C}^{\,\mathrm{op}}$. This motivates the following definition.

Definition 5.19. Let cocCat be the category of cartesian closed categories and $\iota^{\,\mathrm{op}}\colon \mathsf{cocCat} \hookrightarrow \mathsf{Cat}$ be the embedding described above. Then the category $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\,\mathrm{op}})$ it defined by the following data:

- The *objects* of $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\,\mathrm{op}})$ are given by pairs $(\mathscr{C}, \gamma)$, where $\mathscr{C}$ is a cocartesian closed category and
$$\gamma \in [\iota^{\,\mathrm{op}}]_0(\mathscr{C}) = (\mathscr{C}^{\,\mathrm{op}})_0 = \mathscr{C}_0.$$
- The *arrows* of $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\,\mathrm{op}})$ are given by pairs $(F, \theta)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$, where $F\colon \mathscr{C} \to \mathscr{D}$ is a coproduct preserving functor and $\psi\colon \delta \to F_0(\gamma)$ is an arrow in $\mathscr{D}$.
- The *identity* of $(\mathscr{C}, \gamma)$ is given by $(\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma)$.

Proposition 5.20. *The category* $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\,\mathrm{op}})$ *defined above is indeed a category.*

Proof: We have to check well-definedness, associativity and the existence of identities.
Well-definedness: Suppose we are given two arrows in $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\,\mathrm{op}})$, $(F, \theta)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$ and $(G, \omega)\colon (\mathscr{D}, \delta) \to (\mathscr{E}, \epsilon)$. We have to prove that $(G, \omega) \circ (F, \theta) = (G \circ F, G_1(\theta) \circ \omega)$ is an arrow $(\mathscr{C}, \gamma) \to (\mathscr{E}, \epsilon)$. As $G \circ F\colon \mathscr{C} \to \mathscr{E}$ it remains to show that $\theta \circ G_1(\omega)\colon \epsilon \to G_0(F_0(\gamma))$. But as $\omega\colon \epsilon \to G_0(\delta)$ and $G_1(\theta)\colon G_0(\delta) \to G_0(F_0(\gamma))$ this is immediate.
Associativity: Suppose we are given three arrows,

$$(F, \theta)\colon (\mathscr{C}, \gamma) \to (\mathscr{C}, \delta), \qquad (G, \omega)\colon (\mathscr{D}, \delta) \to (\mathscr{E}, \epsilon),$$
$$(H, \nu)\colon (\mathscr{E}, \epsilon) \to (\mathscr{F}, \mu)$$

in $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\,\mathrm{op}})$. We have to show that

$$\big((H, \nu) \circ (G, \omega)\big) \circ (F, \theta) = (H, \nu) \circ \big((G, \omega) \circ (F, \theta)\big).$$

To this end we simply compute

$$\big((H, \nu) \circ (G, \omega)\big) \circ (F, \theta) = \big((H \circ G, H_1(\omega) \circ \nu\big) \circ (F, \theta)$$

$$
\begin{aligned}
&= \big((H \circ G) \circ F, (H \circ G)_1(\theta) \circ H_1(\omega) \circ \nu\big) \\
&= \big(H \circ (G \circ F), H_1(G_1(\theta) \circ \omega) \circ \nu\big) \\
&= (H, \nu) \circ \big(G \circ F, G_1(\theta) \circ \omega\big) \\
&= (H, \nu) \circ \big((G, \omega) \circ (F, \theta)\big).
\end{aligned}
$$

This proves the associativity.
Existence of identities: Suppose we are given an object $(\mathscr{C}, \gamma)$ of $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\mathrm{op}})$. We want to show that $\mathbf{1}_{(\mathscr{C},\gamma)} = (\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma)$. For this reason let two arrows $(F, \theta)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$ and $(E, \nu)\colon (\mathscr{B}, \omega) \to (\mathscr{C}, \gamma)$ be given. We compute

$$
\begin{aligned}
(F, \theta) \circ (\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma) &= (F \circ \mathrm{id}^{\mathscr{C}}, F_1(\mathbf{1}_\gamma) \circ \theta) = (F, \mathbf{1}_{F_0(\gamma)} \circ \theta) \\
&= (F, \theta), \\
(\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma) \circ (E, \nu) &= (\mathrm{id}^{\mathscr{C}} \circ E, \mathrm{id}^{\mathscr{C}}_1(\nu) \circ \mathbf{1}_\gamma) = (E, \nu \circ \mathbf{1}_\gamma) \\
&= (E, \nu),
\end{aligned}
$$

so $(\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma)$ fulfils the properties of the identity. Q.E.D.

With this category at hand we can make the rule $(\mathscr{C}, \gamma) \mapsto \mathsf{coChu}(\mathscr{C}, \gamma)$ a functor.

Theorem 5.21. *We define the rule* $\mathsf{coChu}\colon \mathsf{Groth}(\mathsf{cocCat}, \iota^{\mathrm{op}}) \to \mathsf{Cat}$ *in the following way.*

- *For all* $(\mathscr{C}, \gamma) \in \mathsf{Groth}(\mathsf{cocCat}, \iota^{\mathrm{op}})_0$ *we set* $\mathsf{coChu}_0(\mathscr{C}, \gamma) = \mathsf{coChu}(\mathscr{C}, \gamma)$.
- *For all arrows* $(F, \theta)\colon (\mathscr{C}, \xi) \to (\mathscr{D}, \delta)$ *in* $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\mathrm{op}})$ *we set* $\mathsf{coChu}_1(F, \theta) = F^*$, *where* F^* *is defined in lemma 5.18.*

Proof: To see that coChu is a functor, we need to prove the compatibility with compositions and identities.
Compatibility with composition: Suppose we are given two arrows $(F, \theta)\colon (\mathscr{C}, \gamma) \to (\mathscr{D}, \delta)$ and $(G, \omega)\colon (\mathscr{D}, \delta) \to (\mathscr{E}, \epsilon)$ in $\mathsf{Groth}(\mathsf{cocCat}, \iota^{\mathrm{op}})$. We have to show that

$$
\mathsf{coChu}_1\big((G, \omega) \circ (F, \theta)\big) = (G \circ F)^* = G^* \circ F^* = \mathsf{coChu}_1(G, \omega) \circ \mathsf{coChu}_1(F, \theta). \tag{16}
$$

To prove this it suffices to check that $(G \circ F)^*_0(a, f, x) = (G^* \circ F^*)_0(a, f, x)$ and $(G \circ F)^*_1(\phi_+, \phi_-) = (G^* \circ F^*)_1(\phi_+, \phi_-)$ for all arrows (ϕ_+, ϕ_-) in $\mathsf{Chu}(\mathscr{C}, \gamma)$.

- We begin by considering the objects. We compute that

$$
\begin{aligned}
&(G \circ F)^*_0(a, f, x) \\
&\quad= \big((G \circ F)_0(a), (G \circ F)^{ab} \circ (G \circ F)_1(f) \circ G_1(\theta) \circ \omega, (G \circ F)_0(x)\big) \\
&\quad= \big(G_0(F_0(a)), G^{F_0(a)F_0(b)} \circ G_1(F^{ab}) \circ G_1(F_1(f)) \circ G_1(\theta) \circ \omega, G_0(F_0(x))\big) \\
&\quad= \big(G_0(F_0(a)), G^{F_0(a)F_0(b)} \circ G_1(F^{ab} \circ F_1(f) \circ \theta) \circ \omega, G_0(F_0(x))\big) \\
&\quad= G^*_0\big(F_0(a), F^{ab} \circ F_1(f) \circ \theta, F_0(x)\big) \\
&\quad= G^*_0\big(F^*_0(a, f, x)\big) = (G^* \circ F^*)_0(a, f, x).
\end{aligned}
$$

  This proves the desired equality.
- Next we consider the arrows. Let $(\phi_+, \phi_-)\colon (a, f, x) \to (b, g, y)$ be an arrow in $\mathsf{coChu}(\mathscr{C}, \gamma)$. Then

$$
\begin{aligned}
(G \circ F)^*_1(\phi_+, \phi_-) &= \big((G \circ F)_1(\phi_+), (G \circ F)_1(\phi_-)\big) = \big(G_1(F_1(\phi_+)), G_1(F_1(\phi_-))\big) \\
&= G^*_1\big(F_1(\phi_+), F_1(\phi_-)\big) = (G^* \circ F^*)_1(\phi_+, \phi_-).
\end{aligned}
$$

This proves the equality (16).
Preservation of Identities: Suppose we have $(\mathscr{C}, \gamma) \in \mathsf{Groth}(\mathsf{cocCat}, \iota^{\mathrm{op}})_0$. We then have to check that $\mathsf{coChu}_1(\mathbf{1}_{(\mathscr{C},\gamma)}) = \mathbf{1}_{\mathsf{coChu}(\mathscr{C},\gamma)}$. It suffices to check this equality on objects and arrows. We first observe that

$$\mathsf{coChu}_1(\mathbf{1}_{(\mathscr{C},\gamma)}) = \mathsf{coChu}_1(\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\gamma) = (\mathrm{id}^{\mathscr{C}})^*.$$

- Suppose we have $(a, f, x) \in \mathsf{coChu}(\mathscr{C}, \gamma)_0$. We then compute

$$\begin{aligned}(\mathrm{id}^{\mathscr{C}})^*_0(a, f, x) &= \big(\mathrm{id}^{\mathscr{C}}_0(a), (\mathrm{id}^{\mathscr{C}})^{ax} \circ \mathrm{id}^{\mathscr{C}}_1(f) \circ \mathbf{1}_\gamma, \mathrm{id}^{\mathscr{C}}_0(x)\big)\\ &= (a, (\mathrm{id}^{\mathscr{C}})^{ax} \circ f, x).\end{aligned}$$

  But we can observe that $(\mathrm{id}^{\mathscr{C}})^{ax} = \mathbf{1}_{a+x}$, as $\mathbf{1}_{a\times x}$ makes the diagram

$$\begin{tikzcd}
\mathrm{id}^{\mathscr{C}}_0(a) \arrow[r, "\mathrm{id}^{\mathscr{C}}_1(i_a)"'] \arrow[d, equal] & \mathrm{id}^{\mathscr{C}}_0(a+x) \arrow[d, equal] & \mathrm{id}^{\mathscr{C}}_0(x) \arrow[l, "\mathrm{id}^{\mathscr{C}}_1(i_x)"] \arrow[d, equal] \\
a \arrow[r, "i_a"] \arrow[dr, bend right, "i_a"'] & a+x \arrow[d, dotted, "\mathbf{1}_{a+x}"] & x \arrow[l, "i_x"'] \arrow[dl, bend left, "i_x"] \\
 & a+x &
\end{tikzcd}$$

  commute. So we have $(\mathrm{id}^{\mathscr{C}})^{ax} \circ f = f$, which finalizes the proof of $(\mathrm{id}^{\mathscr{C}})^*_0(a, f, x) = (a, f, x)$.
- To see the equality on arrows, suppose we have an arrow $(\phi_+, \phi_-)\colon (a, f, x) \to (b, g, y)$ in $\mathsf{coChu}(\mathscr{C}, \gamma)$. We then compute

$$(\mathrm{id}^{\mathscr{C}})^*_1(\phi_+, \phi_-) = \big(\mathrm{id}^{\mathscr{C}}_1(\phi_+), \mathrm{id}^{\mathscr{C}}_1(\phi_-)\big) = (\phi_+, \phi_-).$$

So we have proven that $\mathsf{coChu}_1(\mathbf{1}_{(\mathscr{C},\gamma)}) = \mathbf{1}_{\mathsf{coChu}(\mathscr{C},\gamma)}$. Q.E.D.

5.6 Products of Chu categories

We want to relate the category $\mathsf{Chu}(\mathscr{C}, \gamma) \times \mathsf{Chu}(\mathscr{D}, \delta)$ to the category $\mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma, \delta))$. The objects of the Chu category $\mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma, \delta))$ are triplets $((c, d), f, (x, y))$ where

$$(c, d), (x, y) \in (\mathscr{C} \times \mathscr{D})_0 \quad \text{and} \quad f\colon (c, d) \times (x, y) \to (\gamma, \delta).$$

The objects in the category $\mathsf{Chu}(\mathscr{C}, \gamma) \times \mathsf{Chu}(\mathscr{D}, \delta)$ on the other hand are given as pairs $\big((a, f, x), (b, g, y)\big)$, where $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ and $(b, g, y) \in \mathsf{Chu}(\mathscr{D}, \delta)_0$. The first step towards a isomorphism between the two categories would be a functor between the two. We establish such a functor in the following theorem.

Theorem 5.22. *There exists a functor*

$$F\colon \mathsf{Chu}\big(\mathscr{C} \times \mathscr{D}, (\gamma, \delta)\big) \to \mathsf{Chu}(\mathscr{C}, \gamma) \times \mathsf{Chu}(\mathscr{D}, \delta).$$

Proof: We divide this task into three steps:
- *Step 1:* Mapping the objects.
- *Step 2:* Mapping the arrows.
- *Step 3:* Checking the functoriality.

Regarding step 1: Suppose we are given $\big((a, b), f, (x, y)\big) \in \mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma, \delta))_0$. This means that f is an arrow

$$f\colon (a, b) \times (x, y) \to (\gamma, \delta).$$

We wish to extrapolate two arrows $f_1\colon a \times x \to \gamma$ and $f_2\colon b \times y \to \delta$. For this we remember the definition of the product category.

Definition 5.23 (Product categories). Let $\mathscr{C}, \mathscr{D}$ be categories. Then the *product category* is defined as such:

- Its *objects* are pairs (c, d) where $c \in \mathscr{C}_0$ and $d \in \mathscr{D}_0$.
- Its *arrows* are pairs $(f_1, f_2)\colon (a, b) \to (c, d)$, where $f_1\colon a \to c$ and $f_2\colon b \to d$.

To make this useful in our context, we have to relate the product $(a, b) \times (x, y)$ to $(a \times x, b \times y)$. For this we make the following observation.

Lemma 5.24 (Product categories of cartesian closed categories). *Let $\mathscr{C}, \mathscr{D}$ be cartesian closed categories. Consider the product category $\mathscr{C} \times \mathscr{D}$, and elements $(c, d), (c', d') \in (\mathscr{C} \times \mathscr{D})_0$. Then $(c, d) \times (c', d') \cong (c \times c', d \times d')$.*

Proof: For this we first visualize that the product category $\mathscr{C} \times \mathscr{D}$ is actually cartesian closed. Suppose we are given $(c, d), (c', d') \in (\mathscr{C} \times \mathscr{D})_0$. To define the product we need an object $P = (p_1, p_2) \in (\mathscr{C} \times \mathscr{D})_0$ and projections $\mathtt{pr}_1\colon P \to (c, d), \mathtt{pr}_2\colon P \to (c', d')$ such that for ever object $(e, f) \in (\mathscr{C} \times \mathscr{D})_0$ with morphisms $q = (q_1, q_2)\colon (e, f) \to (c, d), t = (t_1, t_2)\colon (e, f) \to (c', d')$ there exists an unique arrow $u = (u_1, u_2)\colon (e, f) \to (p_1, p_2)$ such that

$$\begin{tikzcd}
 & (e,f) \arrow[dl, bend right, "{(q_1,q_2)}"'] \arrow[dr, bend left, "{(t_1,t_2)}"] \arrow[d, dashed, "{(u_1,u_2)}"] & \\
(c,d) & (p_1,p_2) \arrow[l, "\mathtt{pr}_1"] \arrow[r, "pr_2"'] & (c',d')
\end{tikzcd}$$

commutes. We show that we can use $(p_1, p_2) = (c \times c', d \times d')$ and

$$\mathtt{pr}_1 = (\mathtt{pr}_1^{\mathscr{C}}, \mathtt{pr}_1^{\mathscr{D}}), \qquad \mathtt{pr}_2 = (\mathtt{pr}_2^{\mathscr{C}}, \mathtt{pr}_2^{\mathscr{D}}),$$

where $\mathtt{pr}_1^{\mathscr{C}}\colon c \times c' \to c, \mathtt{pr}_2^{\mathscr{C}}\colon c \times c' \to c'$ are given through the product in $\mathscr{C}$ and $\mathtt{pr}_1^{\mathscr{D}}\colon d \times d' \to d, \mathtt{pr}_2^{\mathscr{D}}\colon d \times d' \to d'$ are given through the product in $\mathscr{D}$. One can see that this satisfies the required conditions as it makes

$$\begin{tikzcd}
e \arrow[dr, "t_1"] \arrow[dd, "q_1"'] \arrow[ddr, dotted] & & (e,f) \arrow[ll, "\pi_{\mathscr{C}}"'] \arrow[rr, "\pi_{\mathscr{D}}"] \arrow[dr, "{(t_1,t_2)}"] \arrow[dd, "{(q_1,q_2)}"'] \arrow[ddr, dotted] & & f \arrow[dr, "t_2"] \arrow[dd, "q_2"'] \arrow[ddr, dotted] & \\
 & c' & & (c',d') \arrow[ll, "\pi_{\mathscr{C}}"'] \arrow[rr, "\pi_{\mathscr{D}}"] & & d' \\
c & & (c,d) \arrow[ll, "\pi_{\mathscr{C}}"'] \arrow[rr, "\pi_{\mathscr{D}}"] & & d & \\
 & c \times c' \arrow[ul, "\mathtt{pr}_1^{\mathscr{C}}"] \arrow[uu, "\mathtt{pr}_2^{\mathscr{C}}"] & & (c \times c', d \times d') \arrow[ll, "\pi_{\mathscr{C}}"'] \arrow[rr, "\pi_{\mathscr{D}}"] \arrow[ul, "\mathtt{pr}_1"] \arrow[uu, "\mathtt{pr}_2"] & & d \times d' \arrow[ul, "\mathtt{pr}_1^{\mathscr{D}}"] \arrow[uu, "pr_2^{\mathscr{D}}"']
\end{tikzcd}$$

commute. So as $(c \times c', d \times d')$ satisfies all the requirements of a product, we obtain a canonical isomorphism $(c, d) \times (c', d') \cong (c \times c', d \times d')$. Q.E.D.

So we now have our desired map $F_1\colon \big((a, b), f, (x, y)\big) \to \big((a, f_1, x), (b, f_2, y)\big)$, as we have $(a, b) \times (x, y) = (a \times x, b \times y)$, so the arrow f is by definition a pair (f_1, f_2) of arrows $f_1\colon a \times x \to \gamma, f_2\colon b \times y \to \delta$. So we can proceed to the next step.

Regarding step 2: We are given $\big((a, b), f, (x, y)\big), \big((a', b'), f', (x', y')\big) \in \mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma, \delta))_0$ and an arrow $\phi\colon \big((a, b), f, (x, y)\big) \to \big((a', b'), f', (x', y')\big)$. We want to map this arrow to an arrow

$$(\Phi_1, \Phi_2)\colon \big((a, f_1, x), (b, f_2, y)\big) \to \big((a', f_1', x'), (b', f_2', y')\big).$$

We start by dissecting ϕ. This arrow is actually a tuple (ϕ^+, ϕ^-) of arrows

$$\phi^+ \colon (a,b) \to (a',b'), \qquad \phi^- \colon (x',y') \to (x,y),$$

making the diagram

$$\begin{tikzcd} (a,b)\times(x',y') \arrow[r, "\mathbf{1}_{(a,b)}\times\phi^-"] \arrow[d, "\phi^+\times\mathbf{1}_{(x',y')}"] & (a,b)\times(x,y) \arrow[d, "f"] \\ (a',b')\times(x',y') \arrow[r, "f'"] & (\gamma,\delta) \end{tikzcd} \tag{17}$$

commute. We can further dissect ϕ^+, ϕ^- into

$$\phi_1^+ \colon a \to a', \qquad \phi_2^+ \colon b \to b',$$
$$\phi_1^- \colon x' \to x, \qquad \phi_2^- \colon y' \to y.$$

This allows us to define $\Phi_1 := (\phi_1^+, \phi_1^-), \Phi_2 := (\phi_2^+, \phi_2^-)$. We have to check the commutativity of the diagrams

$$\begin{tikzcd} a\times x' \arrow[r, "\mathbf{1}_a\times\phi_1^-"] \arrow[d, "\phi_1^+\times\mathbf{1}_{x'}"'] & a\times x \arrow[d, "f_1"] \\ a'\times x' \arrow[r, "f_1'"] & \gamma \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} b\times y' \arrow[r, "\mathbf{1}_b\times\phi_2^+"] \arrow[d, "\phi_2^-\times\mathbf{1}_{y'}"'] & b\times y \arrow[d, "f_2"] \\ b'\times y' \arrow[r, "f_2'"] & \delta. \end{tikzcd}$$

This means to check that $f_1 \circ (\mathbf{1}_a \times \phi_1^-) = f_1' \circ (\phi_1^+ \times \mathbf{1}_{x'})$ and $f_2 \circ (\mathbf{1}_b \times \phi_2^+) = f_2' \circ (\phi_2^- \times \mathbf{1}_{y'})$. But these two conditions are equivalent to the commutativity of the diagram (17), as we shall see. The commutativity of the diagram (17) means that

$$f \circ (\mathbf{1}_{(a,b)} \times \phi^-) = f' \circ (\phi^+ \times \mathbf{1}_{(x',y')}). \tag{18}$$

But as we have discussed before, $f = (f_1, f_2)$ so this equation can equivalently be formulated as

$$(f_1, f_2) \circ (\mathbf{1}_{(a,b)} \times \phi^-) = (f_1', f_2') \circ (\phi^+ \times \mathbf{1}_{(x',y')}).$$

Our next goal shall be to show that

$$(\mathbf{1}_{(a,b)} \times \phi^-) = (\mathbf{1}_a \times \phi_1^-, \mathbf{1}_b \times \phi_2^-) \text{ and } (\phi^+ \times \mathbf{1}_{(x',y')}) = (\phi_1^+ \times \mathbf{1}_{x'}, \phi_2^+ \times \mathbf{1}_{y'}). \tag{19}$$

We do this in a more general context.

Lemma 5.25. *Let $\mathscr{C}, \mathscr{D}$ be cartesian closed categories. Let $(a,b), (c,d), (a',b'), (c',d') \in (\mathscr{C} \times \mathscr{D})_0$ and $\phi = (\phi_1, \phi_2) \colon (a,b) \to (a',b'), \psi = (\psi_1, \psi_2) \colon (c,d) \to (c',d')$ be arrows. Then*

$$\phi \times \psi = (\phi_1, \phi_2) \times (\psi_1, \psi_2) = (\phi_1 \times \psi_1, \phi_2 \times \psi_2).$$

In a sense this lemma can be seen as an extension of lemma 5.24 to arrows. Furthermore the equalities only hold up to unique isomorphisms, but as the product of objects is only chosen up to unique isomorphisms, this distinction will be dropped to not artificially inflate the notation and complexity.

Proof: Suppose we are given arrows and objects as in the lemma. By the uniqueness of the arrow $\phi \times \psi$ making

$$\begin{tikzcd} (a,b) \arrow[d, "\phi"] & (a,b)\times(c,d) \arrow[l, "\mathtt{pr}_1"'] \arrow[r, "\mathtt{pr}_2"] \arrow[d, dotted, "\phi\times\psi"] & (c,d) \arrow[d, "\psi"] \\ (a',b') & (a',b')\times(c',d') \arrow[l, "\mathtt{pr}_1'"'] \arrow[r, "\mathtt{pr}_2'"] & (c',d') \end{tikzcd}$$

commute, it suffices to prove that $(\phi_1 \times \psi_1, \phi_2 \times \psi_2)$ also makes the diagram commute. From lemma 5.24 we obtain canonical isomorphisms making the diagram

$$\begin{tikzcd}
 & (a \times c, b \times d) \arrow[dl, "\mathtt{pr}_1"'] \arrow[dr, "\mathtt{pr}_2"] \arrow[d, phantom, "\cong"] & \\
(a,b) \arrow[d, "\phi"] & (a,b) \times (c,d) \arrow[l, "\mathtt{pr}_1"] \arrow[r, "\mathtt{pr}_2"'] \arrow[d, dotted, "\phi\times\psi"] & (c,d) \arrow[d, "\psi"] \\
(a',b') & (a',b') \times (c',d') \arrow[l, "\mathtt{pr}'_1"'] \arrow[r, "\mathtt{pr}'_2"] & (c',d') \\
 & (a' \times c', b' \times d') \arrow[ul, "\mathtt{pr}_1"] \arrow[ur, "\mathtt{pr}_2"'] \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

commute. By pre- and postcomposing with these isomorphisms $(\phi_1 \times \psi_1, \phi_2 \times \psi_2)$ also makes this diagrams commute, but as we remarked before, we drop these isomorphisms for convenience, so

$$(\phi_1 \times \psi_1, \phi_2 \times \psi_2) = \phi \times \psi.$$

Q.E.D.

This lemma gives the desired equalities of (19), hence we can compute

$$\begin{aligned}
(f_1, f_2) \circ (\mathbf{1}_{(a,b)} \times \phi^-) &= (f_1, f_2) \circ (\mathbf{1}_a \times \phi_1^-, \mathbf{1}_b \times \phi_2^-) \\
&= (f_1 \circ (\mathbf{1}_a \times \phi_1^-), f_2 \circ (\mathbf{1}_b \times \phi_2^-)), \\
(f'_1, f'_2) \circ (\phi^+ \times \mathbf{1}_{(x',y')}) &= (f'_1, f'_2) \circ (\phi_1^+ \times \mathbf{1}_{x'}, \phi_2^+ \times \mathbf{1}_{y'}) \\
&= (f'_1 \circ (\phi_1^+ \times \mathbf{1}_{x'}), f_2 \circ (\phi_2^+ \times \mathbf{1}_{y'})).
\end{aligned}$$

This means that the equality (18) can be reformulated as

$$(f_1 \circ (\mathbf{1}_a \times \phi_1^-), f_2 \circ (\mathbf{1}_b \times \phi_2^-)) = (f'_1 \circ (\phi_1^+ \times \mathbf{1}_{x'}), f'_2 \circ (\phi_2^+ \times \mathbf{1}_{y'})).$$

This means nothing else than

$$\begin{aligned}
f_1 \circ (\mathbf{1}_a \times \phi_1^-) &= f'_1 \circ (\phi_1^+ \times \mathbf{1}_{x'}), \\
f_2 \circ (\mathbf{1}_b \times \phi_2^-) &= f_2 \circ (\phi_2^+ \times \mathbf{1}_{y'}),
\end{aligned}$$

our desired two equalities. Before we move on to the last step, we summarize what we have already collected about our functor. Suppose we are given

$$((a,b), (f_1, f_2), (x,y)), ((a',b'), (f'_1, f'_2), (x',y')) \in \mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma, \delta))_0$$

and an arrow

$$\phi = (\phi^+, \phi^-) = ((\phi_1^+, \phi_2^+), (\phi_1^-, \phi_1^-))\colon ((a,b), (f_1, f_2), (x,y)) \to ((a',b'), (f'_1, f'_2), (x',y')).$$

Then

$$\begin{aligned}
F_0((a,b), (f_1, f_2), (x,y)) &= ((a, f_1, x), (b, f_2, y)), \\
F_1(\phi) &= ((\phi_1^+, \phi_1^-), (\phi_2^+, \phi_2^-)).
\end{aligned}$$

Regarding Step 3: We now check the functoriality of the map. For this we check the following:

- *Preservation of identities:* We now check that

$$F_0(\mathbf{1}_{((a,b),(f_1,f_2),(x,y))}) = \mathbf{1}_{F_0((a,b),(f_1,f_2),(x,y))}.$$

We first observe that

$$\mathbf{1}_{((a,b),(f_1,f_2),(x,y))} = (\mathbf{1}_{(a,b)}, \mathbf{1}_{(x,y)}) = ((\mathbf{1}_a, \mathbf{1}_b), (\mathbf{1}_x, \mathbf{1}_y)).$$

By the laws defined above we then have

$$F_0(\mathbf{1}_{((a,b),(f_1,f_2),(x,y))}) = ((\mathbf{1}_a, \mathbf{1}_x), (\mathbf{1}_b, \mathbf{1}_y)). \tag{20}$$

On the other hand we have $F_0((a,b),(f_1,f_2),(x,y)) = ((a,f_1,x),(b,f_2,y))$, so

$$\begin{aligned} \mathbf{1}_{F_0((a,b),(f_1,f_2),(x,y))} &= \mathbf{1}_{((a,f_1,x),(b,f_2,y))} \\ &= (\mathbf{1}_{(a,f_1,x)}, \mathbf{1}_{(b,f_2,y)}) \\ &= ((\mathbf{1}_a, \mathbf{1}_x), (\mathbf{1}_b, \mathbf{1}_y)), \end{aligned}$$

which gives us the desired equality.
- *Compatibility with composition:* Suppose we are given three objects

$$((a,b),(f_1,f_2),(x,y)), \qquad ((a',b'),(f_1',f_2'),(x',y')), \qquad ((a'',b''),(f_1'',f_2''),(x'',y''))$$

in $\mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma,\delta))_0$ and two arrows

$$\begin{aligned} \phi &= ((\phi_1^+, \phi_2^+), (\phi_1^-, \phi_2^-))\colon ((a,b),(f_1,f_2),(x,y)) \to ((a',b'),(f_1',f_2'),(x',y')), \\ \psi &= ((\psi_1^+, \psi_2^+), (\psi_1^-, \psi_2^-))\colon ((a',b'),(f_1',f_2'),(x',y')) \to ((a'',b''),(f_1'',f_2''),(x'',y'')). \end{aligned}$$

Then we compute

$$\begin{aligned} F_1(\psi \circ \phi) &= F_1\big(((\psi_1^+, \psi_2^+), (\psi_1^-, \psi_2^-)) \circ ((\phi_1^+, \phi_2^+), (\phi_1^-, \phi_2^-))\big) \\ &= F_1\big(((\psi_1^+ \circ \phi_1^+, \psi_2^+ \circ \phi_2^+), (\phi_1^- \circ \psi_1^-, \phi_2^- \circ \psi_2^-))\big) \\ &= \big((\psi_1^+ \circ \phi_1^+, \phi_1^- \circ \psi_1^-), (\psi_2^+ \circ \phi_2^+, \phi_2^- \circ \psi_2^-)\big) \\ &= \big((\psi_1^+, \psi_1^-), (\psi_2^+, \psi_2^-)\big) \circ \big((\phi_1^+, \phi_1^-), (\phi_2^+, \phi_2^-)\big) \\ &= F_1(\psi) \circ F_1(\phi). \end{aligned}$$

This finishes the proof that F is a functor. Q.E.D.

Our next goal is to define an inverse functor. For this we first define our functor G in an "inverse" way, so if we are given $((a,f,x),(b,g,y)), ((a',f',x'),(b',g',y')) \in (\mathsf{Chu}(\mathscr{C},\gamma) \times \mathsf{Chu}(\mathscr{D},\delta))_0$ and

$$\phi = \big((\phi_1^+, \phi_1^-), (\phi_2^+, \phi_2^-)\big)\colon ((a,f,x),(b,g,y)) \to ((a',f',x'),(b',g',y')),$$

then we set

$$\begin{aligned} G_0\Big(\big((a,f,x),(b,g,y)\big)\Big) &= \big((a,b),(f,g),(x,y)\big), \\ G_1(\phi) &= \big((\phi_1^+, \phi_2^+), (\phi_1^-, \phi_2^-)\big). \end{aligned}$$

Theorem 5.26. *The rule G defined above is a functor*

$$G\colon \mathsf{Chu}(\mathscr{C},\gamma) \times \mathsf{Chu}(\mathscr{D},\delta) \to \mathsf{Chu}(\mathscr{C} \times \mathscr{D}, (\gamma,\delta))$$

and is the inverse of F.

Proof: By the lemmata shown above one can see that this is well-defined. We check the axioms of a functor.

- *Preservation of identities:* We have $\mathbf{1}_{((a,f,x),(b,g,y))} = (\mathbf{1}_a, \mathbf{1}_x), (\mathbf{1}_b, \mathbf{1}_y))$, and

$$\begin{aligned} G_1\Big(((\mathbf{1}_a, \mathbf{1}_x), (\mathbf{1}_b, \mathbf{1}_y))\Big) &= ((\mathbf{1}_a, \mathbf{1}_b), (\mathbf{1}_x, \mathbf{1}_y)) \\ &= \mathbf{1}_{((a,b),(f,g),(x,y))} = \mathbf{1}_{G_0(((a,f,x),(b,g,y)))}. \end{aligned}$$

- *Compatibility with composition:* Suppose we are given arrows

$$\begin{aligned} \Phi &= ((\Phi_1^+, \Phi_1^-), (\Phi_2^+, \Phi_2^-)) \colon \big((a,f,x),(b,g,y)\big) \to \big((a',f',x'),(b',g',y')\big), \\ \Psi &= ((\Psi_1^+, \Psi_1^-), (\Psi_2^+, \Psi_2^-)) \colon \big((a',f',x'),(b',g',y')\big) \to \big((a'',f'',x''),(b'',g'',y'')\big). \end{aligned}$$

Then we can compute

$$\begin{aligned} G_1(\Psi \circ \Phi) =& G_1\big((\Psi_1^+ \circ \Phi_1^+, \Phi_1^- \circ \Psi_1^-), (\Psi_2^+ \circ \Phi_2^+, \Phi_2^- \circ \Psi_2^-)\big) \\ &= \big((\Psi_1^+ \circ \Phi_1^+, \Psi_2^+ \circ \Phi_2^+), (\Phi_1^- \circ \Psi_1^-, \Phi_2^- \circ \Psi_1^-)\big) \\ &= \big((\Psi_1^+, \Psi_2^+), (\Psi_1^-, \Psi_2^-)\big) \circ \big((\Phi_1^+, \Phi_2^+), (\Phi_1^-, \Phi_2^-)\big) \\ &= G_1(\Psi) \circ G_1(\Phi). \end{aligned}$$

It remains to show that $G \circ F = \mathrm{id}_{\mathsf{Chu}(\mathscr{C}\times\mathscr{D},(\gamma,\delta))}$ and $F \circ G = \mathrm{id}_{\mathsf{Chu}(\mathscr{C},\gamma)} \times \mathrm{id}_{\mathsf{Chu}(\mathscr{D},\delta)}$. But this is immediate, as for any $\big((a,b),(f,g),(x,y)\big) \in \mathsf{Chu}(\mathscr{C}\times\mathscr{D},(\gamma,\delta))_0$ and

$$\phi = ((\phi_1^+, \phi_2^+), (\phi_1^-, \phi_2^-)) \in \mathsf{Chu}(\mathscr{C}\times\mathscr{D},(\gamma,\delta))$$

one can verify

$$\begin{aligned} (G\circ F)_0\big((a,b),(f,g),(x,y)\big) &= G_0((a,f,x),(b,g,y)) = \big((a,b),(f,g),(x,y)\big), \\ (G\circ F)_1\big(\phi\big) &= G_1\big((\phi_1^+, \phi_1^-), (\phi_2^+, \phi_2^-)\big) = \big((\phi_1^+, \phi_2^+), (\phi_1^-, \phi_2^-)\big). \end{aligned}$$

Analogously, given $\big((a,f,x),(b,g,y)\big) \in (\mathsf{Chu}(\mathscr{C},\gamma)\times\mathsf{Chu}(\mathscr{D},\delta))_0$ and

$$\psi = \big((\psi_1^+, \psi_1^-), (\psi_2^+, \psi_2^-)\big) \in (\mathsf{Chu}(\mathscr{C},\gamma)\times\mathsf{Chu}(\mathscr{D},\delta))_1$$

one can immediately verify

$$\begin{aligned} (F\circ G)_0\big((a,f,x),(b,g,y)\big) &= F_1\big((a,b),(f,g),(x,y)\big) = \big((a,f,x),(b,g,y)\big), \\ (F\circ G)_1(\psi) &= F_1\big((\psi_1^+, \psi_2^+), (\psi_1^-, \psi_2^-)\big) = \big((\psi_1^+, \psi_1^-), (\psi_2^+, \psi_2^-)\big). \end{aligned}$$

This shows that F, G are inverse functors. Q.E.D.

We shall sum up our results in a theorem.

Theorem 5.27. *Let $\mathscr{C}, \mathscr{D}$ be cartesian closed categories with $\gamma \in \mathscr{C}_0, \delta \in \mathscr{D}_0$. Then the category $\mathsf{Chu}(\mathscr{C},\gamma) \times \mathsf{Chu}(\mathscr{C},\delta)$ consists of the following data:*

- *The objects are given by triplets $((a,a'),(f,f),(b,b'))$ where $(a,a'),(b\times b') \in (\mathscr{C}\times\mathscr{D})_0$ and $(f,f') \colon (a\times b, a'\times b') \to (\gamma,\delta)$.*
- *The arrows are given by $\big((\phi^+,\psi^+),(\phi^-,\psi^-)\big) \colon \big((a,a'),(f,f'),(b,b')\big) \to \big((c,c'),(g,g'),(d,d')\big)$ where*

$$\begin{aligned} &\phi^+ \colon a \to c, && \phi^- \colon d \to b, \\ &\psi^+ \colon a' \to c' && \psi^- \colon d' \to b'. \end{aligned}$$

Chapter 6

The Chu construction and limits

6.1 Chu and products

We want to compute the product in $\mathsf{Chu}(\mathscr{C}, \gamma)$ of a cartesian closed category $\mathscr{C}$ and an object $\gamma \in \mathscr{C}_0$. Unfortunately we need that $\mathscr{C}$ has binary coproducts if we want to construct the product in the Chu category.

Lemma 6.1. *Let $\mathscr{C}$ be a cartesian closed category with binary coproducts and $\gamma \in \mathscr{C}_0$. Then the category $\mathsf{Chu}(\mathscr{C}, \gamma)$ has products and the product of two objects $(a, f, x), (b, g, y)$ in $\mathsf{Chu}(\mathscr{C}, \gamma)$ is given by*

$$(a, f, x) \times (b, g, y) = (a \times b, F, x + y)$$

for a to be determined arrow $F\colon (a \times b) \times (x + y) \to \gamma$.

Before we prove this lemma we find an equivalent expression for $a \times b \times (x + y)$.

Lemma 6.2. *Let $\mathscr{C}$ be a cartesian closed category with all finite coproducts and $a, b, c, d \in \mathscr{C}_0$. Then*

$$a \times (b + c) \cong (a \times b) + (a \times c).$$

Proof: It suffices to show that $a \times (b + c)$ satisfies the universal property of the coproduct $(a \times b) + (a \times c)$. So we first have to find arrows $i_{a\times b}\colon a \times b \to a \times (b + c)$ and $i_{a\times c}\colon a \times c \to a \times (b + c)$. We define $i_{a\times b} := \mathbf{1}_a \times i_b$ and $i_{a\times c} := \mathbf{1}_a \times i_c$. It remains to show that for every object $O \in \mathscr{C}_0$ with arrows $l_1\colon a \times b \to O$ and $l_2\colon a \times c \to O$ we find an arrow $l\colon a \times (b+c) \to O$ making the diagram

$$\begin{tikzcd}
 & O & \\
a \times b \arrow[ur, bend left, "l_1"] \arrow[r, "i_{a\times b}"'] & a \times (b + c) \arrow[u, dashed, "l"] & a \times c \arrow[l, "i_{a\times c}"] \arrow[ul, bend right, "l_2"']
\end{tikzcd}$$

commute. We can transpose the arrows l_1, l_2 to obtain arrows $\widehat{l_1}\colon b \to O^a, \widehat{l_2}\colon b \to O^a$. By the universal property of the coproduct $b + c$ we obtain a unique arrow $\widehat{l_1} + \widehat{l_2}$ making the diagram

$$\begin{tikzcd}
 & O^a & \\
b \arrow[ur, bend left, "\widehat{l_1}"] \arrow[r, "i_b"'] & b + c \arrow[u, dashed, "\widehat{l_1}+\widehat{l_2}"] & c \arrow[l, "i_c"] \arrow[ul, bend right, "\widehat{l_2}"']
\end{tikzcd}$$

commute. Hence we obtain an arrow $l := \mathtt{eval}_{O,a} \circ \big((\widehat{l_1} + \widehat{l_2}) \times \mathbf{1}_a\big)\colon a \times (b + c) \to O$. This is the desired arrow making the first diagram commute. Q.E.D.

Proof of lemma 6.1: So let $\mathscr{C}$ be a cartesian closed category with coproducts and $\gamma \in \mathscr{C}_0$. Let $(a, f, x), (b, g, y) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ be given. We first want to define an arrow $a\times b\times(x+y) \to \gamma$. By the preceding lemma we know that $a \times b \times (x + y) \cong (a \times b \times x) + (a \times b \times y)$, so it suffices to find two arrows $\rho_1\colon (a \times b \times x) \to \gamma$ and $\rho_2\colon (a \times b \times y) \to \gamma$ to obtain an arrow $F\colon (a \times b \times x) + (a \times b \times y)$. As the product is associative, we have $a \times b \times x \cong (a \times x) \times b$, therefore we obtain an arrow $\mathtt{pr}_{a\times x}\colon a \times b \times x \to a \times x$. So we define $\rho_1 := f \circ \mathtt{pr}_{a\times x}$. Analogously we have $a \times b \times y \cong (b \times y) \times a$, and we obtain $\mathtt{pr}_{b\times y}\colon a \times b \times y \to b \times y$, hence we can set $\rho_2 := g \circ \mathtt{pr}_{b\times y}$. Therefore we obtain the desired arrow $F\colon a \times b \times (x + y) \to \gamma$. So $(a \times b, F, x + y) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$.

It remains to show that $(a \times b, F, x+y)$ is a product of (a, f, x) and (b, g, y). First we have to define the projections. We set

$$\begin{aligned}
\mathtt{pr}_{(a,f,x)} &= (\mathtt{pr}^+_{(a,f,x)}, \mathtt{pr}^-_{(a,f,x)}) := (\mathtt{pr}_a, i_x)\colon (a \times b, F, x + y) \to (a, f, x),\\
\mathtt{pr}_{(b,g,y)} &= (\mathtt{pr}^+_{(b,g,y)}, \mathtt{pr}^-_{(b,g,y)}) := (\mathtt{pr}_b, i_y)\colon (a \times b, F, x + y) \to (b, g, y),
\end{aligned}$$

where the projections $\mathtt{pr}_a, \mathtt{pr}_b$ and the inclusions i_x, i_y on the right hand side of the definition stem from the given product and coproduct. Next we prove the universal property of the product. For this let an object $(u, q, w) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ together with arrows

$$(\phi^+, \phi^-)\colon (u, q, w)\colon (a, f, x), \qquad\qquad (\psi^+, \psi^-)\colon (u, q, w) \to (b, g, y)$$

be given. We want to find a unique arrow $(\Phi^+, \Phi^-)\colon (u, q, w) \to (a \times x, F, x + y)$ making the diagram

$$\begin{tikzcd}
 & (u,q,w) \arrow[dl, bend right, "(\phi^+{,}\phi^-)"'] \arrow[d, dashed, "(\Phi^+{,}\Phi^-)"] \arrow[dr, bend left, "(\psi^+{,}\psi^-)"] & \\
(a,f,x) & (a \times b, F, x+y) \arrow[l, "\mathtt{pr}_{(a,f,x)}"] \arrow[r, "\mathtt{pr}_{(b,g,y)}"'] & (b,g,y)
\end{tikzcd} \tag{21}$$

commute. But as we have $\phi^+\colon u \to a, \psi^+\colon u \to b$ and $\phi^-\colon x \to w, \psi^-\colon y \to w$ we obtain arrows $\Phi^+\colon u \to a \times b, \Phi^-\colon x + y \to w$ making the diagrams

$$\begin{tikzcd}
 & u \arrow[dl, bend right, "\phi^+"'] \arrow[d, dashed, "\Phi^+"] \arrow[dr, bend left, "\psi^+"] & \\
a & a \times b \arrow[l, "\mathtt{pr}_a"] \arrow[r, "\mathtt{pr}_b"'] & b
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & w & \\
x \arrow[ur, bend left, "\phi^-"] \arrow[r, "i_x"'] & x + y \arrow[u, dashed, "\Phi^-"] & y \arrow[l, "i_y"] \arrow[ul, bend right, "\psi^-"']
\end{tikzcd}$$

commute. So by definition (Φ^+, Φ^-) makes the diagram (21) commute. It remains to check that (Φ^+, Φ^-) is a Chu morphism, i.e. the diagram

$$\begin{tikzcd}
u \times (x+y) \arrow[r, "\Phi^+ \times \mathbf{1}_{x+y}"] \arrow[d, "\mathbf{1}_u \times \Phi^-"] & (a \times b) \times (x+y) \arrow[d, "F"] \\
u \times w \arrow[r, "q"'] & \gamma
\end{tikzcd}$$

commutes. As $u \times (x + y) \cong (u \times x) + (u \times y)$, we have a coproduct structure on this object, therefore it suffices to check the commutativity on the summands of the coproduct, so we only have to check the commutativity of the diagrams

$$\begin{tikzcd}
u \times x \arrow[r, "\Phi^+ \times \mathbf{1}_x"] \arrow[d, "\mathbf{1}_u \times \phi^-"'] & a \times b \times x \arrow[d, "\rho_1"] \\
u \times w \arrow[r, "q"] & \gamma
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
u \times y \arrow[r, "\Phi^+ \times \mathbf{1}_y"] \arrow[d, "\mathbf{1}_u \times \psi^-"'] & a \times b \times y \arrow[d, "\rho_2"] \\
u \times w \arrow[r, "q"] & \gamma.
\end{tikzcd}$$

But this is immediate from the definitions. So $(a \times b, F, x + y)$ is the desired product. Q.E.D.

6.2 The Chu construction preserves bicompleteness

As we have seen in the preceding section, it does not suffice that $\mathscr{C}$ has products if $\mathsf{Chu}(\mathscr{C},\gamma)$ is to have products too. Now the next question is whether the existence of a limit and its dual suffices. To answer this question in a positive way we present a variation of the approach of [Man17]. We begin with the fundamental definitions.

Definition 6.3 (Diagrams of type $\mathscr{I}$). Let $\mathscr{I}$ be a small category and $\mathscr{C}$ be an arbitrary category. A *diagram of type* $\mathscr{I}$ is a functor $D\colon \mathscr{I} \to \mathscr{C}$.

Definition 6.4 (Cones over diagrams, morphisms of cones). Let $D\colon \mathscr{I} \to \mathscr{C}$ be a diagram of type $\mathscr{I}$. Then a *cone over* D is an object C together with a family of morphisms $f_i\colon C \to D_0(i)$ for every object $i \in \mathscr{I}_0$ such that for all $i, j \in \mathscr{I}_0$ and all arrows $\alpha\colon i \to j$ in $\mathscr{I}$ the diagram

$$\begin{tikzcd} & C \arrow[dl, bend right, "f_i"'] \arrow[dr, bend left, "f_j"] & \\ D_0(i) \arrow[rr, "D_1(\alpha)"'] & & D_0(j) \end{tikzcd}$$

commutes. We denote the cone by $(C,(f_i)_{i\in\mathscr{I}_0})$. Let two cones $(C,(f_i)_{i\in\mathscr{I}_0}), (E,(g_i)_{i\in\mathscr{I}_0})$ be given. A *morphism of cones* is a morphism $\alpha\colon C \to E$ such that for every $i \in \mathscr{I}_0$ the diagram

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, bend right, "f_i"'] & & E \arrow[dl, bend left, "g_i"] \\ & D_0(i) & \end{tikzcd}$$

commutes.

Definition 6.5 (Limits of diagrams). Let $\mathscr{D}\colon \mathscr{I} \to \mathscr{C}$ be a diagram of type $\mathscr{I}$. A *limit of* D is a cone $(L,(l_i)_{i\in\mathscr{I}_0})$ such that for every cone $(C,(f_i)_{i\in\mathscr{I}_0})$ there exists a unique morphism of cones $F_C\colon L \to C$.

Remark 6.6. To make the notation less clunky we shall write D_i instead of $D_0(i)$ for a diagram $D\colon \mathscr{I} \to \mathscr{C}$.

Remark 6.7. Many fundamental concepts of category theory can be expressed as limits or colimits. Take for example the product of two objects $a \times b$ in an arbitrary category $\mathscr{C}$. This product can be interpreted as the following limit: Let $\mathscr{I}$ be the category with two objects $\bullet_1, \bullet_2$ and only the identity morphisms. We can then define a functor $D\colon \mathscr{I} \to \mathscr{C}$ by setting $D_0(\bullet_1) = a$ and $D_0(\bullet_2) = b$. Then a limit of D is an object P together with arrows $\mathtt{pr}_a\colon P \to a, \mathtt{pr}_2\colon P \to b$ such that for every $L \in \mathscr{C}_0$ and arrows $l_1\colon L \to a, l_2\colon L \to b$, there exists a unique arrow $l\colon L \to P$ such that

$$\begin{tikzcd} & L \arrow[dl, bend right, "l_1"'] \arrow[d, dashed, "l"] \arrow[dr, bend left, "l_2"] & \\ a & P \arrow[l, "\mathtt{pr}_a"] \arrow[r, "\mathtt{pr}_b"'] & b \end{tikzcd}$$

commutes. So P is the product $a \times b$. Terminal objects can also be considered as limits. For this we consider the diagram $\mathbf{0}\colon 0 \to \mathscr{C}$, where 0 is the category without objects and morphisms. Now the limit of $\mathbf{0}$ is an object $\top$ together with exactly one morphism $!_c\colon c \to \top$ for every $c \in \mathscr{C}_0$, hence $\top$ is the terminal object.

Definition 6.8 (Cocones under diagrams, morphisms of cocones). Let $D\colon \mathscr{I} \to \mathscr{C}$ be a diagram of type $\mathscr{I}$. Then a *cocone under* D is an object C together with a family of morphisms $f_i\colon D_i \to C$ for every object $i \in \mathscr{I}_0$ such that for all $i, j \in \mathscr{I}_0$ and all arrows $\alpha\colon i \to j$ in $\mathscr{I}$ the diagram

$$\begin{tikzcd}
 & C & \\
D_0(i) \arrow[ur, bend left, "f_i"] \arrow[rr, "D_1(\alpha)"'] & & D_0(j) \arrow[ul, bend right, "f_j"']
\end{tikzcd}$$

commutes. We denote the cocone by $(C, (f_i)_{i\in\mathscr{I}_0})$. Let two cocones $(C, (f_i)_{i\in\mathscr{I}_0}), (E, (g_i)_{i\in\mathscr{I}_0})$ be given. A *morphism of cocones* is a morphism $\alpha\colon C \to E$ such that for every $i \in \mathscr{I}_0$ the diagram

$$\begin{tikzcd}
C \arrow[rr, "\alpha"] & & E \\
 & D_i \arrow[ul, bend left, "f_i"] \arrow[ur, bend right, "g_i"'] &
\end{tikzcd}$$

commutes.

Definition 6.9 (Colimits of diagrams). Let $\mathscr{D}\colon \mathscr{I} \to \mathscr{C}$ be a diagram of type $\mathscr{I}$. A *colimit of* D is a cocone $(L, (l_i)_{i\in\mathscr{I}_0})$ such that for every cocone $(C, (f_i)_{i\in\mathscr{I}_0})$ there exists a unique morphism of cones, $G_C\colon C \to L$.

Remark 6.10. One can show that limits and colimits of diagrams $D\colon \mathscr{I} \to \mathscr{C}$ are unique if they exists, so we denote the limit of D by $\varprojlim_{i\in\mathscr{I}} D_i$ and the colimit by $\varinjlim_{i\in\mathscr{I}} D_i$.

Definition 6.11 ((Bi-/Co-)complete categories). Let $\mathscr{C}$ be a category.

- The category $\mathscr{C}$ is *complete* if for all small categories $\mathscr{I}$ with finite $\mathscr{I}_0$ and every diagram $D\colon \mathscr{I} \to \mathscr{C}$ the limit $\varprojlim_{i\in\mathscr{I}} D_i$ exists.
- The category $\mathscr{C}$ is *cocomplete* if for all small categories $\mathscr{I}$ with finite $\mathscr{I}_0$ and every diagram $D\colon \mathscr{I} \to \mathscr{C}$ the colimit $\varinjlim_{i\in\mathscr{I}} D_i$ exists.
- The category $\mathscr{C}$ is *bicomplete* if it is both complete and cocomplete.

A very important class of limits are the so called pullbacks.

Definition 6.12 (Pullbacks, Pushouts). Let $\mathscr{C}$ be an arbitrary category and $\mathscr{I}$ be the category given by

$$\begin{tikzcd}
\bullet_1 \arrow[loop above, "\mathbf{1}_{\bullet_1}"] \arrow[r, "f"'] & \bullet_2 \arrow[loop above, "\mathbf{1}_{\bullet_2}"] & \bullet_3 . \arrow[loop above, "\mathbf{1}_{\bullet_3}"] \arrow[l, "g"]
\end{tikzcd}$$

A *pullback* is a limit of a diagram $D\colon \mathscr{I} \to \mathscr{C}$. A *pushout* is a colimit of a diagram $D\colon \mathscr{I}^{\,\mathrm{op}} \to \mathscr{C}$.

Now the following theorem allows us to further identify the notion of bicompleteness.

Theorem 6.13. *Let $\mathscr{C}$ be a category. Then the following statements are equivalent.*

1. *The category $\mathscr{C}$ is bicomplete.*
2. *The category $\mathscr{C}$ has a terminal and initial objects as well as all pullbacks and pushouts.*
3. *The category $\mathscr{C}$ has all equalizers, coequalizers as well a finite products and coproducts.*

Proof: The equivalence of the first and the third statement is proven in [Awo10, Proposition 5.21]. The equivalence of the second and the third statement is proven in [Awo10, Proposition 5.14]. Q.E.D.

With these notions we can state the main theorem of this section.

Theorem 6.14. *Let $\mathscr{C}$ be a cartesian closed category and $\gamma \in \mathscr{C}_0$. Consider the following two statements*

1. *The category $\mathscr{C}$ is bicomplete.*
2. *The category $\mathsf{Chu}(\mathscr{C}, \gamma)$ is bicomplete.*

Then the first statement implies the second.

Before we prove this theorem we need the following lemmata.

Lemma 6.15. *Let $\mathscr{C}$ be a cartesian closed category and $a \in \mathscr{C}$. Let $\top$ be the terminal object of $\mathscr{C}$. Then $a \times \top \cong a$.*

Proof: Let $\mathscr{C}$ be a cartesian closed category, and $\top$ its terminal object. We wish to show that $\mathtt{pr}_a\colon a \times \top \to a$ is an isomorphism for every object a of $\mathscr{C}$. We have the universal property, that for every object d of $\mathscr{C}$ and every arrow $f\colon d \to a$ (we need not choose an arrow $d \to \top$, as this arrow is unique) there exists $\tilde{f}\colon d \to a \times \top$ such that

$$\begin{tikzcd} & d \arrow[dl, bend right, "f"'] \arrow[d, dotted, "\tilde{f}"] \arrow[dr, bend left, "\exists!"] & \\ a & a \times 1 \arrow[l, "\mathtt{pr}_a"'] \arrow[r, "\mathtt{pr}_\top", "\exists!"'] & \top \end{tikzcd}$$

commutes. We apply this to the special case $f = \mathbf{1}_a$. Here we obtain an arrow $\tilde{\mathbf{1}}_a\colon a \to a \times \top$, such that $\mathtt{pr}_a \circ \tilde{\mathbf{1}}_a = \mathbf{1}_a$. Furthermore, $\tilde{\mathbf{1}}_a \circ \mathtt{pr}_a\colon a \times \top \to a \times \top$, and $\tilde{\mathbf{1}}_a \circ \mathtt{pr}_a = \mathbf{1}_{a\times\top} = \mathbf{1}_a \times \mathbf{1}_\top$, a proposition we shall now prove. We know that $\mathbf{1}_{a\times\top}$ is the unique arrow that makes the diagram

$$\begin{tikzcd} & a \times \top \arrow[dl, bend right, "\mathtt{pr}_a"'] \arrow[d, dashed, "\mathbf{1}_{a\times\top}"] \arrow[dr, bend left, "\mathtt{pr}_\top"] & \\ a & a \times \top \arrow[l, "\mathtt{pr}_a"'] \arrow[r, "\mathtt{pr}_\top"] & \top \end{tikzcd}$$

commute. But $\tilde{\mathbf{1}}_a \circ \mathtt{pr}_a$ also fulfils that requirement as

$$\begin{aligned} \mathtt{pr}_a \circ \tilde{\mathbf{1}}_a \circ \mathtt{pr}_a &= (\mathtt{pr}_a \circ \tilde{\mathbf{1}}_a) \circ \mathtt{pr}_a && \text{(associativity)} \\ &= \mathbf{1}_a \circ \mathtt{pr}_a && \text{(definition of } \tilde{\mathbf{1}}_a\text{)} \\ &= \mathtt{pr}_a, \\ \mathtt{pr}_\top \circ \tilde{\mathbf{1}}_a \circ \mathtt{pr}_a &= \mathtt{pr}_\top, \end{aligned}$$

since $\mathtt{pr}_\top \circ \tilde{\mathbf{1}}_a \circ \mathtt{pr}_a\colon a \times \top \to \top$, and the arrow $a \times \top \to \top$ is unique. So by uniqueness of $\mathbf{1}_{a\times\top}$ we have

$$\tilde{\mathbf{1}}_a \circ \mathtt{pr}_a = \mathbf{1}_{a\times\top}.$$

Therefore $\mathtt{pr}_a$ is an isomorphism. Q.E.D.

Lemma 6.16. *Let $\mathscr{C}$ be a cartesian closed category with an initial object $\perp$. For every $x \in \mathscr{C}_0$ let $¡_x\colon \perp \to x$ be the unique arrow. Then $\perp \times x \cong \perp$ via $\mathtt{pr}_\perp$.*

Proof: Let $x \in \mathscr{C}_0$. Consider the product $\perp \times x$ with projections $\mathtt{pr}_x\colon \perp \times x \to x$ and $\mathtt{pr}_\perp\colon \perp \times x \to \perp$. So $\mathtt{pr}_\perp\colon \perp \times x \to \perp$ is an element of $\mathrm{Hom}_{\mathscr{C}}(\perp \times x \to \perp)$. But by the definition of the exponential and the initial element we have

$$\mathrm{Hom}_{\mathscr{C}}(\perp \times x, \perp) \cong \mathrm{Hom}_{\mathscr{C}}(\perp, \perp^x) = \{¡_{\perp^x}\},$$

so $\mathtt{pr}_\perp$ is the only element of $\mathrm{Hom}(\perp \times x, \perp)$. Analogously we have

$$\mathrm{Hom}_{\mathscr{C}}(\perp \times x, \perp \times x) \cong \mathrm{Hom}_{\mathscr{C}}(\perp, (\perp \times x)^x) = \{¡_{(\perp\times x)^x}\},$$

so there exists only one arrow $\perp \times x \to \perp \times x$, the identity arrow. But as $¡_{\perp\times x} \circ \mathtt{pr}_\perp \colon \perp \times x \to \perp \times x$, we necessarily have $¡_{\perp\times x} \circ \mathtt{pr}_\perp = \mathbf{1}_{\perp\times x}$. Similarly we have $\mathtt{pr}_\perp \circ ¡_{\perp\times x} = \mathbf{1}_\perp$. Q.E.D.

Lemma 6.17. *Let $\mathscr{C}, \mathscr{D}$ be cartesian closed categories and $F \colon \mathscr{C} \to \mathscr{D}$ be an isomorphism. Let*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "h"] \\ B \arrow[r, "i"] & C \end{tikzcd}$$

be a commutative diagram with $R, A, B, C \in \mathscr{C}_0, f, g, h, i \in \mathscr{C}_1$ and where R is a pullback. Then

$$\begin{tikzcd} F_0(R) \arrow[r, "F_1(f)"] \arrow[d, "F_1(g)"'] & F_0(A) \arrow[d, "F_1(h)"] \\ F_0(B) \arrow[r, "F_1(i)"] & F_0(C) \end{tikzcd}$$

is a commutative diagram and $F_0(R)$ is a pullback.

Proof: Let $\mathscr{C}, \mathscr{D}$ as well as the isomorphism $F \colon \mathscr{C} \to \mathscr{D}$ and

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "h"] \\ B \arrow[r, "i"] & C \end{tikzcd}$$

as in the lemma be given. The diagram

$$\begin{tikzcd} F_0(R) \arrow[r, "F_1(f)"] \arrow[d, "F_1(g)"'] & F_0(A) \arrow[d, "F_1(h)"] \\ F_0(B) \arrow[r, "F_1(i)"] & F_0(C) \end{tikzcd}$$

commutes as

$$F_1(h) \circ F_1(f) = F_1(h \circ f) = F_1(i \circ g) = F_1(i) \circ F_1(g).$$

To see that $F_0(R)$ is a pullback, let $L \in \mathscr{D}$ and $l_1 \colon L \to F_0(A), l_2 \colon L \to F_0(B)$ be given such that

$$\begin{tikzcd} L \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_2"'] & & \\ & F_0(R) \arrow[r, "F_1(f)"] \arrow[d, "F_1(g)"] & F_0(A) \arrow[d, "F_1(h)"] \\ & F_0(B) \arrow[r, "F_1(i)"] & F_0(C) \end{tikzcd}$$

commutes. Using F^{-1}, we obtain the commutative diagram

$$\begin{tikzcd}
F_0^{-1}(L) \arrow[drr, bend left, "F_1^{-1}(l_1)"] \arrow[ddr, bend right, "F_1^{-1}(l_2)"'] & & \\
& R \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "h"] \\
& B \arrow[r, "i"] & C.
\end{tikzcd}$$

One computes

$$\begin{aligned}
h \circ F_1^{-1}(l_1) &= F_1^{-1}\big(F_1(h)\big) \circ F_1^{-1}(l_1) = F_1^{-1}\big(F_1(h) \circ l_1\big) = F_1^{-1}\big(F_1(i) \circ l_2\big) \\
&= F_1^{-1}\big(F_1(i)\big) \circ F_1^{-1}(l_2).
\end{aligned}$$

As R is a pullback we obtain a unique $\rho\colon F_0^{-1}(L) \to R$ making

$$\begin{tikzcd}
F_0^{-1}(L) \arrow[drr, bend left, "F_1^{-1}(l_1)"] \arrow[ddr, bend right, "F_1^{-1}(l_2)"'] \arrow[dr, dotted, "\rho"] & & \\
& R \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "h"] \\
& B \arrow[r, "i"] & C.
\end{tikzcd}$$

commute. Now $F_1(\rho)\colon L \to F_0(R)$ and

$$\begin{tikzcd}
L \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_2"'] \arrow[dr, dotted, "F_1(\rho)"] & & \\
& F_0(R) \arrow[r, "F_1(f)"] \arrow[d, "F_1(g)"] & F_0(A) \arrow[d, "F_1(h)"] \\
& F_0(B) \arrow[r, "F_1(i)"] & F_0(C)
\end{tikzcd}$$

commutes as

$$F_1(f) \circ F_1(\rho) = F_1(f \circ \rho) = F_1\big(F_1^{-1}(l_1)\big) = l_1$$

and

$$F_1(g) \circ F_1(\rho) = F_1(g \circ \rho) = F_1\big(F_1^{-1}(l_2)\big) = l_2.$$

The arrow $F_1(\rho)$ is necessarily unique, as ρ is unique. So $F_0(R)$ is a pullback. Q.E.D.

Lemma 6.18. *Let $\mathscr{C}$ be an arbitrary category and*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\
C \arrow[r, "i"] & D
\end{tikzcd}$$

be a commutative diagram with $A, B, C, D \in \mathscr{C}_0$, $f, g, h, i \in \mathscr{C}_1$ where A is a pullback. Then

$$\begin{tikzcd}
A & B \arrow[l, "f"] \\
C \arrow[u, "g"] & D \arrow[l, "i"] \arrow[u, "h"']
\end{tikzcd}$$

is a commutative diagram where $f, g, h, i \in {\mathscr{C}_1}^{\mathrm{op}}$ and A is a pushout in $\mathscr{C}^{\mathrm{op}}$.

Proof: Let $\mathscr{C}$ and the diagram as in the lemma be given. It is immediate from the definition of $\mathscr{C}^{\,\mathrm{op}}$, that

$$\begin{tikzcd}
A & B \arrow[l, "f"] \\
C \arrow[u, "g"] & D \arrow[l, "i"] \arrow[u, "h"']
\end{tikzcd}$$

is a commutative diagram if the arrows are considered to be in $\mathscr{C}^{\,\mathrm{op}}$. To see that it is a pushout, Let $L \in \mathscr{C}^{\,\mathrm{op}}$ with $l_i \colon B \to L, l_2 \colon C \to L$ such that

$$\begin{tikzcd}
L & & \\
& A & B \arrow[l, "f"] \arrow[ull, bend right, "l_1"'] \\
& C \arrow[u, "g"] \arrow[uul, bend left, "l_2"] & D \arrow[l, "i"] \arrow[u, "h"']
\end{tikzcd}$$

commutes. By definition of $\mathscr{C}^{\,\mathrm{op}}$ this is the diagram

$$\begin{tikzcd}
L \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_2"'] & & \\
& A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\
& C \arrow[r, "i"] & D
\end{tikzcd}$$

when considered in $\mathscr{C}$. As A is a pullback, we obtain a unique $\rho \colon L \to A$ such that

$$\begin{tikzcd}
L \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_2"'] \arrow[dr, dotted, "\rho"] & & \\
& A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\
& C \arrow[r, "i"] & D
\end{tikzcd}$$

commutes. But if we consider ρ as an arrow in $\mathscr{C}^{\,\mathrm{op}}$, we obtain the commutative diagram

$$\begin{tikzcd}
L & & \\
& A \arrow[ul, dotted, "\rho"'] & B \arrow[l, "f"] \arrow[ull, bend right, "l_1"'] \\
& C \arrow[u, "g"] \arrow[uul, bend left, "l_2"] & D. \arrow[l, "i"] \arrow[u, "h"']
\end{tikzcd}$$

As ρ is unique, we obtain that A is a pushout in $\mathscr{C}^{\,\mathrm{op}}$. Q.E.D.

Remark 6.19. A similar result can be shown if we consider a pushout instead of a pullback in $\mathscr{C}$. The proof of this case is nearly identical to the case shown above. We shall denote this result in the following corollary.

Corollary 6.20. *Let $\mathscr{C}$ be an arbitrary category and*

$$\begin{tikzcd} A & B \arrow[l, "f"'] \\ C \arrow[u, "g"] & D \arrow[l, "i"] \arrow[u, "h"'] \end{tikzcd}$$

be a commutative diagram with $A, B, C, D \in \mathscr{C}_0$, $f, g, h, i \in \mathscr{C}_1$ where A is a pushout. Then

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$

is a commutative diagram where $f, g, h, i \in \mathscr{C}_1{}^{\mathrm{op}}$ and P is a pullback in $\mathscr{C}^{\mathrm{op}}$.

Lemma 6.21. *Let $\mathscr{C}$ be a cartesian closed category and $A, B, C, \gamma \in \mathscr{C}_0$. Suppose we are given two arrows*

$$f \colon A \to B, \qquad\qquad g \colon B \to C$$

in $\mathscr{C}$, and set

$$h_1 := \mathtt{eval}_{\gamma,C} \times (\mathbf{1}_{\gamma^C} \times (g \circ f)).$$

If we set

$$h_2 := \mathtt{eval}_{\gamma,C} \circ (\mathbf{1}_{\gamma^C} \times g) \text{ and } h_3 := \mathtt{eval}_{\gamma,B} \circ (\mathbf{1}_{\gamma^B} \times f),$$

then $\widehat{h_1} = \widehat{h_3} \circ \widehat{h_2}$.

Proof: By definition we have the commutative diagrams

$$\begin{tikzcd} \gamma^A \times A \arrow[r, "\mathtt{eval}_{\gamma,A}"] & \gamma \\ \gamma^B \times A \arrow[u, "\widehat{h_3} \times \mathbf{1}_A"] \arrow[r, "\mathbf{1}_{\gamma^B} \times f"] & \gamma^B \times B, \arrow[u, "\mathtt{eval}_{\gamma,B}"'] \end{tikzcd} \qquad \begin{tikzcd} \gamma^B \times B \arrow[r, "\mathtt{eval}_{\gamma,B}"] & \gamma \\ \gamma^C \times B \arrow[u, "\widehat{h_2} \times \mathbf{1}_B"] \arrow[r, "\mathbf{1}_{\gamma^C} \times g"] & \gamma^C \times C, \arrow[u, "\mathtt{eval}_{\gamma,C}"'] \end{tikzcd}$$

$$\begin{tikzcd} \gamma^A \times A \arrow[r, "\mathtt{eval}_{\gamma,A}"] & \gamma \\ \gamma^C \times A \arrow[u, "\widehat{h_1} \times \mathbf{1}_A"] \arrow[r, "\mathbf{1}_{\gamma^C} \times (g \circ f)"] & \gamma^C \times C, \arrow[u, "\mathtt{eval}_{\gamma,C}"'] \end{tikzcd}$$

These can be fitted in a larger diagram,

$$\begin{tikzcd} \gamma^C \times A \arrow[r, "\mathbf{1}_{\gamma^C} \times f"] \arrow[rr, bend left=50, "\mathbf{1}_{\gamma^C} \times (g \circ f)"] \arrow[d, "\widehat{h_2} \times \mathbf{1}_A"] \arrow[dd, bend right=60, "\widehat{h_1} \times \mathbf{1}_A"'] & \gamma^C \times B \arrow[r, "\mathbf{1}_{\gamma^C} \times g"] \arrow[d, "\widehat{h_2} \times \mathbf{1}_B"] & \gamma^C \times C \arrow[dd, "\mathtt{eval}_{\gamma,C}"] \\ \gamma^B \times A \arrow[r, "\mathbf{1}_{\gamma^B} \times f"] \arrow[d, "\widehat{h_3} \times \mathbf{1}_A"] & \gamma^B \times B \arrow[dr, "\mathtt{eval}_{\gamma,B}"] & \\ \gamma^A \times A \arrow[rr, "\mathtt{eval}_{\gamma,A}"'] & & \gamma. \end{tikzcd}$$

To prove that $\widehat{h_1} = \widehat{h_3} \circ \widehat{h_2}$, it suffices to prove that

$$\mathtt{eval}_{\gamma,A} \circ \big((\widehat{h_3} \circ \widehat{h_2}) \times \mathbf{1}_A\big) = \mathtt{eval}_{\gamma,C} \circ (\mathbf{1}_{\gamma^C} \times (g \circ f)).$$

To see this we compute

$$\begin{aligned}
\mathtt{eval}_{\gamma,C} \circ (\mathbf{1}_{\gamma^C} \times (g \circ f)) &= \mathtt{eval}_{\gamma,C} \circ (\mathbf{1}_{\gamma^C} \times g) \circ (\mathbf{1}_{\gamma^C} \times f) \\
&= \mathtt{eval}_{\gamma,B} \circ (\widehat{h_2} \times \mathbf{1}_B) \circ (\mathbf{1}_{\gamma^C} \times f) \\
&= \mathtt{eval}_{\gamma,B} \circ (\mathbf{1}_{\gamma^B} \times f) \circ (\widehat{h_2} \times \mathbf{1}_A) \\
&= \mathtt{eval}_{\gamma,A} \circ (\widehat{h_3} \times \mathbf{1}_A) \circ (\widehat{h_2} \times \mathbf{1}_A) \\
&= \mathtt{eval}_{\gamma,A} \circ \big((\widehat{h_3} \circ \widehat{h_2}) \times \mathbf{1}_A\big).
\end{aligned}$$

Q.E.D.

Lemma 6.22. *Let $\mathscr{C}$ be a cartesian closed category and $A, B, C, D \in \mathscr{C}_0$. Suppose we are given $f\colon B \times A \to C$ and $g\colon D \to B$. If we consider $h := f \circ (g \times \mathbf{1}_A)$, then $\widehat{h} = \widehat{f} \circ g\colon D \to C^A$.*

Proof: Let $A, B, C, D \in \mathscr{C}_0$ and $f, g \in \mathscr{C}_1$ as in the lemma be given. Set $h := f \circ (\mathbf{1}_A \times g)$, then $\widehat{h}$ is defined as the unique arrow making

$$\begin{tikzcd}
C^A \times A \arrow[r, "\mathtt{eval}_{C,A}"] & C \\
D \times A \arrow[u, "\widehat{h} \times \mathbf{1}_A"] \arrow[ur, "f \circ (g \times \mathbf{1}_A)"'] &
\end{tikzcd}$$

commute. Now we see that this can be dissected into the larger commutative diagram

$$\begin{tikzcd}
C^A \times A \arrow[r, "\mathtt{eval}_{C,A}"] & C \\
B \times A \arrow[u, "\widehat{f} \times \mathbf{1}_A"] \arrow[ur, "f"] & \\
D \times A. \arrow[u, "g \times \mathbf{1}_A"] \arrow[uu, bend left=60, "\widehat{h} \times \mathbf{1}_A"] &
\end{tikzcd}$$

This immediately shows that $\widehat{h} = \widehat{f} \circ g$. Q.E.D.

Lemma 6.23. *Let $\mathscr{C}$ be a cartesian closed category with $\gamma \in \mathscr{C}_0$ and let the diagram*

$$\begin{tikzcd}
P & x_1 \arrow[l, "\pi_1"'] \\
x_2 \arrow[u, "\pi_2"] & y \arrow[l, "\psi_2"] \arrow[u, "\psi_1"']
\end{tikzcd}$$

with $P, x_1, x_2, y \in \mathscr{C}_0$ be given, where P is a pushout. Then we obtain a diagram

$$\begin{tikzcd}
\gamma^P \arrow[r] \arrow[d] & \gamma^{x_1} \arrow[d] \\
\gamma^{x_2} \arrow[r] & \gamma^y,
\end{tikzcd}$$

where γ^P is a pullback.

Proof: Let $\mathscr{C}$ and γ as in the lemma be given. Suppose we have the diagram

$$\begin{tikzcd} P & x_1 \arrow[l, "\pi_1"'] \\ x_2 \arrow[u, "\pi_2"] & y \arrow[l, "\psi_2"] \arrow[u, "\psi_1"'] \end{tikzcd}$$

with $P, x_1, x_2, y \in \mathscr{C}_0$ where P is a pushout. Our first task is to find arrows $p_1\colon \gamma^P \to \gamma^{x_1}, p_2\colon \gamma^P \to \gamma^{x_2}, s_1\colon \gamma^{x_1} \to \gamma^y, s_2\colon \gamma^{x_2} \to \gamma^y$ such that

$$\begin{tikzcd} \gamma^P \arrow[r, "p_1"] \arrow[d, "p_2"] & \gamma^{x_1} \arrow[d, "s_1"] \\ \gamma^{x_2} \arrow[r, "s_2"] & \gamma^y \end{tikzcd}$$

commutes. As we have $\pi_i\colon x_i \to P$ for $i = 1, 2$, we can define p_i as the transpose of $\mathtt{eval}_{\gamma,P} \circ (\mathbf{1}_{\gamma^P} \times \pi_i)\colon \gamma^P \times x_i \to \gamma$. So we already have the arrows

$$\begin{tikzcd} \gamma^P \arrow[r, "p_1"] \arrow[d, "p_2"] & \gamma^{x_1} \\ \gamma^{x_2} & \gamma^y. \end{tikzcd}$$

To define the arrows $s_i\colon \gamma^{x_i} \to \gamma^y$ for $i = 1, 2$ we take s_i to be the transpose of $\mathtt{eval}_{\gamma,x_i} \circ (\mathbf{1}_{\gamma^{x_i}} \times \psi_i)\colon \gamma^{x_i} \times y \to \gamma$. Now we have the diagram

$$\begin{tikzcd} \gamma^P \arrow[r, "p_1"] \arrow[d, "p_2"] & \gamma^{x_1} \arrow[d, "s_1"] \\ \gamma^{x_2} \arrow[r, "s_2"'] & \gamma^y \end{tikzcd}$$

and it remains to check its commutativity. But this is lemma 6.21 for both cases. It remains to check that γ^P is a pullback. For this let $R \in \mathscr{C}_0$ with $r_1\colon R \to \gamma^{x_1}$ and $r_2\colon R \to \gamma^{x_2}$ be given, where $s_1 \circ r_1 = s_2 \circ r_2$. We have to check that there exists a unique arrow $\rho\colon R \to \gamma^P$ such that

$$\begin{tikzcd} R \arrow[dr, dotted, "\rho"] \arrow[drr, bend left, "r_1"] \arrow[ddr, bend right, "r_2"'] & & \\ & \gamma^P \arrow[r, "p_1"] \arrow[d, "p_2"] & \gamma^{x_1} \arrow[d, "s_1"] \\ & \gamma^{x_2} \arrow[r, "s_2"'] & \gamma^y \end{tikzcd}$$

commutes. Such an arrow $\rho\colon R \to \gamma^P$ must be given as the transpose of an arrow $\varpi\colon R \times P \to \gamma$. Now we can transpose ϖ to an arrow $\varrho\colon P \to \gamma^R$. So our next goal is to find arrows $\alpha_1\colon x_1 \to \gamma^R, \alpha_2\colon x_2 \to \gamma^R$ such that

$$\begin{tikzcd} \gamma^R & x_1 \arrow[l, "\alpha_1"'] \\ x_2 \arrow[u, "\alpha_2"] & y \arrow[l, "\psi_2"] \arrow[u, "\psi_1"'] \end{tikzcd} \tag{22}$$

commutes, as we then obtain $\varrho\colon P \to \gamma^R$ from the fact that P is a pushout. We have arrows $r_i\colon R \to \gamma^{x_i}$ for $i = 1, 2$, which have to be given as arrows $R \times x_i \to \gamma$, which themselves can be transposed into arrows $\alpha_i\colon x_i \to \gamma^R$. It remains to check that $\alpha_1 \circ \psi_1 = \alpha_2 \circ \psi_2$. We already know that $s_1 \circ r_1 = s_2 \circ r_2$, so we know that we find an arrow $q\colon R \times y \to \gamma$ such that $\widehat{q} = s_1 \circ r_1 = s_2 \circ r_2$. Now we want to show that if we transpose q with respect to R, we get $\widehat{q} = \alpha_1 \circ \psi_1$ and $\widehat{q} = \alpha_2 \circ \psi_2$. We know that s_1 and s_2 are defined as the unique arrows making

$$\begin{tikzcd}
\gamma^y \times y \arrow[r, "\mathtt{eval}_{\gamma,y}"] & \gamma \\
\gamma^{x_1} \times y \arrow[u, "s_1 \times \mathbf{1}_y"] \arrow[r, "\mathbf{1}_{\gamma^{x_1}} \times \psi_1"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma,x_1}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\gamma^y \times y \arrow[r, "\mathtt{eval}_{\gamma,y}"] & \gamma \\
\gamma^{x_1} \times y \arrow[u, "s_2 \times \mathbf{1}_y"] \arrow[r, "\mathbf{1}_{\gamma^{x_2}} \times \psi_2"'] & \gamma^{x_2} \times x_2 \arrow[u, "\mathtt{eval}_{\gamma,x_2}"']
\end{tikzcd}$$

commute. We can expand these commutative diagrams to commutative diagrams

$$\begin{tikzcd}
\gamma^y \times y \arrow[r, "\mathtt{eval}_{\gamma,y}"] & \gamma \\
\gamma^{x_1} \times y \arrow[u, "s_1 \times \mathbf{1}_y"'] \arrow[r, "\mathbf{1}_{\gamma^{x_1}} \times \psi_1"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma,x_1}"'] \\
R \times y \arrow[u, "r_1 \times \mathbf{1}_y"'] \arrow[uu, bend left=60, "\widehat{q} \times \mathbf{1}_y"] \arrow[ur, bend right, "r_1 \times \psi_1"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\gamma^y \times y \arrow[r, "\mathtt{eval}_{\gamma,y}"] & \gamma \\
\gamma^{x_2} \times y \arrow[u, "s_2 \times \mathbf{1}_y"] \arrow[r, "\mathbf{1}_{\gamma^{x_2}} \times \psi_2"'] & \gamma^{x_2} \times x_2 \arrow[u, "\mathtt{eval}_{\gamma,x_2}"'] \\
R \times y. \arrow[u, "r_2 \times \mathbf{1}_y"'] \arrow[uu, bend left=60, "\widehat{q} \times \mathbf{1}_y"] \arrow[ur, bend right, "r_2 \times \psi_2"'] &
\end{tikzcd}$$

So by using $\mathtt{eval}_{\gamma,y} \circ (\widehat{q} \times \mathbf{1}_y)$ we can conclude that

$$q = \mathtt{eval}_{\gamma,x_2} \circ (r_2 \times \psi_2) = \mathtt{eval}_{\gamma,x_1} \circ (r_1 \times \psi_1). \tag{23}$$

By definition of the α_i we have the commutative diagrams

$$\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times x_1 \arrow[u, "\mathbf{1}_R \times \alpha_1"] \arrow[r, "r_1 \times \mathbf{1}_{x_1}"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma,x_1}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times x_2 \arrow[u, "\mathbf{1}_R \times \alpha_2"] \arrow[r, "r_2 \times \mathbf{1}_{x_2}"'] & \gamma^{x_2} \times x_2. \arrow[u, "\mathtt{eval}_{\gamma,x_2}"']
\end{tikzcd}$$

If we tranpose q with regard to R and use the equality (23), we see that $\widehat{q}$ is the unique arrow making the diagrams

$$\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times y \arrow[u, "\mathbf{1}_R \times \widehat{q}"] \arrow[r, "r_1 \times \psi_1"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma,x_1}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times y \arrow[u, "\mathbf{1}_R \times \widehat{q}"] \arrow[r, "r_2 \times \psi_2"'] & \gamma^{x_2} \times x_2 \arrow[u, "\mathtt{eval}_{\gamma,x_2}"']
\end{tikzcd}$$

commute. As $r_1 \times \psi_1 = (r_1 \times \mathbf{1}_{x_1}) \circ (\mathbf{1}_R \times \psi_1)$ and $r_2 \times \psi_2 = (r_2 \times \mathbf{1}_{x_2}) \circ (\mathbf{1}_R \times \psi_2)$ we can see that $\widehat{q} = \alpha_1 \circ \psi_1 = \alpha_2 \circ \psi_2$ as desired, as the commutativity of the preceding diagrams is equivalent to the commutativity of

$$\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times x_1 \arrow[u, "\mathbf{1}_R \times \alpha_1"'] \arrow[r, "r_1 \times \times \mathbf{1}_{x_1}"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma,x_1}"'] \\
R \times y \arrow[u, dashed, "\mathbf{1}_R \times \psi_1"] \arrow[uu, bend left=60, "\mathbf{1}_R \times \widehat{q}"] \arrow[ur, bend right, "r_1 \times \psi_1"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times y \arrow[u, "\mathbf{1}_R \times \alpha_2"'] \arrow[r, "r_2 \times \psi_2"'] & \gamma^{x_2} \times x_2 \arrow[u, "\mathtt{eval}_{\gamma,x_2}"'] \\
R \times y. \arrow[u, dashed, "\mathbf{1}_R \times \psi_2"] \arrow[uu, bend left=60, "\mathbf{1}_R \times \widehat{q}"] \arrow[ur, bend right, "r_2 \times \psi_2"'] &
\end{tikzcd}$$

So we have shown that (22) commutes, so we obtain an arrow $\varrho\colon P \to \gamma^R$ such that

$$\begin{tikzcd}
\gamma^R & & \\
& P \arrow[ul, dotted, "\varrho"] & x_1 \arrow[l, "\pi_1"] \arrow[ull, bend right, "\alpha_1"'] \\
& x_2 \arrow[u, "\pi_2"'] \arrow[uul, bend left, "\alpha_2"] & y \arrow[l, "\psi_2"] \arrow[u, "\psi_1"']
\end{tikzcd}$$

commutes, as P is a pushout. Now we use this ϱ to obtain our $\rho\colon R \to \gamma^P$. To prove the commutativity of

$$\begin{tikzcd}
R \arrow[dr, dotted, "\rho"] \arrow[drr, bend left, "r_1"] \arrow[ddr, bend right, "r_2"'] & & \\
& \gamma^P \arrow[r, "p_1"] \arrow[d, "p_2"] & \gamma^{x_1} \arrow[d, "s_1"] \\
& \gamma^{x_2} \arrow[r, "s_2"'] & \gamma^y
\end{tikzcd}$$

it suffices to show $r_1 = p_1 \circ \rho$ and $r_2 = p_2 \circ \rho$. The arrow ρ is the unique arrow making

$$\begin{tikzcd}
\gamma^P \times P \arrow[r, "\mathtt{eval}_{\gamma,P}"] & \gamma \\
R \times P \arrow[u, "\rho\times\mathbf{1}_P"] \arrow[r, "\mathbf{1}_R\times\varrho"'] & R \times \gamma^R \arrow[u, "\mathtt{eval}_{\gamma,R}"']
\end{tikzcd}$$

commute, and the arrows p_1, p_2 are the unique arrows making

$$\begin{tikzcd}
\gamma^{x_1} \times x_1 \arrow[r, "\mathtt{eval}_{\gamma,x_1}"] & \gamma \\
\gamma^P \times x_1 \arrow[u, "p_1\times\mathbf{1}_{x_1}"] \arrow[r, "\mathbf{1}_{\gamma^P}\times\pi_1"'] & \gamma^P \times P \arrow[u, "\mathtt{eval}_{\gamma,P}"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\gamma^{x_2} \times x_2 \arrow[r, "\mathtt{eval}_{\gamma,x_2}"] & \gamma \\
\gamma^P \times x_2 \arrow[u, "p_2\times\mathbf{1}_{x_2}"] \arrow[r, "\mathbf{1}_{\gamma^P}\times\pi_2"'] & \gamma^P \times P \arrow[u, "\mathtt{eval}_{\gamma,P}"]
\end{tikzcd}$$

commute. As we have seen before, r_1, r_2 are the unique arrows making

$$\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times x_1 \arrow[u, "\mathbf{1}_R\times\alpha_1"] \arrow[r, "r_1\times\mathbf{1}_{x_1}"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma,x_1}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times x_2 \arrow[u, "\mathbf{1}_R\times\alpha_2"] \arrow[r, "r_2\times\mathbf{1}_{x_2}"'] & \gamma^{x_2} \times x_2 \arrow[u, "\mathtt{eval}_{\gamma,x_2}"']
\end{tikzcd}$$

commute. Hence we obtain the commutative diagrams

$$\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times x_1 \arrow[u, "\mathbf{1}_R\times(\varrho\circ\pi_1)"] \arrow[r, "r_1\times\mathbf{1}_{x_1}"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma,x_1}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
R \times \gamma^R \arrow[r, "\mathtt{eval}_{\gamma,R}"] & \gamma \\
R \times x_2 \arrow[u, "\mathbf{1}_R\times(\varrho\circ\pi_2)"] \arrow[r, "r_2\times\mathbf{1}_{x_2}"'] & \gamma^{x_2} \times x_2 \arrow[u, "\mathtt{eval}_{\gamma,x_2}"']
\end{tikzcd}$$

since $\alpha_i = \varrho \circ \pi_i$ for $i = 1, 2$. Now we can use lemma 6.21 on $\mathbf{1}_R \times (g \circ \pi_1)$ and obtain $r_1 = p_1 \circ \rho$, as well as $r_2 = p_2 \circ \rho$. This shows that γ^P is a pushout. Q.E.D.

Remark 6.24. If $\mathscr{C}$ is a cartesian closed category, $\gamma \in \mathscr{C}_0$ and

$$\begin{tikzcd} P & x_1 \arrow[l, "\pi_1"'] \\ x_2 \arrow[u, "\pi_2"] & y \arrow[l, "\psi_2"] \arrow[u, "\psi_1"'] \end{tikzcd}$$

is a commutative diagram where P is a pushout, then we label the diagram obtained in the previous lemma

$$\begin{tikzcd} \gamma^P \arrow[r, "\gamma^{\pi_1}"] \arrow[d, "\gamma^{\pi_2}"'] & \gamma^{x_1} \arrow[d, "\gamma^{\psi_1}"] \\ \gamma^{x_2} \arrow[r, "\gamma^{x_2}"] & \gamma^y. \end{tikzcd}$$

Lemma 6.25. *Let $\mathscr{C}$ be a cartesian closed category and*

$$\begin{tikzcd} P & A \arrow[l, "i"'] \\ B \arrow[u, "h"] & C \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

be a commutative diagram of objects $P, A, B, C \in \mathscr{C}_0$ such that P is a pushout. Let $R \in \mathscr{C}_0$ be arbitrary. Then

$$\begin{tikzcd} R \times P & R \times A \arrow[l, "\mathbf{1}_R \times i"'] \\ R \times B \arrow[u, "\mathbf{1}_R \times h"] & R \times C \arrow[l, "\mathbf{1}_R \times g"] \arrow[u, "\mathbf{1}_R \times f"] \end{tikzcd}$$

is a commutative diagram where $R \times P$ is a pushout.

Proof: Let the diagram as in the lemma be given and $R \in \mathscr{C}_0$ be arbitrary. To see that

$$\begin{tikzcd} R \times P & R \times A \arrow[l, "\mathbf{1}_R \times i"'] \\ R \times B \arrow[u, "\mathbf{1}_R \times h"] & R \times C \arrow[l, "\mathbf{1}_R \times g"] \arrow[u, "\mathbf{1}_R \times f"] \end{tikzcd}$$

is a commutative diagram it suffices to check that $(\mathbf{1}_R \times i) \circ (\mathbf{1}_R \times f) = (\mathbf{1}_R \times h) \circ (\mathbf{1}_R \times g)$. We can immediately compute that

$$(\mathbf{1}_R \times i) \circ (\mathbf{1}_R \times f) = \mathbf{1}_R \times (i \circ f) \quad \text{and} \quad (\mathbf{1}_R \times h) \circ (\mathbf{1}_R \times g) = \mathbf{1}_R \times (h \circ g).$$

Now $\mathbf{1}_R \times (i \circ f)$ and $\mathbf{1}_R \times (h \circ g)$ are the unique arrows making

$$\begin{tikzcd} R \arrow[d, "\mathbf{1}_R"'] & R \times C \arrow[l, "\mathtt{pr}_R"'] \arrow[r, "\mathtt{pr}_C"] \arrow[d, dotted, "\mathbf{1}_R \times (i \circ f)"] & C \arrow[d, "i \circ f"] \\ R & R \times P \arrow[l, "\mathtt{pr}_R"'] \arrow[r, "\mathtt{pr}_P"] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[d, "\mathbf{1}_R"'] & R \times C \arrow[l, "\mathtt{pr}_R"'] \arrow[r, "\mathtt{pr}_C"] \arrow[d, dotted, "\mathbf{1}_R \times (h \circ g)"] & C \arrow[d, "h \circ g"] \\ R & R \times P \arrow[l, "\mathtt{pr}_R"'] \arrow[r, "\mathtt{pr}_P"] & P \end{tikzcd}$$

commute. But as $\mathtt{pr}_R \circ \big(\mathbf{1}_R \times (h \circ g)\big) = \mathbf{1}_R \circ \mathtt{pr}_R$ and

$$\begin{gathered} \mathtt{pr}_P \circ \big(\mathbf{1}_R \times (i \circ f) = (i \circ f) \circ \mathtt{pr}_C = (h \circ g) \circ \mathtt{pr}_C \\ \mathtt{pr}_P \circ \big(\mathbf{1}_R \times (h \circ g)\big), \end{gathered}$$

we know that

$$\begin{tikzcd}
R \arrow[d, "\mathbf{1}_R"'] & R \times C \arrow[l, "\mathtt{pr}_R"'] \arrow[r, "\mathtt{pr}_C"] \arrow[d, dotted, "\mathbf{1}_R \times (h \circ g)"] & C \arrow[d, "i \circ f"] \\
R & R \times P \arrow[l, "\mathtt{pr}_R"'] \arrow[r, "\mathtt{pr}_P"] & P
\end{tikzcd}$$

commutes, and by uniqueness we obtain $\mathbf{1}_R \times (i \circ f) = \mathbf{1}_R \times (h \circ g)$. Now we want to show that $R \times P$ is a pushout. For this let $\gamma \in \mathscr{C}_0$ as well as $\phi\colon R \times A \to \gamma$ and $\psi\colon R \times B \to \gamma$ be given, so that

$$\begin{tikzcd}
\gamma & & \\
& R \times P & R \times A \arrow[l, "\mathbf{1}_R \times i"'] \arrow[ull, bend right, "\phi"'] \\
& R \times B \arrow[u, "\mathbf{1}_R \times h"'] \arrow[uul, bend left, "\psi"] & R \times C \arrow[l, "\mathbf{1}_R \times g"] \arrow[u, "\mathbf{1}_R \times f"']
\end{tikzcd}$$

commutes. Now by transposing $h_1 := \phi \circ (\mathbf{1}_R \times f)$ and $h_2 := \psi \circ (\mathbf{1}_R \times g)$, we obtain $\widehat{h_1} = \widehat{\phi} \circ f\colon C \to \gamma^R$ and $\widehat{h_2} = \widehat{\psi} \circ g\colon C \to \gamma^R$ by lemma 6.22. Furthermore we have $\widehat{h_1} = \widehat{h_2}$ as $h_1 = h_2$. Hence we obtain the commutative diagram

$$\begin{tikzcd}
\gamma^R & & \\
& P & A \arrow[l, "i"'] \arrow[ull, bend right, "\widehat{\phi}"'] \\
& B \arrow[u, "h"] \arrow[uul, bend left, "\widehat{\psi}"] & C. \arrow[l, "g"] \arrow[u, "f"]
\end{tikzcd}$$

As P is a pushout, this yields a unique arrow $\iota\colon P \to \gamma^R$ such that

$$\begin{tikzcd}
\gamma^R & & \\
& P \arrow[ul, dotted, "\iota"'] & A \arrow[l, "i"'] \arrow[ull, bend right, "\widehat{\phi}"'] \\
& B \arrow[u, "h"] \arrow[uul, bend left, "\widehat{\psi}"] & C. \arrow[l, "g"] \arrow[u, "f"]
\end{tikzcd}$$

commutes. Now this ι is given as the transpose of $r := \mathtt{eval}_{\gamma,R} \circ (\iota \times \mathbf{1}_R)\colon P \times R \to \gamma$. Our next goal is to show that r is the unique arrow making

$$\begin{tikzcd}
\gamma & & \\
& R \times P \arrow[ul, dotted, "r"'] & R \times A \arrow[l, "\mathbf{1}_R \times i"'] \arrow[ull, bend right, "\phi"'] \\
& R \times B \arrow[u, "\mathbf{1}_R \times h"'] \arrow[uul, bend left, "\psi"] & R \times C \arrow[l, "\mathbf{1}_R \times g"] \arrow[u, "\mathbf{1}_R \times f"']
\end{tikzcd}$$

commute. By employing lemma 6.22 again, we obtain, by setting $q_1 := r \circ (\mathbf{1}_R \times i$ and $q_2 := r \circ (\mathbf{1}_R \times h)$, the equalities

$$\widehat{q_1} = \widehat{r} \circ i = \iota \circ i = \widehat{\phi} \quad \text{and} \quad \widehat{q_2} = \widehat{r} \circ h = \iota \circ h = \widehat{\psi},$$

therefore we obtain $q_1 = \phi, q_2 = \psi$. To see that r is unique we use the isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(R \times P, \gamma) \cong \mathrm{Hom}_{\mathscr{C}}(P, \gamma^R).$$

As ι is unique in $\mathrm{Hom}_{\mathscr{C}}(P, \gamma^R)$ and every other arrow r' making the above diagram commute would transpose to ι, we already have $r = r'$. So $R \times P$ is a pushout. Q.E.D.

Proof of theorem 6.14: By the theorem 6.13 it suffices to check only that the existence of pullbacks and pushouts as well as an initial object and a terminal object in one of the categories implies their existence in the other.
Proof of 1.⇒2.: Suppose $\mathscr{C}$ is bicomplete. We first show that $\mathsf{Chu}(\mathscr{C}, \gamma)$ has an initial and a terminal object. As $\mathscr{C}$ is bicomplete, there exists a terminal object $\top \in \mathscr{C}_0$ and an initial object $\perp \in \mathscr{C}_0$. Let $c \in \mathscr{C}_0$. We denote the unique arrow $c \to \top$ with $!_c$ and the unique arrow $\perp \to c$ by ¡_c. Then the initial object of $\mathsf{Chu}(\mathscr{C}, \gamma)$ is $(\perp, T, \top)$, where T is defined as follows. Let $\mathtt{pr}_\perp \colon \top \times \perp \to \perp$ be the isomorphisms from the previous lemma. Then $T := \text{¡}_\gamma \circ \mathtt{pr}_\perp$. Now we have to show that given $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)$ there exists a unique $\text{¡}_{(a,f,x)} \colon (\perp, T, \top) \to (a, f, x)$. As we have $\text{¡}_a \colon \perp \to a$ and $!_x \colon x \to \top$, we set $\text{¡}_{(a,f,x)} := (\text{¡}_a, !_x)$. It remains to check that

$$\begin{tikzcd}
\perp \times x \arrow[r, "\mathbf{1}_\perp \times !_x"] \arrow[d, "\text{¡}_a \times \mathbf{1}_x"'] & \perp \times \top \arrow[d, "T"] \\
a \times x \arrow[r, "f"] & \gamma
\end{tikzcd} \tag{24}$$

commutes and that $\text{¡}_{(a,f,x)}$ is unique. To see that (24) commutes it suffices to show that

$$T \circ (\mathbf{1}_\perp \times !_x) \circ \text{¡}_{\perp \times x} = f \circ (\text{¡}_a \times \mathbf{1}_x) \circ \text{¡}_{\perp \times x},$$

as $\perp \xrightarrow{\cong} \perp \times x$ via $\text{¡}_{\perp \times x}$. But this is immediate as both these arrows have domain γ and codomain $\perp$, but there can only be one arrow $\perp \to \gamma$. To see the uniqueness of $\text{¡}_{(a,f,x)}$, let an arrow $(\phi^+, \phi^-) \colon (b, g, y) \to (a, f, x)$ in $\mathsf{Chu}(\mathscr{C}, \gamma)$ be given. We have to show that $(\phi^+, \phi^-) \circ \text{¡}_{(b,g,y)} = \text{¡}_{(a,f,x)}$. One computes

$$\begin{aligned}
(\phi^+, \phi^-) \circ \text{¡}_{(b,g,y)} &= (\phi^+, \phi^-) \circ (\text{¡}_b, !_y) \\
&= (\phi^+ \circ \text{¡}_b, !_y \circ \phi^-) \\
&= (\text{¡}_a, !_x) = \text{¡}_{(a,f,x)},
\end{aligned}$$

which yields the desired result.

To see that $\mathsf{Chu}(\mathscr{C}, \gamma)$ has a terminal object, we employ that $\mathsf{Chu}(\mathscr{C}, \gamma)^{\mathrm{op}} \cong \mathsf{Chu}(\mathscr{C}, \gamma)$, which was proven in lemma 5.1. Therefore $(\top, T, \perp)$ is the terminal object of $\mathsf{Chu}(\mathscr{C}, \gamma)$.

Now we show that $\mathsf{Chu}(\mathscr{C}, \gamma)$ has pullbacks. For this let

$$\begin{tikzcd}
 & (a_1, f_1, x_1) \arrow[d, "(\phi^+, \phi^-)"] \\
(a_2, f_2, x_2) \arrow[r, "(\psi^+, \psi^-)"] & (b, g, y)
\end{tikzcd} \tag{25}$$

be given. We first have to find $(P, h, R) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ with arrows $(\pi_1^+, \pi_1^-) \colon (R, h, P) \to (a_1, f_1, x_1)$ and $(\pi_2^+, \pi_2^-) \colon (R, h, P) \to (a_2, f_2, x_2)$. For this one first dissects diagram (25) into the diagrams

$$\begin{tikzcd}
 & a_1 \arrow[d, "\phi^+"] \\
a_2 \arrow[r, "\psi^+"] & b
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
 & x_1 \\
x_2 & y. \arrow[l, "\psi^-"'] \arrow[u, "\phi^-"']
\end{tikzcd}$$

Therefore we obtain a pullback $R \in \mathscr{C}_0$ together with morphisms $\pi_1^+ \colon R \to a_1, \pi_2^+ \colon R \to a_2$ as well as a pushout $P \in \mathscr{C}_0$ with morphisms $\pi_1^- : x_1 \to P, \pi_2^- : x_2 \to P$ such that the diagrams

$$\begin{tikzcd} R \arrow[r, "\pi_1^+"] \arrow[d, "\pi_2^+"'] & a_1 \arrow[d, "\phi^+"] \\ a_2 \arrow[r, "\psi^+"] & b \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P & x_1 \arrow[l, "\pi_1^-"'] \\ x_2 \arrow[u, "\pi_2^-"] & y \arrow[l, "\psi^-"] \arrow[u, "\phi^-"'] \end{tikzcd}$$

commute. It remains to find an arrow $h \colon P \times R \to \gamma$. It suffices to find an arrow $P \to \gamma^R$, as we then can take its transpose. By lemma 6.23 we know that

$$\begin{tikzcd} \gamma^P \arrow[r, "\gamma^{\pi_1^-}"] \arrow[d, "\gamma^{\pi_2^-}"'] & \gamma^{x_1} \arrow[d, "\gamma^{\psi^-}"] \\ \gamma^{x_2} \arrow[r, "\gamma^{\psi^-}"] & \gamma^y \end{tikzcd}$$

is a commutative diagram where γ^P is a pullback. Now we can make R a cone over

$$\begin{tikzcd} & \gamma^{x_1} \arrow[d, "\gamma^{\phi^-}"] \\ \gamma^{x_2} \arrow[r, "\gamma^{\psi^-}"] & \gamma^y \end{tikzcd}$$

by defining $R \to \gamma^{x_1}$ by $\widehat{f_1} \circ \pi_1^+$ and $R \to \gamma^{x_2}$ by $\widehat{f_2} \circ \pi_2^+$. So we obtain

$$\begin{tikzcd} R \arrow[r, "\widehat{f_1} \circ \pi_1^+"] \arrow[d, "\widehat{f_2} \circ \pi_2^+"'] & \gamma^{x_1} \arrow[d, "\gamma^{\phi^-}"] \\ \gamma^{x_2} \arrow[r, "\gamma^{\psi^-}"] & \gamma^y . \end{tikzcd} \tag{26}$$

It remains to show that this diagram is commutative. We know that $\widehat{f_1}$ and $\widehat{f_2}$ are defined as the arrows making

$$\begin{tikzcd} \gamma^{x_1} \times x_1 \arrow[r, "\mathtt{eval}_{\gamma, x_1}"] & \gamma \\ a_1 \times x_1 \arrow[u, "\widehat{f_1} \times \mathbf{1}_{x_1}"] \arrow[ur, "f_1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \gamma^{x_2} \times x_2 \arrow[r, "\mathtt{eval}_{\gamma, x_2}"] & \gamma \\ a_2 \times x_2 \arrow[u, "\widehat{f_2} \times \mathbf{1}_{x_2}"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

commute. On the other hand γ^{ϕ^-} and γ^{ψ^-} are defined as the unique arrows making the diagrams

$$\begin{tikzcd} \gamma^y \times y \arrow[r, "\mathtt{eval}_{\gamma, y}"] & \gamma \\ \gamma^{x_1} \times y \arrow[u, "\gamma^{\phi^-} \times \mathbf{1}_y"] \arrow[r, "\mathbf{1}_{\gamma^{x_1}} \times \phi^-"'] & \gamma^{x_1} \times x_1 \arrow[u, "\mathtt{eval}_{\gamma, x_1}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \gamma^y \times y \arrow[r, "\mathtt{eval}_{\gamma, y}"] & \gamma \\ \gamma^{x_2} \times y \arrow[u, "\gamma^{\psi^-} \times \mathbf{1}_y"] \arrow[r, "\mathbf{1}_{\gamma^{x_2}} \times \psi^-"'] & \gamma^{x_2} \times x_2 \arrow[u, "\mathtt{eval}_{\gamma, x_2}"'] \end{tikzcd}$$

commute. As $\pi_1^+ \colon R \to a_1$ and $\pi_2^+ \colon R \to a_2$, we know that $\widehat{f_1} \circ \pi_1^+$ and $\widehat{f_2} \circ \pi_2$ are the unique arrows making

$$\begin{tikzcd} \gamma^{x_1} \times x_1 \arrow[r, "\mathtt{eval}_{\gamma, x_1}"] & \gamma \\ a_1 \times x_1 \arrow[u, "\widehat{f_1} \times \mathbf{1}_{x_1}"] \arrow[ur, "f_1"'] & \\ R \times x_1 \arrow[u, "\pi_1^+ \times \mathbf{1}_{x_1}"] \arrow[uur, bend right, "f_1 \circ (\pi_1^+ \times \mathbf{1}_{x_1})"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \gamma^{x_2} \times x_2 \arrow[r, "\mathtt{eval}_{\gamma, x_2}"] & \gamma \\ a_2 \times x_2 \arrow[u, "\widehat{f_2} \times \mathbf{1}_{x_2}"] \arrow[ur, "f_2"'] & \\ R \times x_2 \arrow[u, "\pi_2^+ \times \mathbf{1}_{x_2}"] \arrow[uur, bend right, "f_2 \circ (\pi_2^+ \times \mathbf{1}_{x_2})"'] & \end{tikzcd}$$

commute. Using ϕ^- and ψ^- we obtain commutative diagrams

$$\begin{tikzcd}
\gamma^{x_1} \times y \arrow[r, "\mathbf{1}_{\gamma^{x_1}} \times \phi^-"] & \gamma^{x_1} \times x_1 \arrow[r, "\mathtt{eval}_{\gamma,x_1}"] & \gamma \\
a_1 \times y \arrow[u, "\widehat{f}_1 \times \mathbf{1}_y"] \arrow[r, dashed, "\mathbf{1}_{a_1} \times \phi^-"] & a_1 \times x_1 \arrow[u, "\widehat{f}_1 \times \mathbf{1}_{x_1}"] \arrow[ur, "f_1"'] & \\
R \times y \arrow[u, "\pi_1^+ \times \mathbf{1}_y"] \arrow[r, "\mathbf{1}_R \times \phi^-"'] & R \times x_1 \arrow[u, "\pi_1^+ \times \mathbf{1}_{x_1}"] \arrow[uur, bend right, "f_1 \circ (\pi_1^+ \times \mathbf{1}_{x_1})"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\gamma^{x_2} \times y \arrow[r, "\mathbf{1}_{\gamma^{x_2}} \times \psi^-"] & \gamma^{x_2} \times x_2 \arrow[r, "\mathtt{eval}_{\gamma,x_2}"] & \gamma. \\
a_2 \times y \arrow[u, "\widehat{f}_2 \times \mathbf{1}_y"] \arrow[r, dashed, "\mathbf{1}_{a_2} \times \psi^-"] & a_2 \times x_2 \arrow[u, "\widehat{f}_2 \times \mathbf{1}_{x_2}"] \arrow[ur, "f_2"'] & \\
R \times y \arrow[u, "\pi_2^+ \times \mathbf{1}_y"'] \arrow[r, "\mathbf{1}_R \times \psi^-"] & R \times x_2 \arrow[u, "\pi_2^+ \times \mathbf{1}_{x_2}"] \arrow[uur, bend right, "f_2 \circ (\pi_2^+ \times \mathbf{1}_{x_2})"'] &
\end{tikzcd}$$

If we can now show that

$$f_1 \circ (\pi_1^+ \times \mathbf{1}_{x_1}) \circ (\mathbf{1}_R \times \phi^-) = f_2 \circ (\pi_2^+ \times \mathbf{1}_{x_2}) \circ (\mathbf{1}_R \times \psi^-),$$

we are finished, as then $\gamma^{\phi^-} \circ \widehat{f}_1 \circ \pi_1^-$ and $\gamma^{\psi^-} \circ \widehat{f}_2 \circ \pi_2^+$ are defined by the same exponential equation, hence are the same by the uniqueness of a solution. So we compute

$$\begin{aligned}
f_1 \circ (\pi_1^+ \times \mathbf{1}_{x_1}) \circ (\mathbf{1}_R \times \phi^-) &= f_1 \circ (\mathbf{1}_{a_1} \times \phi^-) \circ (\pi_1^+ \times \mathbf{1}_y) \\
&= f_2 \circ (\phi^+ \times \mathbf{1}_y) \circ (\pi_1^+ \times \mathbf{1}_y) \\
&= f_2 \circ \big((\phi^+ \circ \pi_1^+) \times \mathbf{1}_y\big) \\
&= f_2 \circ \big((\psi^+ \circ \pi_2^+) \times \mathbf{1}_y\big) \\
&= f_2 \circ (\psi^+ \times \mathbf{1}_y) \circ (\pi_2^+ \times \mathbf{1}_y) \\
&= f_2 \circ (\mathbf{1}_{a_2} \times \psi^-) \circ (\pi_2^+ \times \mathbf{1}_y) \\
&= f_2 \circ (\pi_2^+ \times \mathbf{1}_{x_2}) \circ (\mathbf{1}_R \times \psi^-).
\end{aligned}$$

Therefore (26) commutes, and as γ^P is a pullback, we obtain a unqiue arrow $\rho\colon R \to \gamma^P$ such that

$$\begin{tikzcd}
R \arrow[dr, dotted, "\rho"] \arrow[drr, bend left, "\widehat{f}_1 \circ \pi_1^+"] \arrow[ddr, bend right, "\widehat{f}_2 \circ \pi_2^+"'] & & \\
& \gamma^P \arrow[r, "\gamma^{\pi_1^-}"] \arrow[d, "\gamma^{\pi_2^-}"] & \gamma^{x_1} \arrow[d, "\gamma^{\psi^-}"] \\
& \gamma^{x_2} \arrow[r, "\gamma^{\psi^-}"'] & \gamma^y
\end{tikzcd}$$

commutes. Now we can transpose ρ to obtain an arrow $h\colon P \times R \to \gamma$. So $(R, h, P) \in \mathsf{Chu}(\mathscr{C}, \gamma)$ and by definition the diagram

$$\begin{tikzcd}
(R, h, P) \arrow[r, "(\pi_1^+, \pi_1^-)"] \arrow[d, "(\pi_2^+, \pi_2^-)"'] & (a_1, f_1, x_1) \arrow[d, "(\phi^+, \phi^-)"] \\
(a_2, f_2, x_2) \arrow[r, "(\psi^+, \psi^-)"] & (b, g, y)
\end{tikzcd}$$

commutes. It remains to show that (R, h, P) is indeed a pullback. For this let $(Q, v, W) \in \mathsf{Chu}(\mathscr{C}, \gamma)$ together with morphisms

$$(\xi_1^+, \xi_1^-)\colon (Q, v, W) \to (a_1, f_1, x_1) \quad \text{and} \quad (\xi_2^+, \xi_2^-)\colon (Q, v, W) \to (a_2, f_2, x_2)$$

be given, such that

$$\begin{tikzcd}
(Q,v,W) \arrow[r, "{(\xi_1^+,\xi_1^-)}"] \arrow[d, "{(\xi_2^+,\xi_2^-)}"'] & (a_1,f_1,x_1) \arrow[d, "{(\phi^+,\phi^-)}"] \\
(a_2,f_2,x_2) \arrow[r, "{(\psi^+,\psi^-)}"] & (b,g,y)
\end{tikzcd}$$

commutes. We then have to find a unique arrow $(\mu^+, \mu^-)\colon (Q, v, W) \to (R, h, P)$ such that

$$\begin{tikzcd}
(Q,v,W) \arrow[dr, dotted, "{(\mu^+,\mu^-)}"] \arrow[drr, bend left, "{(\xi_1^+,\xi_1^-)}"] \arrow[ddr, bend right, "{(\xi_2^+,\xi_2^-)}"'] & & \\
& (R,h,P) \arrow[r, "{(\pi_1^+,\pi_1^-)}"] \arrow[d, "{(\pi_2^+,\pi_2^-)}"] & (a_1,f_1,x_1) \arrow[d, "{(\phi^+,\phi^-)}"] \\
& (a_2,f_2,x_2) \arrow[r, "{(\psi^+,\psi^-)}"'] & (b,g,y)
\end{tikzcd}$$

commutes. As R is a pullback and P is a pushout, we obtain arrows $\mu^+\colon Q \to R$, $\mu^-\colon P \to W$ such that the diagrams

$$\begin{tikzcd}
Q \arrow[dr, dotted, "\mu^+"] \arrow[drr, bend left, "\xi_1^+"] \arrow[ddr, bend right, "\xi_2^+"'] & & \\
& R \arrow[r, "\pi_1^+"] \arrow[d, "\pi_2^+"'] & a_1 \arrow[d, "\phi^+"] \\
& a_2 \arrow[r, "\psi^+"] & b
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
W & & \\
& P \arrow[ul, dotted, "\mu^-"'] & x_1 \arrow[l, "\pi_1^-"'] \arrow[ull, bend right, "\xi_1^-"'] \\
& x_2 \arrow[u, "\pi_2^-"] \arrow[uul, bend left, "\xi_2^-"] & y \arrow[l, "\psi^-"] \arrow[u, "\phi^-"']
\end{tikzcd}$$

commute. It remains to check that $(\mu^+, \mu^-)\colon (Q, v, W) \to (R, h, P)$ is an arrow in $\mathsf{Chu}(\mathscr{C}, \gamma)$, i.e. that

$$\begin{tikzcd}
Q \times P \arrow[r, "\mu^+ \times \mathbf{1}_P"] \arrow[d, "\mathbf{1}_Q \times \mu^-"'] & R \times P \arrow[d, "h"] \\
Q \times W \arrow[r, "v"] & \gamma
\end{tikzcd}$$

commutes. We already know that (ξ_1^+, ξ_1^-) and (ξ_2^+, ξ_2^-) are arrows in $\mathsf{Chu}(\mathscr{C}, \gamma)$, so the diagrams

$$\begin{tikzcd}
Q \times x_1 \arrow[r, "\xi_1^+ \times \mathbf{1}_{a_1}"] \arrow[d, "\mathbf{1}_Q \times \xi_1^-"'] & a_1 \times x_1 \arrow[d, "f_1"] \\
Q \times W \arrow[r, "v"] & \gamma
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Q \times x_2 \arrow[r, "\xi_2^+ \times \mathbf{1}_{x_2}"] \arrow[d, "\mathbf{1}_Q \times \xi_2^-"'] & a_2 \times x_2 \arrow[d, "f_2"] \\
Q \times W \arrow[r, "v"] & \gamma
\end{tikzcd}$$

commute. Furthermore we know that $(\xi_1^+, \xi_1^-) = (\pi_1^+, \pi_1^-) \circ (\mu^+, \mu^-)$ as well as $(\xi_2^+, \xi_2^-) = (\pi_2^+, \pi_2^-) \circ (\mu^+, \mu^-)$. This allows us the derive that

$$\begin{aligned}
f_2 \circ (\xi_2^+ \times \mathbf{1}_{x_2}) &= f_2 \circ \big((\pi_2^+ \circ \mu^+) \times \mathbf{1}_{x_2}\big) \\
&= f_2 \circ (\pi_2^+ \times \mathbf{1}_{x_2}) \circ (\mu^+ \times \mathbf{1}_{x_2}) \\
&= h \circ (\mathbf{1}_R \times \pi_2^-) \circ (\mu^+ \times \mathbf{1}_{x_2})
\end{aligned}$$

$$= h \circ (\mu^+ \times \mathbf{1}_P) \circ (\mathbf{1}_Q \times \pi_2^-),$$

because (π_2^+, π_2^-) is an arrow in $\mathsf{Chu}(\mathscr{C}, \gamma)$. Furthermore we can compute

$$\begin{aligned} v \circ (\mathbf{1}_Q \times \xi_2^-) &= v \circ \big(\mathbf{1}_Q \times (\mu^- \circ \pi_2^-)\big) \\ &= v \circ (\mathbf{1}_Q \times \mu^-) \circ (\mathbf{1}_Q \times \pi_2^-). \end{aligned}$$

We can also do these computations with ξ_1^+ in place of ξ_2^+ and ξ_1^- in place of ξ_2^- and obtain

$$\begin{aligned} f_1 \circ (\xi_1^+ \times \mathbf{1}_{x_1}) &= f_1 \circ \big((\pi_1^+ \circ \mu^+) \times \mathbf{1}_{x_1}\big) \\ &= f_1 \circ (\pi_1^+ \times \mathbf{1}_{x_1}) \circ (\mu^+ \times \mathbf{1}_{x_1}) \\ &= h \circ (\mathbf{1}_R \times \pi_1^-) \circ (\mu^+ \times \mathbf{1}_{x_1}) \\ &= h \circ (\mu^+ \times \mathbf{1}_P) \circ (\mathbf{1}_Q \times \pi_1^-) \\ &\text{as well as} \\ v \circ (\mathbf{1}_Q \times \xi_1^-) &= v \circ \big(\mathbf{1}_Q \times (\mu^- \circ \pi_1^-)\big) \\ &= v \circ (\mathbf{1}_Q \times \mu^-) \circ (\mathbf{1}_Q \times \pi_1^-). \end{aligned}$$

Now we can use lemma 6.25 in the following way. First we get the commutative diagram

$$\begin{tikzcd} Q \times P & Q \times x_1 \arrow[l, "\mathbf{1}_Q \times \pi_1^-"'] \\ Q \times x_2 \arrow[u, "\mathbf{1}_Q \times \pi_2^-"] & Q \times y, \arrow[l, "\mathbf{1}_Q \times \psi^-"] \arrow[u, "\mathbf{1}_Q \times \phi^-"'] \end{tikzcd}$$

where $Q \times P$ is a pushout. So we have $f_2 \circ (\xi_2^+ \times \mathbf{1}_{x_2})\colon Q \times x_2 \to \gamma$ and $f_1 \circ (\xi_1^+ \times \mathbf{1}_{x_1})\colon Q \times x_1 \to \gamma$ such that

$$\begin{tikzcd} \gamma & & \\ & Q \times P & Q \times x_1 \arrow[l, "\mathbf{1}_Q \times \pi_1^-"'] \arrow[llu, bend right, "f_1 \circ (\xi_1^+ \times \mathbf{1}_{x_1})"'] \\ & Q \times x_2 \arrow[u, "\mathbf{1}_Q \times \pi_2^-"'] \arrow[luu, bend left, "f_2 \circ (\xi_2^+ \times \mathbf{1}_{x_2})"] & Q \times y, \arrow[l, "\mathbf{1}_Q \times \psi^-"] \arrow[u, "\mathbf{1}_Q \times \phi^-"'] \end{tikzcd}$$

commutes. Now by the preceding computations we know that both

$$\begin{tikzcd} \gamma & & \\ & Q \times P \arrow[lu, dotted, "h \circ (\mu^+ \times \mathbf{1}_P)"'] & Q \times x_1 \arrow[l, "\mathbf{1}_Q \times \pi_1^-"'] \arrow[llu, "f_1 \circ (\xi_1^+ \times \mathbf{1}_{x_1})"'] \\ & Q \times x_2 \arrow[u, "\mathbf{1}_Q \times \pi_2^-"'] \arrow[luu, "f_2 \circ (\xi_2^+ \times \mathbf{1}_{x_2})"] & Q \times y, \arrow[l, "\mathbf{1}_Q \times \psi^-"] \arrow[u, "\mathbf{1}_Q \times \phi^-"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \gamma & & \\ & Q \times P \arrow[lu, dotted, "v \circ (\mathbf{1}_Q \times \mu^-)"'] & Q \times x_1 \arrow[l, "\mathbf{1}_Q \times \pi_1^-"'] \arrow[llu, "f_1 \circ (\xi_1^+ \times \mathbf{1}_{x_1})"'] \\ & Q \times x_2 \arrow[u, "\mathbf{1}_Q \times \pi_2^-"'] \arrow[luu, "f_2 \circ (\xi_2^+ \times \mathbf{1}_{x_2})"] & Q \times y, \arrow[l, "\mathbf{1}_Q \times \psi^-"] \arrow[u, "\mathbf{1}_Q \times \phi^-"'] \end{tikzcd}$$

commute, so we by uniqueness we get the desired equality

$$h \circ (\mu^+ \times \mathbf{1}_P) = v \circ (\mathbf{1}_Q \times \mu^-),$$

so $(\mu^+, \mu^-)\colon (Q, v, W) \to (R, h, P)$ is an arrow in $\mathsf{Chu}(\mathscr{C}, \gamma)$. It is necessarily unique, as μ^+ and μ^- are unique. We have now shown that (R, h, P) is a pullback.

To see that $\mathsf{Chu}(\mathscr{C},\gamma)$ has pushouts, let a diagram[6]

$$\begin{tikzcd} & (a_1,f_1,x_1) \\ (a_2,f_2,x_2) & (b,g,y) \arrow[l, "{(\psi^+,\psi^-)}"'] \arrow[u, "{(\phi^+,\phi^-)}"] \end{tikzcd}$$

with $(a_1,f_1,x_1),(a_2,f_2,x_2),(b,g,y) \in \mathsf{Chu}(\mathscr{C},\gamma)_0$ and $(\phi^+,\phi^-),(\psi^+,\psi^-) \in \mathsf{Chu}(\mathscr{C},\gamma)_1$ be given. Now we can turn this into a diagram

$$\begin{tikzcd} & (x_1,f_1,a_1) \arrow[d, "{(\phi^-,\phi^+)}"] \\ (x_2,f_2,a_2) \arrow[r, "{(\psi^-,\psi^+)}"'] & (y,g,b) \end{tikzcd}$$

as $\mathsf{Chu}(\mathscr{C},\gamma) = \mathsf{Chu}(\mathscr{C},\gamma)^{\,\mathrm{op}}$ by lemma 5.1. By the preceding paragraph we obtain $(R,h,P) \in \mathsf{Chu}(\mathscr{C},\gamma)_0$ and $(\pi_1^+,\pi_1^-),(\pi_2^+,\pi_2^-) \in \mathsf{Chu}(\mathscr{C},\gamma)_1$ such that

$$\begin{tikzcd} (R,h,P) \arrow[r, "{(\pi_1^-,\pi_1^+)}"] \arrow[d, "{(\pi_2^+,\pi_2^-)}"] & (x_1,f_1,a_1) \arrow[d, "{(\phi^-,\phi^+)}"] \\ (x_2,f_2,a_2) \arrow[r, "{(\psi^-,\psi^+)}"'] & (y,g,b) \end{tikzcd}$$

is a commutative diagram and (R,h,P) is a pullback. Now it is immediate that

$$\begin{tikzcd} (P,h,R) & (a_1,f_1,x_1) \arrow[l, "{(\pi_1^+,\pi_1^-)}"'] \\ (a_2,f_2,x_2) \arrow[u, "{(\pi_2^+,\pi_2^-)}"] & (b,g,y) \arrow[l, "{(\psi^+,\psi^-)}"'] \arrow[u, "{(\phi^+,\phi^-)}"] \end{tikzcd}$$

commutes and (P,h,R) is a pushout by lemma 6.17 and corollary 6.20. Q.E.D.

[6] I will now reuse all variables used in the previous paragraph, but these are chosen independently from those.

Chapter 7

Generalizations of the Chu construction

In this chapter we want to generalize the Chu categories, as the title predicts. We will do this in various ways. For this we start by examining what possibilities for generalization we have ad hoc.

- As an object in $\mathsf{Chu}(\mathscr{C}, \gamma)$ is a triplet (a, f, x), where $a, x \in \mathscr{C}_0$ and $f\colon a \times x \to \gamma$, the first thing one could do is to allow different codomains γ of f. This idea is made precise by the generalized Chu category, discussed in section 7.1.
- Another idea is to allow $a \in \mathscr{C}_0$ and $x \in \mathscr{D}_0$ for a different cartesian closed category $\mathscr{D}$. To make this precise we use the identification of the Chu construction with a Grothendieck construction and generalize the equivalent Grothendieck construction accordingly.
- At last one could could wish not only to consider $f\colon a \times x \to \gamma$, but arrows $f\colon \prod_{i=1}^{n} a_i \to \gamma$ for $a_i \in \mathscr{C}$.

7.1 The generalized Chu category

We first generalize the Chu category in the first sense we discussed. This gives rise to the following definition. The following definition is taken from [Pet21].

Definition 7.1. Let $\mathscr{C}$ be a cartesian closed category and $\Gamma\colon \mathscr{C} \to \mathscr{C}$ be an endofunctor. Then $\mathsf{Chu}(\mathscr{C}, \Gamma)$ is the category given by the following data:

- The *objects* of $\mathsf{Chu}(\mathscr{C}, \Gamma)$ are quadruples $(x; a, f, b)$ where $a, b, x \in \mathscr{C}_0$ and $f\colon a \times b \to \Gamma_0(x)$.
- The *arrows* of $\mathsf{Chu}(\mathscr{C}, \Gamma)$ are triplets $(\phi^0, \phi^+, \phi^-)\colon (x; a, f, b) \to (y; c, g, d)$ where $\phi^0\colon x \to y$, $\phi^+\colon a \to c$ and $\phi^-: d \to b$ such that the diagram

$$\begin{tikzcd} a \times d \arrow[rr, "\phi^+ \times \mathbf{1}_d"] \arrow[d, "\mathbf{1}_a \times \phi^-"'] & & c \times d \arrow[d, "g"] \\ a \times b \arrow[r, "f"'] & \Gamma_0(x) \arrow[r, "\Gamma_1(\phi^0)"'] & \Gamma_0(y) \end{tikzcd}$$

  commutes.
- The *composition* of two arrows

$$(\phi^0, \phi^+, \phi^-)\colon (x; a, f, b) \to (y; c, g, d), \qquad (\psi^0, \psi^+, \psi^-)\colon (y; c, g, d) \to (z; s, h, t)$$

  is given by $(\psi^0, \psi^+, \psi^-) \circ (\phi^0, \phi^+, \phi^-) = (\psi^0 \circ \phi^0, \psi^+ \circ \phi^+, \phi^- \circ \psi^-)$.
- The *identities* are $\mathbf{1}_{(x;a,f,b)} = (\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b)$.

Lemma 7.2. *Let $\mathscr{C}$ be a cartesian closed category and $\Gamma\colon \mathscr{C} \to \mathscr{C}$ be an endofunctor. Then $\mathsf{Chu}(\mathscr{C}, \Gamma)$ as defined above is a category.*

Proof: We first show that the composition of two arrows is well-defined. Let

$$(\phi^0, \phi^+, \phi^-)\colon (x; a, f, b) \to (y; c, g, d),$$
$$(\psi^0, \psi^+, \psi^-)\colon (y; c, g, d) \to (z; i, h, j).$$

Then $(\psi^0, \psi^+, \psi^-) \circ (\phi^0, \phi^+, \phi^-) = (\psi^0 \circ \phi^0, \psi^+ \circ \phi^+, \phi^- \circ \psi^-)$ and we have to show the commutativity of the diagram

$$\begin{tikzcd}
a \times j \arrow[r, "\mathbf{1}_a \times (\phi^- \circ \psi^-)"] \arrow[dd, "(\psi^+ \circ \phi^+) \circ \mathbf{1}_j"'] & a \times b \arrow[d, "f"] \\
 & \Gamma_0(x) \arrow[d, "\Gamma_1(\psi^0 \circ \phi^0)"] \\
i \times j \arrow[r, "h"] & \Gamma_0(z).
\end{tikzcd}$$

To achieve this we embed the diagram in a bigger diagram, namely

$$\begin{tikzcd}
 & a \times d \arrow[r, "\mathbf{1}_a \times \phi^-"] \arrow[dd, "\phi^+ \times \mathbf{1}_d"'] & a \times b \arrow[d, "f"] & \\
 & \arrow[r, phantom, "\square_1"] & \Gamma_0(x) \arrow[d, "\Gamma_1(\phi^0)"'] \arrow[dd, bend left=60, "\Gamma_1(\psi^0 \circ \phi^0)"] & \\
a \times j \arrow[uur, bend left, "\mathbf{1}_a \times \psi^-"] \arrow[ddr, bend right, "\phi^+ \times \mathbf{1}_j"'] \arrow[r, phantom, "\square_2"] & c \times d \arrow[r, "g"] & \Gamma_0(y) \arrow[d, "\Gamma_1(\psi^0)"'] \arrow[r, phantom, "\triangle_1"] & {} \\
 & \arrow[r, phantom, "\square_3"] & \Gamma_0(z) & \\
 & c \times j \arrow[r, "\psi^+ \times \mathbf{1}_j"'] \arrow[uu, "\mathbf{1}_c \times \psi^-"] & i \times j, \arrow[u, "h"'] &
\end{tikzcd}$$

where we already know that $\square_1, \square_3, \triangle_1$ commute. But we can deduce that $\square_2$ commutes, as $(\phi^+ \times \mathbf{1}_d) \circ (\mathbf{1}_a \times \psi^-) = (\mathbf{1}_c \times \psi^-) \circ (\phi^+ \times \mathbf{1}_j)$. This shows the desired equality.
Identities: Let $(x; a, f, b) \in \mathsf{Chu}(\mathscr{C}, \Gamma)_0$. We want to show that $\mathbf{1}_{(x;a,f,b)} = (\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b)$. First we observe that this is actually well-defined, as

$$\begin{tikzcd}
a \times b \arrow[r, "\mathbf{1}_a \times \mathbf{1}_b"] \arrow[dd, "\mathbf{1}_a \times \mathbf{1}_b"'] & a \times b \arrow[d, "f"] \\
 & \Gamma_0(x) \arrow[d, "\Gamma_1(\mathbf{1}_x)"] \\
a \times b \arrow[r, "f"] & \Gamma_0(x)
\end{tikzcd}$$

commutes because $\Gamma_1(\mathbf{1}_x) = \mathbf{1}_{\Gamma_0(x)}$. Now let two arrows

$$(\phi^0, \phi^+, \phi^-)\colon (x; a, f, b) \to (y; c, g, d),$$
$$(\psi^0, \psi^+, \psi^-)\colon (z; i, h, j) \to (x; a, f, b)$$

be given. We then compute

$$(\phi^0, \phi^+, \phi^-) \circ \mathbf{1}_{(x;a,f,b)} = (\phi^0, \phi^+, \phi^-) \circ (\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b)$$

$$
\begin{aligned}
&= (\phi^0 \circ \mathbf{1}_x, \phi^+ \circ \mathbf{1}_a, \mathbf{1}_b \circ \phi^-)\\
&= (\phi^0, \phi^+, \phi^-),\\
\mathbf{1}_{(x;a,f,b)} \circ (\psi^0, \psi^+, \psi^-) &= (\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b) \circ (\psi^0, \psi^+, \psi^-)\\
&= (\mathbf{1}_x \circ \psi^0, \mathbf{1}_a \circ \psi^+, \psi^- \circ \mathbf{1}_b)\\
&= (\psi^0, \psi^+, \psi^-),
\end{aligned}
$$

therefore the requirements of an identity are fulfilled. The existence is secured through the existence of the identities in $\mathscr{C}$.
Associativity: Assume we are given objects and arrows

$$(x; a, m, b) \xrightarrow{(\phi^0,\phi^+,\phi^-)} (y; c, n, d) \xrightarrow{(\psi^0,\psi^+,\psi^-)} (z; e, o, f) \xrightarrow{(\theta^0,\theta^+,\theta^-)} (s; g, p, h)$$

in $\mathsf{Chu}(\mathscr{C}, \Gamma)$. Then we can compute

$$
\begin{aligned}
&\big((\theta^0, \theta^+, \theta^-) \circ (\psi^0, \psi^+, \psi^-)\big) \circ (\phi^0, \phi^+, \phi^-)\\
&\quad= (\theta^0 \circ \psi^0, \theta^+ \circ \psi^+, \psi^- \circ \theta^-) \circ (\phi^0, \phi^+, \phi^-)\\
&\quad= \big((\theta^0 \circ \psi^0) \circ \phi^0, (\theta^+ \circ \psi^+) \circ \phi^+, \phi^- \circ (\psi^- \circ \theta^-)\big)\\
&\quad= \big(\theta^0 \circ (\psi^0 \circ \phi^0), \theta^+ \circ (\psi^+ \circ \phi^+), (\phi^- \circ \psi^-) \circ \theta^-\big)\\
&\quad= (\theta^0, \theta^+, \theta^-) \circ (\psi^0 \circ \phi^0, \psi^+ \circ \phi^+, \phi^- \circ \psi^-)\\
&\quad= (\theta^0, \theta^+, \theta^-) \circ \big((\psi^0, \psi^+, \psi^-) \circ (\phi^0, \phi^+, \phi^-)\big),
\end{aligned}
$$

and the associativity is shown. Q.E.D.

Remark 7.3. One can observe that the generalized Chu category is indeed a generalization of the standard Chu category, as we can choose $\Gamma^\gamma \colon \mathscr{C} \to \mathscr{C}$ for a fixed $\gamma \in \mathscr{C}_0$, defined by $\Gamma_0(x) = \gamma, \Gamma_1(f) = \mathbf{1}_\gamma$ for all $x \in \mathscr{C}_0, f \in \mathscr{C}_1$.

Our next goal is to also generalize the functors associated with the standard Chu construction to the generalized Chu category, namely the internal Chu functor and the global Chu functor.

7.2 The internal, generalized Chu functor

We mimic the internal, generalized Chu functor. The following definition is taken from [Pet21, p.79].

Definition 7.4. Let $\mathscr{C}$ be a cartesian closed category. We define $\mathsf{CHU}^{\mathscr{C}} \colon \mathsf{Fun}(\mathscr{C}, \mathscr{C}) \to \mathsf{Cat}$ by the following axioms:

- For all functors $\Gamma \in \mathsf{Fun}(\mathscr{C}, \mathscr{C})_0$ we have $\mathsf{CHU}^{\mathscr{C}}_0(\Gamma) = \mathsf{Chu}(\mathscr{C}, \Gamma)$.
- For all natural transformations $\eta \in \mathsf{Fun}(\mathscr{C}, \mathscr{C})_1, \eta \colon \Gamma \Rightarrow \Delta$ we define $\mathsf{CHU}^{\mathscr{C}}_1$ to be the functor $\mathsf{CHU}^{\mathscr{C}}_1(\eta) \colon \mathsf{Chu}(\mathscr{C}, \Gamma) \to \mathsf{Chu}(\mathscr{C}, \Delta)$ given by the following data:
  - For all $(x; a, f, b)$ we set $\big[\mathsf{CHU}^{\mathscr{C}}_1(\eta)\big]_0(x; a, f, b) = (x; a, \eta_x \circ f, b)$.
  - For all $(\phi^0, \phi^+, \phi^-) \colon (x; a, f, b) \to (y; c, g, d)$ we define $\big[\mathsf{CHU}^{\mathscr{C}}_1(\eta)\big]_1(\phi^0, \phi^+, \phi^-) = (\phi^0, \phi^+, \phi^-)$.

Proposition 7.5. $\mathsf{CHU}^{\mathscr{C}}$ *is a functor for every cartesian closed category.*

Proof: We first check that $\mathsf{CHU}^{\mathscr{C}}$ is well-defined. For this let a natural transformation $\eta \colon \Gamma \Rightarrow \Delta$ be given. We have to show that $\mathsf{CHU}^{\mathscr{C}}_1(\eta)$ is a functor $\mathsf{Chu}(\mathscr{C}, \Gamma) \to \mathsf{Chu}(\mathscr{C}, \Delta)$. For this we show the following:

Firstly we observe that $[\mathsf{CHU}_1^{\mathscr{C}}(\eta)]_0(x;a,f,b) \in \mathsf{Chu}(\mathscr{C},\Delta)_0$, as

$$a \times b \xrightarrow{f} \Gamma_0(x) \xrightarrow{\eta_x} \Delta_0(x).$$

$\mathsf{CHU}_1^{\mathscr{C}}(\eta)$ preserves identities, as for $(x;a,f,b) \in \mathsf{Chu}(\mathscr{C},\Gamma)$ we have $\mathbf{1}_{(x;a,f,b)} = (\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b)$ and

$$\big(\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_1\big(\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b)\big) = (\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b) = (\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b) = \mathbf{1}_{(x;a,\eta_x \circ f,b)},$$

so the preservation of identities is shown.

Analogously we have

$$\begin{aligned}\big(\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_1&\big((\phi^0,\phi^+,\phi^-)\circ(\psi^0,\psi^+,\psi^-)\big)\\ &= (\phi^0,\phi^+,\phi^-)\circ(\psi^0,\psi^+,\psi^-)\\ &= \big(\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_1(\phi^0,\phi^+,\phi^-)\circ\big(\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_1(\psi^0,\psi^+,\psi^-).\end{aligned}$$

Therefore $\mathsf{CHU}_1^{\mathscr{C}}(\eta)$ is a functor.
Preservation of identities: Let $\Gamma \in \mathsf{Fun}(\mathscr{C},\mathscr{C})$. Then $\mathbf{1}_\Gamma$ is the family $(\mathbf{1}_{\Gamma(C)})_{C\in\mathscr{C}_0}$ making the diagrams

$$\begin{tikzcd} \Gamma_0(C) \arrow[r, "\Gamma_1(f)"] \arrow[d, "\mathbf{1}_{\Gamma(C)}"'] & \Gamma_0(C') \arrow[d, "\mathbf{1}_{\Gamma_0(C')}"] \\ \Gamma_0(C) \arrow[r, "\Gamma_1(f)"] & \Gamma_0(C') \end{tikzcd}$$

commute. Now by definition

$$\begin{aligned}\big(\mathsf{CHU}_1^{\mathscr{C}}(\mathbf{1}_\Gamma)\big)_0(x;a,f,b) &= (x;a,(\mathbf{1}_\Gamma)_x \circ f, b) = (x;a,\mathbf{1}_x\circ f,b)\\ &= (x;a,f,b),\\ \big(\mathsf{CHU}_1^{\mathscr{C}}(\mathbf{1}_\Gamma)\big)_1(\phi^0,\phi^+,\phi^-) &= (\phi^0,\phi^+,\phi^-),\end{aligned}$$

therefore $\mathsf{CHU}_1^{\mathscr{C}}(\mathbf{1}_\Gamma) = \mathrm{id}^{\mathsf{Chu}(\mathscr{C},\Gamma)}$.
Compatibility with composition: Assume we are given $\Gamma,\Lambda,\Omega \in \mathsf{Fun}(\mathscr{C},\mathscr{C})_0$ and $\eta\colon \Gamma \Rightarrow \Lambda, \mu\colon \Lambda \Rightarrow \Omega$. We then seek to show that

$$\mathsf{CHU}_1^{\mathscr{C}}(\mu\circ\eta) = \mathsf{CHU}_1^{\mathscr{C}}(\mu)\circ\mathsf{CHU}_1^{\mathscr{C}}(\eta).$$

As $\big(\mathsf{CHU}_1^{\mathscr{C}}(\mu\circ\eta)\big)_0(\phi^0,\phi^+,\phi^-) = (\phi^0,\phi^+,\phi^-) = \big(\mathsf{CHU}_1^{\mathscr{C}}(\mu)\circ\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_0(\phi^0,\phi^+,\phi^-)$, it suffices to examine the action on the objects. We compute

$$\begin{aligned}\big(\mathsf{CHU}_1^{\mathscr{C}}(\mu)\circ\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_0(x;a,f,b) &= \big(\mathsf{CHU}_1^{\mathscr{C}}(\mu)\big)_0\Big(\big(\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_0(x;a,f,b)\Big)\\ &= \big(\mathsf{CHU}_1^{\mathscr{C}}(\mu)\big)_0(x;a,\eta_x\circ f,b)\\ &= (x;a,\mu_x\circ(\eta_x\circ f),b) = (x;a,(\mu_x\circ\eta_x)\circ g,b)\\ &= (x;a,(\mu\circ\eta)_x\circ f,b) = \big(\mathsf{CHU}_1^{\mathscr{C}}(\mu\circ\eta)\big)_0(x;a,f,b).\end{aligned}$$

Here we used that $(\mu_x\circ\eta_x) = (\mu\circ\eta)_x$, a fact found in [Pet20b, Definition 1.15.3]. Therefore $\mathsf{CHU}^{\mathscr{C}}$ is a functor. Q.E.D.

Proposition 7.6. *Let $\Gamma,\Delta \in \mathsf{Fun}(\mathscr{C},\mathscr{C})$ for a cartesian closed category $\mathscr{C}$. Let $\eta\colon\Gamma\Rightarrow\Delta$ be a natural transformation such that $\eta_x\colon \Gamma_0(x)\hookrightarrow\Delta_0(x)$ is a monomorphism for every $x\in\mathscr{C}_0$. Then $\mathsf{CHU}_1^{\mathscr{C}}(\eta)$ is a full embedding of $\mathsf{Chu}(\mathscr{C},\Gamma)$ into $\mathsf{Chu}(\mathscr{C},\Delta)$.*

Proof: We have already seen that $\mathsf{CHU}_1^{\mathscr{C}}(\eta)$ is a functor. It remains to show that it is injective on objects and that the rule

$$\begin{aligned}\big(\mathsf{CHU}_1^{\mathscr{C}}(\eta)\big)_{(X,Y)}\colon \mathsf{Chu}(\mathscr{C},\Gamma)_1\big(X,Y\big) &\to \mathsf{Chu}(\mathscr{C},\Delta)_1\big((x;a,\eta_x\circ f,b),(y;c,\eta_y\circ g,d)\big),\\ (\phi^0,\phi^+,\phi^-) &\mapsto \mathsf{CHU}_1^{\mathscr{C}}(\eta)(\phi^0,\phi^+,\phi^-) \end{aligned} \tag{27}$$

is a bijection for every $X=(x;a,f,b), Y=(y;c,g,d)$.

- To see that $\mathsf{CHU}_1^{\mathscr{C}}(\eta)$ is injective on objects, assume we have $(x;a,\eta_x\circ f,b)=(y,c,\eta_y\circ g,d)$. Then $x=y, a=c, b=d$, therefore $\eta_x=\eta_y$. It remains to show that $f=g$. This follows from the assumption that η_x is a monomorphism
 for every $x\in\mathscr{C}_0$.
- It is immediate that the rule given by (27) is injective. To see that it is surjective, let $(\phi^0,\phi^+,\phi^-)\colon (x;a,\eta_x\circ f,b)\to(y;c,g,d)$ be given. This means that $\phi^0\colon x\to y, \phi^+\colon a\to c$ and $\phi^-\colon d\to b$ such that the diagram

$$\begin{tikzcd} a\times d \arrow[rr, "\phi^+\times\mathbf{1}_d"] \arrow[d, "\mathbf{1}_a\times\phi^-"'] & \Box_1 & c\times d \arrow[d, "g"] \\ a\times b \arrow[r, "f"] & \Gamma_0(x) \arrow[r, "\Gamma_1(\phi^0)"] \arrow[d, "\eta_x"'] & \Gamma_0(y) \arrow[d, "\eta_y"] \\ & \Delta_0(x) \arrow[r, "\Delta_1(\phi^0)"] & \Delta_0(y) \end{tikzcd}$$

 commutes. But as the commutativity of $\Box_1$ is enough that (ϕ^0,ϕ^+,ϕ^-) is an arrow $(x;a,f,b)\to(y;c,g,d)$, the surjectivity is shown.

Therefore $\mathsf{CHU}_1^{\mathscr{C}}(\eta)$ is a full embedding. Q.E.D.

7.3 The generalized, global Chu functor

We now want to generalize the notion of the global Chu functor. So we have to define a functor $\mathsf{CHU}\colon \mathsf{Groth}(\mathscr{D},\mathcal{F})\to\mathsf{Cat}$ for a to be determined functor $\mathcal{F}$ and a to be determined category $\mathscr{D}$ such that $\mathsf{CHU}_0(\mathscr{C},\Gamma)=\mathsf{Chu}(\mathscr{C},\Gamma)$. On the way to such a functor we have to a address the following problem.

> *How can we generalize the lemma given in [Pet21, p. 27] concerning product preserving functors to* $\mathsf{Chu}(\mathscr{C},\Gamma)$*?*

To this end we make the following lemma.

Lemma 7.7. *Let $\mathscr{C},\mathscr{D}$ be cartesian closed categories, $\Gamma\colon\mathscr{C}\to\mathscr{C}, \Delta\colon\mathscr{D}\to\mathscr{D}$ and $F\colon\mathscr{C}\to\mathscr{D}$ be a product preserving functor with canonical isomorphism F_{ab} for $a,b\in\mathscr{C}_0$. Furthermore, let $\eta\colon F\circ\Gamma\Rightarrow\Delta\circ F$ be given. Then the rule $F_\vee\colon \mathsf{Chu}(\mathscr{C},\Gamma)\to\mathsf{Chu}(\mathscr{D},\Delta)$ defined by*

$$\begin{aligned}(F_\vee)_0(x;a,f,b)&=(F_0(x);F_0(a),\eta_x\circ F_1(f)\circ F_{ab},F_0(b)),\\ (F_\vee)_1((\phi^0,\phi^+,\phi^-)\colon(x;a,f,b)\to(y;c,g,d))&=(F_1(\phi^0),F_1(\phi^+),F_1(\phi^-))\end{aligned}$$

is a functor.

Proof: We first check the well-definedness. For this we have to show that

$$(F_\vee)_0(x;a,f,b)\in\mathsf{Chu}(\mathscr{D},\Delta)_0, \tag{28}$$
$$(F_\vee)_1(\phi^0,\phi^+,\phi^-)\in\mathsf{Chu}(\mathscr{D},\Delta)_1. \tag{29}$$

We first show (28). As $F_0(x), F_0(a), F_0(b) \in \mathscr{D}_0$ and

$$\eta_x \circ F_1(f) \circ F_{ab} \colon F_0(a) \times F_0(b) \to \Delta_0(F_0(x)),$$

this is immediate.

So it remains to check (29). To this end, let $(\phi^0, \phi^+, \phi^-)\colon (x; a, f, b) \to (y; c, g, d)$ be given. We have to check that the diagram

$$\begin{tikzcd}
F_0(a) \times F_0(d) \arrow[rr, "F_1(\phi^+)\times \mathbf{1}_{F_0(d)}"] \arrow[d, "\mathbf{1}_{F_0(a)}\times F_1(\phi^-)"'] & & F_0(c) \times F_0(d) \arrow[d, "\eta_y\circ F_1(g)\circ F_{cd}"] \\
F_0(a) \times F_0(b) \arrow[r, "\eta_x\circ F_1(f)\circ F_{ab}"] & \Delta_0(F_0(x)) \arrow[r, "\Delta_1(F_1(\phi^0))"] & \Delta_0(F_0(y))
\end{tikzcd}$$

commutes. We compute

$$\begin{aligned}
\eta_y \circ F_1(g) \circ F_{cd} \circ (F_1(\phi^+) \times \mathbf{1}_{F_0(d)}) &= \eta_y \circ F_1(g) \circ F_{cd} \circ (F_1(\phi^+) \times F_1(\mathbf{1}_d)) \\
&= \eta_y \circ F_1(g) \circ F_1(\phi^+ \times \mathbf{1}_d) \circ F_{ad} \\
&= \eta_y \circ F_1(g \circ (\phi^+ \times \mathbf{1}_d) \circ F_{ad} \\
&= \eta_y \circ F_1(\Gamma_1(\phi^0) \circ f \circ (\mathbf{1}_a \times \phi^-)) \circ F_{ad} \\
&= \eta_y \circ F_1(\Gamma_1(\phi^0)) \circ F_1(f) \circ F_1(\mathbf{1}_a \times \phi^-) \circ F_{ad} \\
&= \eta_y \circ F_1(\Gamma_1(\phi^0)) \circ F_1(f) \circ F_{ab} \circ (F_1(\mathbf{1}_a) \times F_1(\phi^-)) \\
&= \eta_y \circ F_1(\Gamma_1(\phi^0)) \circ F_1(f) \circ F_{ab} \circ (\mathbf{1}_{F_0(a)} \times F_1(\phi^-)) \\
&= \Delta_1(F_1(\phi^0)) \circ \eta_x \circ F_1(f) \circ F_{ab} \circ (\mathbf{1}_{F_0(a)} \times F_1(\phi^-))
\end{aligned}$$

Now we check the axioms of a functor.

- *Compatibility with composition:* Let

  $$(\phi^0, \phi^+, \phi^-)\colon (x; a, f, b) \to (y; c, g, d) \text{ and } (\theta^0, \theta^+, \theta^-)\colon (y; c, g, d) \to (z; s, h, t)$$

  be given. We then compute

  $$\begin{aligned}
  (F_\vee)_1\big((\theta^0, \theta^+, \theta^-) \circ (\phi^0, \phi^+, \phi^-)\big) &= \big(F_1(\theta^0 \circ \phi^0), F_1(\theta^+ \circ \phi^+), F_1(\phi^- \circ \theta^-)\big) \\
  &= \big(F_1(\theta^0) \circ F_1(\phi^0), F_1(\theta^+) \circ_1 (\phi^+), F_1(\phi^-) \circ_1 (\theta^-)\big) \\
  &= (F_\vee)_1\big(\theta^0, \theta^+, \theta^-\big) \circ (F_\vee)_1\big(\phi^0, \phi^+, \phi^-\big),
  \end{aligned}$$

  which proves the desired equality.
- *Preservation of identities:* It is immediate that

  $$(F_\vee)_1(\mathbf{1}_x, \mathbf{1}_a, \mathbf{1}_b) = (\mathbf{1}_{F_0(x)}, \mathbf{1}_{F_0(a)}, \mathbf{1}_{F_0(b)}),$$

  as F is a functor and preserves identities. Q.E.D.

Definition 7.8. We set $\mathsf{Groth}(\mathsf{ccCat}, \mathsf{Fun}(\text{-}, \text{-}))$ to be the category defined in the following way:

- The *objects* of $\mathsf{Groth}(\mathsf{ccCat}, \mathsf{Fun}(\text{-}, \text{-}))$ are pairs $(\mathscr{C}, \Gamma)$, where $\mathscr{C} \in \mathsf{ccCat}_0$ and $\Gamma \in \mathsf{Fun}(\mathscr{C}, \mathscr{C})_0$.
- The *arrows* of $\mathsf{Groth}(\mathsf{ccCat}, \mathsf{Fun}(\text{-}, \text{-}))$ are pairs $(F, \eta)\colon (\mathscr{C}, \Gamma) \to (\mathscr{D}, \Delta)$, such that $F\colon \mathscr{C} \to \mathscr{D}$ is a product preserving functor and $\eta\colon F \circ \Gamma \Rightarrow \Delta \circ F$ is a natural transformation.

- If $(H,\mu)\colon (\mathscr{D},\Delta) \to (\mathscr{E},\Xi)$ is another arrow in $\mathsf{Groth}(\mathsf{ccCat},\mathsf{Fun}(\text{-},\text{-}))$, then the *composition* $(H,\mu)\circ(F,\eta)\colon (\mathscr{C},\Gamma) \to (\mathscr{E},\Xi)$ is given by

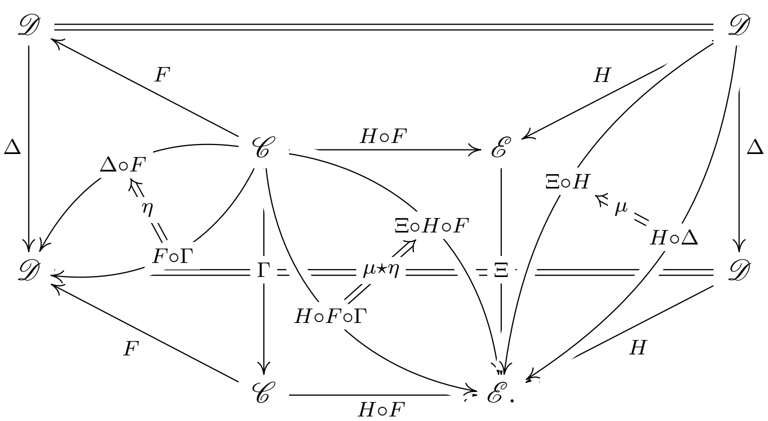
$$(H,\mu)\circ(F,\eta) = (H\circ F, \mu\star\eta),$$

where $\mu\star\eta$ is the natural transformation $\mu\star\eta\colon H\circ F\circ\Gamma \Rightarrow \Xi\circ H\circ F$ defined through the commutativity of the diagram

$$\begin{tikzcd}
H_0(F\circ\Gamma)_0(x) \arrow[r, "(H\circ F\circ\Gamma)_1(f)"] \arrow[d, "H_1(\eta_x)"'] \arrow[dd, bend right=80, "(\mu\star\eta)_x"'] & H_0(F\circ\Gamma)_0(y) \arrow[d, "H_1(\eta_y)"] \arrow[dd, bend left=80, "(\mu\star\eta)_y"] \\
H_0(\Delta\circ F)_0(x) \arrow[r, "(H\circ\Delta\circ F)_1(f)"] \arrow[d, "\mu_{F_0(x)}"'] & H_0(\Delta\circ F)_0(y) \arrow[d, "\mu_{F_0(y)}"] \\
(\Xi\circ H\circ F)_0(x) \arrow[r, "(\Xi\circ H\circ F)_1(f)"] & (\Xi\circ H\circ F)_0(y),
\end{tikzcd}$$

which means that $(\mu\star\eta)_x = \mu_{F_0(x)} \circ H_1(\eta_x)$ for all $x \in \mathscr{C}_0$.

One can further visualize $\mu\star\eta$ as



Lemma 7.9. $\mathsf{Groth}(\mathsf{ccCat},\mathsf{Fun}(\text{-},\text{-}))$ *as defined above is a category.*

Proof: We check the well-definedness of the composition. Let $(H,\mu)\colon (\mathscr{D},\Delta) \to (\mathscr{E},\Xi)$ and $(F,\nu)\colon (\mathscr{C},\Gamma) \to (\mathscr{D},\Delta)$. Then $H\circ F\colon \mathscr{C} \to \mathscr{E}$ and $\mu\star\eta$ is a natural transformation $H\circ F\circ\Gamma \Rightarrow \Xi\circ H\circ F$, as both the upper and the lower rectangle in the diagram

$$\begin{tikzcd}
H_0(F\circ\Gamma)_0(x) \arrow[r, "(H\circ F\circ\Gamma)_1(f)"] \arrow[d, "H_1(\eta_x)"'] & H_0(F\circ\Gamma)_0(y) \arrow[d, "H_1(\eta_y)"] \\
H_0(\Delta\circ F)_0(x) \arrow[r, "(H\circ\Delta\circ F)_1(f)"] \arrow[d, "\mu_{F_0(x)}"'] & H_0(\Delta\circ F)_0(y) \arrow[d, "\mu_{F_0(y)}"] \\
(\Xi\circ H\circ F)_0(x) \arrow[r, "(\Xi\circ H\circ F)_1(f)"] & (\Xi\circ H\circ F)_0(y),
\end{tikzcd}$$

commute, hence

$$\begin{aligned}
(\Xi\circ H\circ F)_1(f)\circ(\mu\star\eta)_x &= (\Xi\circ H\circ F)_1(f)\circ\mu_{F_0(x)}\circ H_1(\eta_x) \\
&= \mu_{F_0(y)}\circ(H\circ\Delta\circ F)_1(f)\circ H_1(\eta_x) \\
&= \mu_{F_0(y)}\circ H_1(\eta_y)\circ(H\circ F\circ\Gamma)_1(f)
\end{aligned}$$

$$= (\mu \star \eta)_y \circ (H \circ F \circ \Gamma)_1(f).$$

To see the existence of identities, one checks that $(\mathbf{1}_{\mathscr{C}}, \mathbf{1}_\Gamma) = (\mathrm{id}^{\mathscr{C}}, (\mathbf{1}_{\Gamma_0(x)})_{x\in\mathscr{C}_0})$. For this, let $(F,\eta)\colon (\mathscr{B},\Lambda) \to (\mathscr{C},\Gamma)$ and $(G,\mu)\colon (\mathscr{C},\Gamma) \to (\mathscr{D},\Delta)$ be given. Then

$$(\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\Gamma) \circ (F,\eta) = (\mathrm{id}^{\mathscr{C}} \circ F, \mathbf{1}_\Gamma \star \eta) = (F, \mathbf{1}_\Gamma \star \eta).$$

As $(\mathbf{1}_\Gamma \star \eta)_x = \mathbf{1}_{(\Gamma\circ F)_0(x)} \circ \mathrm{id}_1^{\mathscr{C}}(\eta_x) = \eta_x$, because $\eta_x\colon F_0(\Lambda_0(x)) \to \Gamma_0(F_0(x))$, we see that $(\mathbf{1}_\Gamma \star \eta) = \eta$. On the other hand, one computes

$$(G,\mu) \circ (\mathrm{id}^{\mathscr{C}}, \mathbf{1}_\Gamma) = (G \circ \mathrm{id}^{\mathscr{C}}, \mu \circ \mathbf{1}_\Gamma) = (G, \mu \circ \mathbf{1}_\Gamma),$$

and $(\mu\star\mathbf{1}_\Gamma)_x = \mu_{\mathrm{id}_0^{\mathscr{C}}(x)} \circ G_1(\mathbf{1}_{\Gamma_0(x)}) = \mu_{\mathrm{id}_0^{\mathscr{C}}(x)} \circ \mathbf{1}_{(G\circ\Gamma)_0(x)} = \mu_x$, as $\mu_x\colon G_0(\Gamma_0(x)) \to \Delta_0(F_0(x))$. So $\mathsf{Groth}(\mathsf{ccCat}, \mathsf{Fun}(\text{-}, \text{-}))$ is indeed a category. Q.E.D.

Lemma 7.10. *The rule* $\mathsf{CHU}\colon \mathsf{Groth}(\mathsf{ccCat}, \mathsf{Fun}(\text{-}, \text{-})) \to \mathsf{Cat}$ *defined by*

$$\mathsf{CHU}_0(\mathscr{C},\Gamma) = \mathsf{Chu}(\mathscr{C},\Gamma),$$
$$\mathsf{CHU}_1((F,\eta)\colon (\mathscr{C},\Gamma) \to (\mathscr{D},\Delta)) = F_\vee$$

is a functor.

Proof: We can see that CHU is well-defined, as $\mathsf{CHU}(\mathscr{C},\Gamma) \in \mathsf{Cat}$ for all $\mathscr{C} \in \mathsf{ccCat}_0$ and $\Gamma \in \mathsf{Fun}(\mathscr{C},\mathscr{C})$. Furthermore is $(F_\vee)\colon \mathsf{CHU}_0(\mathscr{C},\Gamma) \to \mathsf{CHU}_0(\mathscr{D},\Delta)$ for $(F,\eta)\colon (\mathscr{C},\Gamma) \to (\mathscr{D},\Delta)$. So we need to check the axioms of a functor. To this end, we show that $(G \circ F)_\vee = G_\vee \circ F_\vee$ for $(F,\eta)\colon (\mathscr{C},\Gamma) \to (\mathscr{D},\Delta)$ and $(G,\mu)\colon (\mathscr{D},\Delta) \to (\mathscr{E},\Xi)$. We check this equality of functors on the objects and on the arrows.

- *On objects:* Let $(x;a,f,b) \in \mathsf{Chu}(\mathscr{C},\Gamma)_0$ be given. We compute

  $$\begin{aligned}
  &((G\circ F)_\vee)_0(x;a,f,b)\\
  &= \big((G\circ F)_0(x), (G\circ F)_0(a), (\mu\star\eta)_x \circ (G\circ F)_1(f) \circ (G\circ F)_{ab}, (G\circ F)_0(b)\big)\\
  &= \big(G_0(F_0(x)), G_0(F_0(a)), \mu_{F_0(x)} \circ G_1(\eta_x) \circ (G_1(F_1(f))) \circ G_1(F_{ab}) \circ G_{F_0(a)F_0(b)}, G_0(F_0(b))\big)\\
  &= \big(G_0(F_0(x)), G_0(F_0(a)), \mu_{F_0(x)} \circ G_1(\eta_x \circ F_1(f) \circ F_{ab}) \circ G_{F_0(a)F_0(b)}, G_0(F_0(b))\big)\\
  &= (G_\vee)_0\big(F_0(x), F_0(a), \eta_x \circ F_1(f) \circ F_{ab}, F_0(b)\big)\\
  &= (G_\vee)_0(F_\vee)_0(x;a,f,b) = (G_\vee \circ F_\vee)_0(x;a,f,b),
  \end{aligned}$$

  which gives the desired equality.
- *On arrows:* Let $(\phi^0,\phi^+,\phi^-)\colon (x;a,f,b) \to (y;c,g,d)$ be an arrow in $\mathsf{Chu}(\mathscr{C},\Gamma)$. We then compute

  $$\begin{aligned}
  ((G\circ F)_\vee)_1(\phi^0,\phi^+,\phi^-) &= \big((G\circ F)_1(\phi^0), (G\circ F)_1(\phi^+), (G\circ F)_1(\phi^-)\big)\\
  &= \big(G_1(F_1(\phi^0)), G_1(F_1(\phi^+)), G_1(F_1(\phi^-))\big)\\
  &= (G_\vee)_1(F_1(\phi^0), F_1(\phi^+), F_1(\phi^-))\\
  &= (G_\vee)_1 \circ (F_\vee)_1(\phi^0,\phi^+,\phi^-).
  \end{aligned}$$

So the compatibility with composition is proven, as the above shows that

$$\mathsf{CHU}_1\big((G,\mu)\circ(F,\eta)\big) = (G\circ F)_\vee = G_\vee \circ F_\vee = \mathsf{CHU}_1(G,\mu) \circ \mathsf{CHU}_1(F,\eta).$$

It remains to show the preservation of identities. But this is immediate, as one can compute that $(\mathrm{id}^{\mathscr{C}})_\vee(x;a,f,b) = (x,a,f,b)$ for all $(x;a,f,b) \in \mathsf{Chu}(\mathscr{C},\gamma)_0$ and $((\mathrm{id}^{\mathscr{C}})_\vee)_1(\phi^0,\phi^+,\phi^-) = (\phi^0,\phi^+,\phi^-)$ for all $(\phi^0,\phi^+,\phi^-)\colon (x;a,f,b) \to (y;c,g,d)$. So $\mathsf{CHU}\colon \mathsf{Groth}(\mathsf{ccCat}, \mathsf{Fun}(\text{-}, \text{-})) \to \mathsf{Cat}$ is a functor. Q.E.D.

Theorem 7.11. *If $F\colon \mathscr{C} \to \mathscr{D}$ is a full embedding and η_x is a monomorphism for every $x \in \mathscr{C}_0$, then $F_\vee$ is a representation of $\mathsf{Chu}(\mathscr{C}, \Gamma)$ in $\mathsf{Chu}(\mathscr{D}, \Delta)$.*

Proof: We first check that $F_\vee$ is injective on objects. To this end, let $(x; a, f, b)$ and $(y; c, g, d)$ with

$$(F_\vee)_0(x; a, f, b) = (F_0(x); F_0(a), \eta_x \circ F_1(f) \circ F_{ab}, F_0(b)) = (F_0(y); F_0(c), \eta_y \circ F_1(g) \circ F_{cd}, F_0(d))$$

be given. It follows immediately, that $F_0(x) = F_0(y), F_0(a) = F_0(c), F_0(b) = F_0(d)$, and as F is a full embedding, we have $x = y, a = c, b = d$ and therefore $F_{ab} = F_{cd}, \eta_x = \eta_y$. It remains to prove $f = g$. As η_x is a monomorphism, we have $F_1(f) \circ F_{ab} = F_1(g) \circ F_{ab}$. As F_{ab} is always an isomorphism and F is faithful, we have $f = g$.

Next we prove that $(F_\vee)_{((x;a,f,b),(y;c,g,d))}$ is bijective for all $(x; a, f, b), (y; c, g, d) \in \mathsf{Chu}(\mathscr{C}, \Gamma)_0$. Let $(\phi^0, \phi^+, \phi^+), (\theta^0, \theta^+, \theta^-)\colon (x; a, f, b) \to (y; c, g, d)$ be given, such that

$$(F_\vee)_1(\phi^0, \phi^+, \phi^-) = (F_1(\phi^0), F_1(\phi^+), F_1(\phi^-)) = (F_1(\theta^0), F_1(\theta^+), F_1(\theta^-)) = (F_\vee)_1(\theta^0, \theta^+, \theta^-).$$

Then $F_1(\phi^0) = F_1(\theta^0), F_1(\phi^+) = F_1(\theta^+), F_1(\phi^-) = F_1(\theta^-)$, and as F is faithful, we have $\phi^0 = \theta^0, \phi^+ = \theta^+, \phi- = \theta^-$.

To see that $(F_\vee)_{((x;a,f,b),(y;c,g,d))}$ is surjective, let

$$(\theta^0, \theta^+, \theta^-)\colon\ \mathsf{Chu}(\mathscr{D}, \Delta)_1\big((F_\vee)_0(x; a, f, b), (F_\vee)_0(y; c, g, d)\big)$$

be given. This means that $\theta^0\colon F_0(x) \to F_0(y), \theta^+\colon F_0(a) \to F_0(c), \theta^-\colon F_0(d) \to F_0(b)$. As F is full, we obtain ϕ^0, ϕ^+, ϕ^- such that

$$F_1(\phi^0) = \theta^0, \quad F_1(\phi^+) = \theta^+, \quad F_1(\phi^-) = \theta^-.$$

So $F_\vee$ is full.

To sum our results up, we have shown that F is injective on objects, full and faithful, so F is a representation. Q.E.D.

7.4 A Grothendieck construction equivalent to the Chu category

S. Abramsky found a Grothendieck construction equivalent to the category of Chu spaces over a set K in [Abr18]. We want to generalize this result to all Chu categories. So we first find an appropriate Grothendieck construction and then a fitting functor.

Our Grothendieck-type construction $\overleftrightarrow{\mathsf{Groth}}((\mathscr{C} \times \mathscr{C})^{\mathrm{op}}, F)$ should have as objects pairs $((a, x), f)$, where $a, x \in \mathscr{C}_0$ and $f \in F_0(a, x)$. This would suit our goal, as this would produce tuples $((a, x), f)$, where $a, x \in \mathscr{C}_0$ and $f\colon a \times x \to \gamma$, if we take F to be the functor which sends a pair (a, x) of objects to the set $\mathrm{Hom}_{\mathscr{C}}(a \times x, \gamma)$. Next onto the arrows. In the Chu category the arrows are given by pairs (ϕ^+, ϕ^-), where $\phi^+\colon a \to b$ and $\phi^-\colon y \to x$. So if we want to define arrows $(\Phi, \phi)\colon ((a, x), f) \to ((b, y), g)$, we have to let $\Phi\colon a \to b$. For the arrow ϕ we want to use the conditions placed on arrows in $\mathsf{Chu}(\mathscr{C}, \gamma)$. We have the commutative diagram

$$\begin{tikzcd} a \times y \arrow[r, "\phi^+ \times \mathbf{1}_y"] \arrow[d, "\mathbf{1}_a \times \phi^-"'] & b \times y \arrow[d, "g"] \\ a \times x \arrow[r, "f"] & \gamma. \end{tikzcd}$$

As $f \in \mathrm{Hom}(a \times x, \gamma)$ and $g \in \mathrm{Hom}(b \times d, \gamma)$, this can be rephrased through the diagram

$$
\begin{tikzcd}
& f \circ (\mathbf{1}_a \times \phi^-) \arrow[r, equal] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & g \circ (\phi^+ \times \mathbf{1}_y) \arrow[dl, phantom, "\rotatebox{-90}{$\ni$}"] & \\
& \mathrm{Hom}(a \times y, \gamma) & & \\
f \arrow[uur, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & & g \arrow[uul, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\mathrm{Hom}(a \times x, \gamma) \arrow[uur, "- \circ (\mathbf{1}_a \times \phi^-)"'] & & & \mathrm{Hom}(b \times y, \gamma). \arrow[uull, "- \circ (\phi^+ \times \mathbf{1}_y)"]
\end{tikzcd}
$$

This motivates the following definition.

Definition 7.12 (The antiparallel Grothendieck construction). Let $\mathscr{C}$ be a cartesian closed category and $F\colon (\mathscr{C} \times \mathscr{C})^{\mathrm{op}} \to \mathsf{Set}$ be a covariant functor. The category $\overleftrightarrow{\mathsf{Groth}}(\mathscr{C}, F)$ is defined in the following way:

- The *objects* of $\overleftrightarrow{\mathsf{Groth}}(\mathscr{C}, F)$ are triplets (A_1, A_2, a) such that $A = (A_1, A_2) \in (\mathscr{C} \times \mathscr{C})_0$ and $a \in F_0(A)$.
- The *arrows* of $\overleftrightarrow{\mathsf{Groth}}(\mathscr{C}, F)$ are pairs $(\Phi, \phi)\colon (A_1, A_2, a) \to (B_1, B_2, b)$, such that $\Phi\colon A_1 \to B_1$ and $\phi\colon B_2 \to A_2$ such that

$$F_1(\Phi \times \mathbf{1}_{B_2})(b) = F_1(\mathbf{1}_{A_1} \times \phi)(a)$$

- The composition of two arrows $(\Phi, \phi)\colon (A_1, A_2, a) \to (B_1, B_2, b)$ and $(\Psi, \psi)\colon (B_1, B_2, b) \to (C_1, C_2, c)$ is given by

$$(\Psi, \psi) \circ (\Phi, \phi) = (\Psi \circ \Phi, \phi \circ \psi).$$

One can visualize the arrows in the antiparallel Grothendieck construction as in figure 7.1.

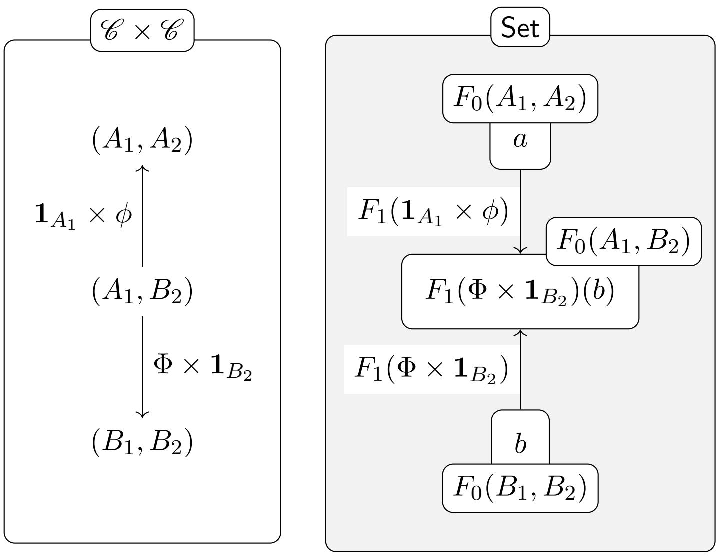


Figure 7.1: An illustration regarding the arrows in $\overleftrightarrow{\mathsf{Groth}}(\mathscr{C}, F)$

Lemma 7.13. $\overleftrightarrow{\mathsf{Groth}}(\mathscr{C}, F)$ *as in the preceding definition is a category.*

Proof: We show that the composition is well-defined. To this end let $(\Phi, \phi)\colon (A_1, A_2, a) \to (B_1, B_2, b)$ and $(\Psi, \psi)\colon (B_1, B_2, b) \to (C_1, C_2, c)$ be given. As $\Psi \circ \Phi\colon A_1 \to C_1$ and $\phi \circ \psi\colon C_2 \to A_2$, it remains to check the condition

$$F_1\big((\Psi \circ \Phi) \times \mathbf{1}_{C_2}\big)(c) = F_1\big(\mathbf{1}_{A_1} \times (\phi \circ \psi)\big)(a).$$

We already have the equalities

$$F_1(\Phi \times \mathbf{1}_{B_2})(b) = F_1(\mathbf{1}_{A_1} \times \phi)(a) \quad \text{and} \quad F_1(\Psi \times \mathbf{1}_{C_2})(c) = F_1(\mathbf{1}_{B_1} \times \psi)(b).$$

Thus we compute

$$\begin{aligned} F_1\big((\Psi \circ \Phi) \times \mathbf{1}_{C_2}\big)(c) &= F_1\big((\Psi \times \mathbf{1}_{C_2}) \circ (\Phi \times \mathbf{1}_{C_2})\big)(c) = \big(F_1(\Phi \times \mathbf{1}_{C_2}) \circ F_1(\Psi \times \mathbf{1}_{C_2})\big)(c) \\ &= F_1(\Phi \times \mathbf{1}_{C_2})\big(F_1(\mathbf{1}_{B_1} \times \psi)(b)\big) = F_1\big((\mathbf{1}_{B_1} \times \psi) \circ (\Phi \times \mathbf{1}_{C_2})\big)(b) \\ &= F_1\big((\Phi \times \mathbf{1}_{B_2}) \circ (\mathbf{1}_{A_1} \times \psi)\big)(b) = F_1(\mathbf{1}_{A_1} \times \psi)\big(F_1(\Phi \times \mathbf{1}_{B_2})(b)\big) \\ &= F_1(\mathbf{1}_{A_1} \times \psi)\big(F_1(\mathbf{1}_{A_1} \times \phi)(a)\big) = F_1\big((\mathbf{1}_{A_1} \times \phi) \circ (\mathbf{1}_{A_1} \times \psi)\big)(a) \\ &= F_1\big(\mathbf{1}_{A_1} \times (\phi \circ \psi)\big)(a). \end{aligned}$$

Therefore the composite arrow of two arrows exists. Furthermore we have an identity arrow for each $(A_1, A_2, a) \in \overleftrightarrow{\mathsf{Groth}}(\mathscr{C}, F)$, as $(\mathbf{1}_{A_1}, \mathbf{1}_{A_2})\colon (A_1, A_2) \to (A_1, A_2)$ and

$$F_1(\mathbf{1}_{A_1} \times \mathbf{1}_{A_2})(a) = F_1(\mathbf{1}_{A_1} \times \mathbf{1}_{A_2})(a).$$

Hence $\overleftrightarrow{\mathsf{Groth}}(\mathscr{C}, F)$ is a category. Q.E.D.

The missing ingredient is the functor which sends the pair to the appropriate Hom-set. This functor is realised through the following rule.

Definition 7.14 (The functor $\mathbb{S}^\gamma$). Let $\gamma \in \mathscr{C}$. We define $\mathbb{S}^\gamma\colon (\mathscr{C} \times \mathscr{C})^{\mathrm{op}} \to \mathsf{Set}$ in the following way:

$$\begin{aligned} \mathbb{S}^\gamma_0(x, y) &= \mathrm{Hom}_{\mathscr{C}}(x \times y, \gamma), \\ \mathbb{S}^\gamma_1\big((f_1, f_2)\colon (x, y) \to (x', y')\big) &= \big((\text{-} \circ (f_1 \times f_2))\colon \mathrm{Hom}_{\mathscr{C}}(x' \times y', \gamma) \to \mathrm{Hom}_{\mathscr{C}}(x \times y, \gamma)\big). \end{aligned}$$

Proposition 7.15. *The rule $\mathbb{S}^\gamma$ defined above is a functor $(\mathscr{C} \times \mathscr{C})^{\mathrm{op}} \to \mathsf{Set}$.*

Proof: The well-definedness is immediate, as $\mathrm{Hom}_{\mathscr{C}}(x \times y, \gamma) \in \mathsf{Set}_0$ for all $x, y \in \mathscr{C}$. Furthermore, if $(f_1, f_2)\colon (x, y) \to (x', y')$, then for every $g\colon x \times y \to \gamma$ we have that $g \circ (f_1 \times f_2)\colon x' \times y' \to \gamma$, so $g \circ (f_1 \times f_2) \in \mathrm{Hom}_{\mathscr{C}}(x \times y, \gamma)$. We check the axioms of a functor.

- *Compatibility with composition:* Let $(f_1, f_2)\colon (x, y) \to (x', y')$ and $(g_1, g_2)\colon (x', y') \to (x'', y'')$. We have to check that

$$\mathbb{S}^\gamma_1\big((g_1, g_2) \circ (f_1, f_2)\big) = \mathbb{S}^\gamma_1(f_1, f_2) \circ \mathbb{S}^\gamma(g_1, g_2).$$

  We can do this element-wise, so let $h \in \mathrm{Hom}_{\mathscr{C}}(x'' \times y'', \gamma)$ be given. One computes

$$\begin{aligned} \mathbb{S}^\gamma_1\big((g_1, g_2) \circ (f_1, f_2)\big)(h) &= \mathbb{S}^\gamma_1(g_1 \circ f_1, g_2 \circ f_2)(h) \\ &= h \circ \big((g_1 \circ f_1) \times (g_2 \circ f_2)\big) \\ &= h \circ (g_1 \times g_2) \circ (f_1 \times f_2) \\ &= \mathbb{S}^\gamma_1(f_1, f_2)\big(h_1 \circ (g_1 \times g_2)\big) \\ &= \mathbb{S}^\gamma_1(f_1, f_2)\Big(\mathbb{S}^\gamma_1(g_1, g_2)(h)\Big) \\ &= \big(\mathbb{S}^\gamma_1(f_1, f_2) \circ \mathbb{S}^\gamma_1(g_1, g_2)\big)(h). \end{aligned}$$

- *Preservation of identities:* We have $(\mathbf{1}_x, \mathbf{1}_y)\colon (x, y) \to (x, y)$. Let $f \in \mathrm{Hom}_{\mathscr{C}}(x \times y, \gamma)$. We compute

$$\mathbb{S}_1^\gamma(\mathbf{1}_x, \mathbf{1}_y)(f) = f \circ (\mathbf{1}_x \times \mathbf{1}_y) = f$$

  so $\mathbb{S}_1^\gamma(\mathbf{1}_x, \mathbf{1}_y) = \mathbf{1}_{\mathrm{Hom}_{\mathscr{C}}(x \times y, \gamma)}$.

Hence $\mathbb{S}^\gamma$ is a functor. Q.E.D.

Theorem 7.16. *We have* $\underleftarrow{\overrightarrow{\mathsf{Groth}}}\left(\mathscr{C}, \mathbb{S}^\gamma\right) = \mathsf{Chu}(\mathscr{C}, \gamma)$.

Proof: We first check how the objects of $\underleftarrow{\overrightarrow{\mathsf{Groth}}}\left(\mathscr{C}, \mathbb{S}^\gamma\right)$ look. They are tuples $((a, x), f)$ where $(a, x) \in \mathscr{C} \times \mathscr{C}$ and $f \in \mathbb{S}_0^\gamma(a, x) = \mathrm{Hom}_{\mathscr{C}}(a \times x, \gamma)$. This already looks promising, as we can identify $((a, x), f)$ with $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$. Now let us examine the arrows. How does $(\Phi, \phi)\colon ((a, x), f) \to ((b, y), g)$ look? We have $\Phi\colon a \to b$ and $\phi\colon y \to x$, such that

$$g \circ (\Phi \times \mathbf{1}_y) = \mathbb{S}_1^\gamma(\Phi \times \mathbf{1}_y)(g) = \mathbb{S}_1^\gamma(\mathbf{1}_a \times \phi)(f) = f \circ (\mathbf{1}_a \times \phi).$$

This is exactly the desired commutativity of

$$\begin{tikzcd}
a \times y \arrow[r, "\mathbf{1}_a \times \Psi"] \arrow[d, "\Phi \times \mathbf{1}_y"'] & a \times x \arrow[d, "f"] \\
b \times y \arrow[r, "g"] & \gamma.
\end{tikzcd}$$

Now this allows us to define a rule $\underleftarrow{\overrightarrow{\mathsf{Groth}}}\left(\mathscr{C}, \mathbb{S}^\gamma\right) \to \mathsf{Chu}(\mathscr{C}, \gamma)$ by

$$\begin{aligned}
\underleftarrow{\overrightarrow{\mathsf{Groth}}}\left(\mathscr{C}, \mathbb{S}^\gamma\right)_0 \ni \big((a, x), f)\big) &\mapsto (a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0, \\
\underleftarrow{\overrightarrow{\mathsf{Groth}}}\left((\mathscr{C}, \mathbb{S}^\gamma\right)_1 \ni (\Phi, \phi) &\mapsto (\Phi, \phi) \in \mathsf{Chu}(\mathscr{C}, \gamma)_1.
\end{aligned}$$

Next we show the bijectivity of this rule on objects. First, injectivity. Let $((a, x), f), (b, y), g) \in \underleftarrow{\overrightarrow{\mathsf{Groth}}}\left(\mathscr{C}, \mathbb{S}^\gamma\right)_0$ be given such that $(a, f, x) = (b, g, y)$. Then $((a, x), f) = ((b, y), g)$ follows immediately. The surjectivity is also immediate. Let $(a, f, x) \in \mathsf{Chu}(\mathscr{C}, \gamma)_0$ be given. Then $((a, x), f) \in \underleftarrow{\overrightarrow{\mathsf{Groth}}}\left(\mathscr{C}, \mathbb{S}^\gamma\right)_0$ as $(a, x) \in (\mathscr{C} \times \mathscr{C})_0$ and $f \in \mathrm{Hom}(a \times x, f) = \mathbb{S}_0^\gamma(a, x)$.

Now onto the bijectivity of the arrows. To see that the rule is injective on arrows, let

$$(\Phi, \phi), (\Phi', \phi')\colon ((a, x), f) \to ((b, y), g)$$

be given such that $(\Phi, \phi) = (\Phi', \phi')$ as arrows in $\underleftarrow{\overrightarrow{\mathsf{Groth}}}(\mathscr{C}, \mathbb{S}^\gamma)$. Then the equality $(\Phi, \phi) = (\Phi', \phi')$ as arrows in $\mathsf{Chu}(\mathscr{C}, \gamma)$ follows immediately. To see the surjectivity, let $(\Phi, \Psi)\colon (a, f, x) \to (b, g, y)$ be given. As $\Phi\colon a \to b$, it remains to check that $\text{-} \circ (\mathbf{1}_a \times \Psi)\colon \mathrm{Hom}(a \times x, \gamma) \to \mathrm{Hom}(a \times y, \gamma)$ and $g \circ (\Phi \times \mathbf{1}_y) = f \circ (\mathbf{1}_a \times \Psi)$. But these two conditions are immediate from the definition of arrows in $\mathsf{Chu}(\mathscr{C}, \gamma)$. This allows us to identify $\mathsf{Chu}(\mathscr{C}, \gamma)$ with $\underleftarrow{\overrightarrow{\mathsf{Groth}}}\left((\mathscr{C} \times \mathscr{C})^{\mathrm{op}}, \mathbb{S}^\gamma\right)$.
Q.E.D.

7.5 Generalizing the Grothendieck construction equivalent to Chu

We have seen that $\mathsf{Chu}(\mathscr{C}, \gamma) = \underleftarrow{\overrightarrow{\mathsf{Groth}}}\left(\mathscr{C}, \mathbb{S}\right)$. Our next goal is to generalize the notion of the Grothendieck construction involved. The obvious step is to allow $(\mathscr{C} \times \mathscr{D})$ for two arbitrary closed categories.

Definition 7.17 (The antiparallel Grothendieck construction $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C},\mathscr{D},\mathcal{F})$). Let $\mathscr{C},\mathscr{D}$ be cartesian closed categories and $\mathcal{F}\colon (\mathscr{C}\times\mathscr{D})^{\mathrm{op}} \to \mathsf{Set}$ be a covariant functor. The category $\overrightarrow{\underleftarrow{\mathsf{Groth}}}\big(\mathscr{C},\mathscr{D},\mathcal{F}\big)$ is defined in the following way:

- The *objects* of $\overrightarrow{\underleftarrow{\mathsf{Groth}}}\big(\mathscr{C},\mathscr{D},\mathcal{F}\big)$ are triplets $\big(C,D,c\big)$, such that $(C,D)\in(\mathscr{C}\times\mathscr{D})_0$ and $c\in\mathcal{F}_0(C,D)$.
- The *arrows* of $\overrightarrow{\underleftarrow{\mathsf{Groth}}}\big(\mathscr{C},\mathscr{D},\mathcal{F}\big)$ are pairs $(\Phi,\phi)\colon \big(C,D,c\big)\to\big(C',D',c'\big)$ such that $\Phi\colon C\to C'$ and $\phi\colon D'\to D$ such that
$$\mathcal{F}_1(\Phi,\mathbf{1}_{D'})(c') = \mathcal{F}_1(\mathbf{1}_C,\phi)(c).$$
- The *composition* of two arrows $(\Phi,\phi)\colon \big(C,D,c\big)\to\big(C',D',c'\big)$ and $(\Psi,\psi)\colon \big(C'',D'',c''\big)$ is given by
$$(\Psi,\psi)\circ(\Phi,\phi) = (\Psi\circ\Phi,\phi\circ\psi).$$

Remark 7.18. As we have already seen, there exists a natural isomorphism between $\mathscr{C}\times\mathscr{C}$ and the subcategory of $\mathscr{C}$ consisting of products $a\times x$, so we will identify these two categories in the case $\mathscr{D}=\mathscr{C}$ in order to make the notation a little more bearable.

Lemma 7.19. $\overrightarrow{\underleftarrow{\mathsf{Groth}}}\big(\mathscr{C},\mathscr{D},\mathcal{F}\big)$ *as defined above is a category.*

Proof: We check the well-definedness of the composition, as the well-definedness of the objects and arrows is immediate. So let two arrows $(\Phi,\phi)\colon \big(C,D,c\big)\to\big(C',D',c'\big)$ and $(\Psi,\psi)\colon \big(C'',D'',c''\big)$ be given. As $\Psi\circ\Phi\colon C\to C''$ and $\psi\circ\phi\colon D''\to D$, it remains to prove that
$$\mathcal{F}_1\big((\Psi\circ\Phi),\mathbf{1}_{D''})(c'') = \mathcal{F}_1\big(\mathbf{1}_C,(\phi\circ\psi)\big)(c).$$
We already have
$$\mathcal{F}_1(\Phi,\mathbf{1}_{D'})(c') = \mathcal{F}_1(\mathbf{1}_C,\phi)(c) \quad\text{and}\quad \mathcal{F}_1(\Psi,\mathbf{1}_{D''})(c'') = \mathcal{F}_1(\mathbf{1}_{C'},\psi)(c'),$$
so we can simply compute
$$\begin{aligned}
\mathcal{F}_1\big((\Psi\circ\Phi),\mathbf{1}_{D''})(c'') &= \mathcal{F}_1\big((\Psi,\mathbf{1}_{D''})\circ(\Phi,\mathbf{1}_{D''})\big)(c'')\\
&= \big(\mathcal{F}_1(\Phi,\mathbf{1}_{D''})\circ\mathcal{F}_1(\Psi,\mathbf{1}_{D''})\big)(c'')\\
&= \mathcal{F}_1(\Phi,\mathbf{1}_{D''})\Big(\mathcal{F}_1(\mathbf{1}_{C'},\psi)(c')\Big)\\
&= \mathcal{F}_1\big((\mathbf{1}_{C'},\psi)\circ(\Phi,\mathbf{1}_{D''})\big)(c')\\
&= \mathcal{F}_1\big((\Phi,\mathbf{1}_{D'})\circ(\mathbf{1}_C,\psi)\big)(c')\\
&= \mathcal{F}_1(\mathbf{1}_C,\psi)\Big(\mathcal{F}_1(\Phi,\mathbf{1}_{D'})(c')\Big)\\
&= \mathcal{F}_1(\mathbf{1}_C,\psi)\Big(\mathcal{F}_1(\mathbf{1}_C,\phi)(c)\Big)\\
&= \mathcal{F}_1\big(\mathbf{1}_C,\phi)\circ(\mathbf{1}_C,\psi)\big)(c)\\
&= \mathcal{F}_1\big(\mathbf{1}_C,(\phi\circ\psi)\big)(c).
\end{aligned}$$
To see that identities exists, we simply observe that $\mathbf{1}_{(C,D,c)} = (\mathbf{1}_C,\mathbf{1}_D)$ does the trick. Q.E.D.

7.6 The Grothendieck construction over finite products

As we have seen in the preceding section, we can generalize the Grothendieck category equivalent to the Chu construction by allowing products with factors in different categories. But what if we want to generalize the Grothendieck construction to allow an arbitrary but finite number of products? But this leaves us with a choice of the direction of the arrows. During the time in which we only considered $\mathscr{C} \times \mathscr{C}$, this choice was limited, as there only existed the possibilities

$$
\begin{tikzcd}[column sep=tiny]
(a, \arrow[d, "\phi^+"'] & x \arrow[d, "\phi^-"] & , f) \\
(b, & y & , g)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=tiny]
(a, \arrow[d, "\phi^+"'] & x & , f) \\
(b, & y \arrow[u, "\phi^-"'] & , g).
\end{tikzcd}
$$

But now as we consider three elements, we are left with 8 possibilities, which are

$$
\begin{tikzcd}[column sep=tiny]
(a, \arrow[d, "\phi^1"'] & x_1 \arrow[d, dashed, "\phi^2"] & , x_2 \arrow[d, "\phi^3"] & , f) \\
(b, & y_1 & , y_2 & , g),
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
(a, & x_1 \arrow[d, dashed, "\phi^2"] & , x_2 \arrow[d, "\phi^2"] & , f) \\
(b, \arrow[u, "\phi^1"] & y_1 & , y_2 & , g),
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
(a, \arrow[d, "\phi^1"'] & x_1 & , x_2 \arrow[d, "\phi^3"] & , f) \\
(b, & y_1 \arrow[u, dashed, "\phi^2"] & , y_2 & , g),
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
(a, \arrow[d, "\phi^1"'] & x_1 \arrow[d, dashed, "\phi^2"] & , x_2 & , f) \\
(b, & y_1 & , y_2 \arrow[u, "\phi^3"'] & , g),
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
(a, & x_1 & , x_2 \arrow[d, "\phi^3"] & , f) \\
(b, \arrow[u, "\phi^1"] & y_1 \arrow[u, dashed, "\phi^2"] & , y_2 & , g),
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
(a, & x_1 \arrow[d, dashed, "\phi^2"] & , x_2 & , f) \\
(b, \arrow[u, "\phi^1"] & y_1 & , y_2 \arrow[u, "\phi^3"'] & , g),
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
(a, \arrow[d, "\phi^1"'] & x_1 & , x_2 & , f) \\
(b, & y_1 \arrow[u, dashed, "\phi^2"] & , y_2 \arrow[u, "\phi^3"'] & , g),
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=tiny]
(a, & x_1 & , x_2 & , f) \\
(b, \arrow[u, "\phi^1"] & y_1 \arrow[u, dashed, "\phi^2"] & , y_2 \arrow[u, "\phi^3"'] & , g).
\end{tikzcd}
$$

To make this more formal, we define an appropriate set of categories.

Definition 7.20 (The set n-Tuples). Let $n \in \mathbb{N}_{>0}$. We define the set n-Tuples to be the set of categories $\mathscr{C}$ which fulfil the following conditions:

- $\mathscr{C}$ has $2n$ objects which we name $x_1, \dots, x_n, y_1, \dots, y_n$.
- $\mathscr{C}$ has n arrows $\mathtt{ar}_1, \dots, \mathtt{ar}_n$ in addition to the identities, such that for each $i \in \{1, \dots, n\}$, one of the following cases holds:
  - $\mathrm{dom}(\mathtt{ar}_i) = x_i$ and $\mathrm{codom}(\mathtt{ar}_i) = y_i$,
  - $\mathrm{dom}(\mathtt{ar}_i) = y_i$ and $\mathrm{codom}(\mathtt{ar}_i) = x_i$.

Remark 7.21. For the following work we would want a more "metaphorical" name for each of the categories in n-Tuples. As each category in n-Tuples is determined by its arrows, we will simply stack these arrows to identify the corresponding category, so if we for example consider

$$
\begin{tikzcd}
x_1 \bullet \arrow[r] & \bullet y_1 \\
x_2 \bullet & \bullet y_2 \arrow[l] \\
x_3 \bullet \arrow[r] & \bullet y_3
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
x_1 \bullet \arrow[r] & \bullet y_1 \\
x_2 \bullet & \bullet y_2, \arrow[l]
\end{tikzcd}
$$

where we omit the identity arrows, we would write - $\substack{\rightarrow \\ \leftarrow \\ \rightarrow}$ - to refer to the first category and - $\substack{\rightarrow \\ \leftarrow}$ - for the second category.

Now we come to the desired generalization of $\overset{\longrightarrow}{\underset{\longleftarrow}{\mathsf{Groth}}}(\mathscr{C}, \mathcal{F})$

Definition 7.22 (The Grothendieck category $\mathsf{Groth}(\mathscr{I}, \mathscr{C}, \mathcal{F})$). Let $\mathscr{C}$ be a category, $\mathcal{F} \colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$ and $\mathscr{T} \in n$-Tuples with arrows $\mathtt{ar}_1, \dots, \mathtt{ar}_n$ for a $n \in \mathbb{N}_{>0}$. We define $\mathsf{Groth}(\mathscr{T}, \mathscr{C}, \mathcal{F})$ to be the category given by the following data:

- The *objects* of $\mathsf{Groth}(\mathscr{T},\mathscr{C},\mathcal{F})$ are tuples $(A_1,\dots,A_n,a)$ such that $A_1,\dots,A_n \in \mathscr{C}_0$ and $a \in \mathcal{F}_0(A_1,\dots,A_n)$.
- The *arrows* are given as such: Let $(A_1,\dots,A_n,a),(B_1,\dots,B_n,b)$ as above we given. Then an arrow $(A_1,\dots,A_n,a) \to (B_1,\dots,B_n,b)$ is a tuple $(\phi^1,\dots,\phi^n)$ such that there exists a functor $\mathcal{T} \in \mathsf{Fun}(\mathscr{T},\mathscr{C})$ such that $\mathcal{T}_0(x_i) = A_i, \mathcal{T}_0(y_i) = B_i$ and $\mathcal{T}_1(\mathtt{ar}_i) = \phi^i$ for all $i = 1,\dots,n$. Furthermore we place one more condition on $(\phi^1,\dots,\phi^n)$. We set

$$\Phi^{+,i} := \begin{cases} \phi^i & \text{if } \mathrm{dom}(\mathtt{ar}_i) = x_i, \\ \mathbf{1}_{\mathrm{dom}(\phi^i)} & \text{if } \mathrm{dom}(\mathtt{ar}_i) = y_i \end{cases}$$
$$\text{and} \quad \Phi^{-,i} := \begin{cases} \mathbf{1}_{\mathrm{dom}(\phi^i)} & \text{if } \mathrm{dom}(\mathtt{ar}^i) = x_i, \\ \phi^i & \text{if } \mathrm{dom}(\mathtt{ar}^i) = y_i. \end{cases}$$

  These arrows must fulfil the condition

$$\mathcal{F}_1(\Phi^{+,1},\dots,\Phi^{+,n})(b) = \mathcal{F}_1(\Phi^{-,1},\dots,\Phi^{-,n})(a).$$

- The *composition* is defined as follows. Let

$$(\phi^1,\dots,\phi^n)\colon (A_1,\dots,A_n,a) \to (B_1,\dots,B_n,b),$$
$$(\psi^1,\dots,\psi^n)\colon (B_1,\dots,B_n,b) \to (C_1,\dots,C_n,c)$$

  be given. Then the composition $(\psi^1,\dots,\psi^n)\circ(\phi^1,\dots,\phi^n)$ is an arrow $(\theta^1,\dots,\theta^n)$ such that

$$\theta^i = \begin{cases} \psi^i \circ \phi^i & \text{if } \mathrm{dom}(\mathtt{ar}_i) = x_i, \\ \phi^i \circ \psi^i & \text{if } \mathrm{dom}(\mathtt{ar}_i) = y_i. \end{cases}$$

Before we prove that this indeed gives rise to a category, we start with an example.

Example 7.23. We examine the category $\mathsf{Groth}(\text{-}\,\substack{\rightarrow\\ \leftarrow\\ \rightarrow}\,\text{-},\mathscr{C},\mathcal{F})$ for an arbitrary category $\mathscr{C}$ and a functor $\mathcal{F}\colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$. Let two objects (A_1,A_2,A_3,a) and (B_1,B_2,B_3,b) in $\mathsf{Groth}(\text{-}\,\substack{\rightarrow\\ \leftarrow\\ \rightarrow}\,\text{-},\mathscr{C},\mathcal{F})$ be given. An arrow $(\phi^1,\phi^2,\phi^3)\colon (A_1,A_2,A_3,a) \to (B_1,B_2,B_3,b)$ consists of arrows ϕ^1,ϕ^2,ϕ^3 in $\mathscr{C}$ such that

$$A_1 \xrightarrow{\phi^1} B_1,$$
$$A_2 \xleftarrow{\phi^2} B_2,$$
$$A_3 \xrightarrow{\phi^3} B_3.$$

This aids us to visualize how the composition should look. Suppose we are given another arrow $(\psi^1,\psi^2,\psi^3)\colon (B_1,B_2,B_3,b) \to (C_1,C_2,C_3,c)$. Then we have

$$A_1 \xrightarrow{\phi^1} B_1 \xrightarrow{\psi^1} C_1$$
$$A_2 \xleftarrow{\phi^2} B_2 \xleftarrow{\psi^2} C_2$$
$$A_3 \xrightarrow{\phi^3} B_3 \xrightarrow{\psi^3} C_3.$$

Ergo $(\psi^1,\psi^2,\psi^3)\circ(\phi^1,\phi^2,\phi^3) = (\psi^1\circ\phi^1,\phi^2\circ\psi^2,\psi^3\circ\phi^3)$.

Proposition 7.24. $\mathsf{Groth}(\mathscr{T},\mathscr{C},\mathcal{F})$ *is a category for every cartesian closed category* $\mathscr{C}$*, each* $\mathscr{T} \in n\text{-}\mathsf{Tuples}$ *and each functor* $\mathcal{F}\colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$*.*

Proof: We have to prove that the composition is well-defined. We seek to employ the equality obtained in lemma 7.30 in the proof of the equality

$$\mathcal{F}_1(\Theta^{+,1},\dots,\Theta^{+,n})(b) = \mathcal{F}_1(\Theta^{-,1},\dots,\Theta^{-,n})(a).$$

But for this we need closer knowledge about $\Theta^{+,i}$ and $\Theta^{-,i}$. As we have

$$\Theta^{+,i} = \begin{cases} \psi^i \circ \phi^i & \text{if } \operatorname{dom}(\mathtt{ar}^i) = x_i, \\ \mathbf{1}_{C_i} & \text{else} \end{cases}$$
$$\text{and} \quad \Theta^{-,i} = \begin{cases} \phi^i \circ \psi^i & \text{if } \operatorname{codom}(\mathtt{ar}^i) = x_i, \\ \mathbf{1}_{A_i} & \text{else,} \end{cases}$$

by definition, we can always find a factorization $\Theta^{+,i} = \theta^{+,i,2} \circ \theta^{+,i,1}$ and $\Theta^{-,i} = \theta^{-,i,2} \circ \theta^{-,i,1}$ where

$$\begin{aligned} &\theta^{+,i,2} = \psi^i, \theta^{+,i,1} = \phi^i && \text{if } \operatorname{dom}(\mathtt{ar}^i) = x_i, \\ &\theta^{+,i,2} = \theta^{+,i,1} = \mathbf{1}_{C_i} && \text{else,} \\ &\theta^{-,i,2} = \phi^i, \theta^{-,i,1} = \psi^i && \text{if } \operatorname{codom}(\mathtt{ar}^i) = x_i, \\ &\theta^{-,i,2} = \theta^{-,i,1} = \mathbf{1}_{A_i} && \text{else.} \end{aligned}$$

Now we simply observe that $\theta^{+,i,2} = \Psi^{+,i}, \theta^{+,i,1} = \Phi^{+,1}$ and $\theta^{-,i,2} = \Psi^{-,i}, \theta^{-,i,1} = \Phi^{-,i}$, so we compute

$$\begin{aligned} \mathcal{F}_1(\Theta^{+,1},\dots,\Theta^{+,n})(c) &= \mathcal{F}_1\big((\theta^{+,1,2},\dots,\theta^{+,n,2}) \circ (\theta^{+,1,1},\dots,\theta^{+,n,1})\big)(c) \\ &= \mathcal{F}_1(\theta^{+,1,1},\dots,\theta^{+,n,1}) \circ \mathcal{F}_1(\theta^{+,1,2},\dots,\theta^{+,n,2})(c) \\ &= \mathcal{F}_1(\Phi^{+,1},\dots,\Phi^{+,n}) \circ \mathcal{F}_1(\Psi^{+,1},\dots,\Psi^{+,n})(c) \\ &= \mathcal{F}_1(\Phi^{+,1},\dots,\Phi^{+,n}) \circ \mathcal{F}_1(\Psi^{-,1},\dots,\Psi^{-,n})(b) \\ &= \mathcal{F}_1\big((\Psi^{-,1},\dots,\Psi^{-,n}) \circ (\Phi^{+,1},\dots,\Phi^{+,n})\big)(b). \end{aligned}$$

We want to examine $(\Psi^{-,1},\dots,\Psi^{-,n}) \circ (\Phi^{+,1},\dots,\Phi^{+,n})$, to be more precise we want to show the equality

$$(\Psi^{-,1},\dots,\Psi^{-,n}) \circ (\Phi^{+,1},\dots,\Phi^{+,n}) = (\Phi^{+,1},\dots,\Phi^{+,n}) \circ (\Psi^{-,1},\dots,\Psi^{-,n}).$$

To see this, it suffices to prove $\Phi^{+,i} \circ \Psi^{-,i} = \Psi^{-,i} \circ \Phi^{+,i}$ for all $i = 1,\dots,n$. So let $i = 1,\dots,n$ be given. By definition we have

$$\Psi^{-,i} \circ \Phi^{+,i} = \begin{cases} \Psi^{-,i} \circ \phi^i = \mathbf{1}_{\operatorname{dom}(\psi^i)} \circ \phi^i & \text{if } \operatorname{dom}(\mathtt{ar}^i) = x_i, \\ \Psi^{-,i} \circ \mathbf{1}_{\operatorname{dom}(\phi^i)} = \psi^i \circ \mathbf{1}_{\operatorname{dom}(\phi^i)} & \text{if } \operatorname{dom}(\mathtt{ar}^i) = y_i. \end{cases}$$

But by definition we have

$$\Phi^{+,i} \circ \Psi^{-,i} = \begin{cases} \phi^i \circ \mathbf{1}_{\operatorname{dom}(\psi^i)} & \text{if } \operatorname{dom}(\mathtt{ar}^i) = x_i, \\ \mathbf{1}_{\operatorname{dom}(\phi^i)} \circ \psi^i & \text{if } \operatorname{dom}(\mathtt{ar}^i) = y_i, \end{cases}$$

so $\Psi^{-,i} \circ \Phi^{+,i} = \Phi^{+,i} \circ \Psi^{-,i}$. This allows us to complete our computation to

$$\begin{aligned} \mathcal{F}_1\big((\Psi^{-,1},\dots,\Psi^{-,n}) \circ (\Phi^{+,1},\dots,\Phi^{+,n})\big)(b) &= \mathcal{F}_1\big((\Phi^{+,1},\dots,\Phi^{+,n}) \circ (\Psi^{-,1},\dots,\Psi^{-,n})\big)(b) \\ &= \mathcal{F}_1(\Psi^{-,1},\dots,\Psi^{-,n}) \circ \mathcal{F}_1(\Phi^{+,1},\dots,\Phi^{+,n})(b) \\ &= \mathcal{F}_1(\Psi^{-,1},\dots,\Psi^{-,n}) \circ \mathcal{F}_1(\Phi^{-,1},\dots,\Phi^{-,n})(a) \end{aligned}$$

$$\begin{aligned}&= \mathcal{F}_1\big((\Phi^{-,1},\dots,\Phi^{-,n})\circ(\Psi^{-,1},\dots,\Psi^{-,n})\big)(a)\\ &= \mathcal{F}_1(\Theta^{-,1},\dots,\Theta^{-,n})(a).\end{aligned}$$

To see that identities exist, we remark that $\mathbf{1}_{(A_1,\dots,A_n,a)} = (\mathbf{1}_{A_1},\dots,\mathbf{1}_{A_n})$, which trivially fulfils the condition. This completes the proof that $\mathsf{Groth}(\mathscr{T},\mathscr{C},\mathcal{F})$ is a category. Q.E.D.

Now we want to eliminate "redundant" categories, i.e. we want to identify categories who are naturally isomorphic. This is to reduce the number of Grothendieck type categories we have to consider, as we a priori have far to many, 8 in the case that $n = 3$. For this we want to consider actions by the symmetric group $\mathfrak{S}_n$.

Lemma 7.25. *Let $n \in \mathbb{N}_{>0}$ and n-Tuples be given. We define a left-action of $\mathfrak{S}_n$ on n-Tuples in the following way: Let $\mathscr{T} \in n$-Tuples and $\sigma \in \mathfrak{S}_n$. Then $\sigma\cdot\mathscr{T}$ is the category given by the following data:*

- *The objects are the same, $(\sigma\cdot\mathscr{T})_0 = \mathscr{T}_0$.*
- *The arrows are $\sigma\cdot\mathtt{ar}_1,\dots,\sigma\cdot\mathtt{ar}_n$, where*

$$\begin{aligned}&\sigma\cdot\mathtt{ar}_i\colon x_i\to y_i, &&\text{if } \mathtt{ar}_{\sigma^{-1}(i)}\colon x_{\sigma^{-1}(i)}\to y_{\sigma^{-1}(i)},\\ &\sigma\cdot\mathtt{ar}_i\colon y_i\to x_i, &&\text{if } \mathtt{ar}_{\sigma^{-1}(i)}\colon y_{\sigma^{-1}(i)}\to x_{\sigma^{-1}(i)}.\end{aligned}$$

Before we prove that this is indeed a left-action, we examine a concrete case.

Example 7.26. Consider the category $\text{-}\,\substack{\to\\ \leftarrow\\ \to}\,\text{-} \in 3$-$\mathsf{Tuples}$ and $\sigma = (123) \in \mathfrak{S}_3$. We then have

$$\sigma\cdot\left(\begin{array}{ccc} x_1\bullet & \longrightarrow & \bullet y_1\\ x_2\bullet & \longleftarrow & \bullet y_2\\ x_3\bullet & \longrightarrow & \bullet y_3\end{array}\right) = \begin{array}{ccc} x_1\bullet & \longrightarrow & \bullet y_1\\ x_2\bullet & \longrightarrow & \bullet y_2\\ x_3\bullet & \longleftarrow & \bullet y_3,\end{array}$$

or short $\sigma\cdot(\text{-}\,\substack{\to\\ \leftarrow\\ \to}\,\text{-}) = (\text{-}\,\substack{\to\\ \to\\ \leftarrow}\,\text{-})$. So one can imagine the action as "replacing the arrow $\mathtt{ar}_i$ by $\mathtt{ar}_{\sigma^{-1}(i)}$ in the picture".

Proof of lemma 7.25: We have to prove the following two statements:

- For each $\mathscr{T} \in n$-Tuples and $\sigma,\tau \in \mathfrak{S}_n$, we have
$$(\sigma\circ\tau)\cdot\mathscr{T} = \sigma\cdot(\tau\circ\mathscr{T}),$$
where $\circ$ is the binary operation in $\mathfrak{S}_n$.
- Let id be the neutral element in $\mathfrak{S}_n$, then $\mathrm{id}\cdot\mathscr{T} = \mathscr{T}$.

To see the first condition, we simply examine the arrows in $(\sigma\circ\tau)\cdot\mathscr{T}$ and $\sigma\cdot(\tau\cdot\mathscr{T})$. It suffices to show that for each pair (x_i, y_i) we have either $\sigma\cdot\tau\cdot\mathtt{ar}_i\colon x_i\to y_i$ and $(\sigma\circ\tau)\cdot\mathtt{ar}_i\colon x_i\to y_i$ or $\sigma\cdot\tau\cdot\mathtt{ar}_i\colon y_i\to x_i$ and $(\sigma\circ\tau)\cdot\mathtt{ar}_i\colon y_i\to x_i$. We have to distinguish the following cases:

- We have $\sigma\cdot\tau\cdot\mathtt{ar}_i\colon x_i\to y_i$. By definition this means that $\tau\cdot\mathtt{ar}_{\sigma^{-1}(i)}\colon x_{\sigma^{-1}(i)}$. Using the definition again, this means that $\mathtt{ar}_{\tau^{-1}(\sigma^{-1}(i))}\colon x_{\tau^{-1}(\sigma^{-1}(i))}\to y_{\tau^{-1}(\sigma^{-1}(i))}$. But as $\mathfrak{S}_n$ is a group, we have $\tau^{-1}(\sigma^{-1}(i)) = (\sigma\circ\tau)^{-1}(i)$ for all $i = 1,\dots,n$, so we obtain $(\sigma\cdot\tau)\cdot\mathtt{ar}_i\colon x_i\to y_i$.
- Now we assume $\sigma\cdot\tau\cdot\mathtt{ar}_i\colon y_i\to x_i$. Analogously to the first case we obtain
$$\mathtt{ar}_{\tau^{-1}(\sigma^{-1}(i))}\colon y_{\tau^{-1}(\sigma^{-1}(i))}\to x_{\tau^{-1}(\sigma^{-1}(i))}.$$
This means that $(\sigma\circ\tau)\cdot\mathtt{ar}_i\colon y_i\to x_i$.

To see the second condition we simply remark that $\mathrm{id}\cdot\mathtt{ar}_i\colon x_i\to y_i$ if $\mathtt{ar}_i\colon x_i\to y_i$, and the other way round, hence $\mathrm{id}\cdot\mathscr{T} = \mathscr{T}$. This yields that the rule is indeed a left-action. Q.E.D.

Remark 7.27. One can observe that σ is also a functor $\sigma\colon \mathscr{T} \to \sigma \cdot \mathscr{T}$.

This allows us to identify a few Grothendieck-type categories via isomorphisms in case $\mathscr{C}$ is a cartesian closed category and $\mathcal{F}$ fulfils a special requirement, which we shall now specify. First we correlate the category $\mathscr{C}^n$ with a subcategory of $\mathscr{C}$.

Definition 7.28. Let $\mathscr{C}$ be a cartesian closed category and $n \in \mathbb{N}$. We define $\mathscr{C}^{\times n}$ to be the subcategory of $\mathscr{C}$ consisting of the following data:
- The *objects* of $\mathscr{C}^{\times n}$ are products $\prod_{i=1}^n c_i$ for $c_i \in \mathscr{C}_0$.
- The *arrows* of $\mathscr{C}^{\times n}$ are products $\prod_{i=1}^n f_i\colon \prod_{i=1}^n c_i \to \prod_{i=1}^n d_i$.

It is immediate that this definitions gives rise to a category, as $\mathscr{C}$ is a category. With this definition at hand we can identify $\mathscr{C}^n$ with this subcategory.

Lemma 7.29. *Let $\mathscr{C}$ be a cartesian closed category, then $\mathscr{C}^n \cong \mathscr{C}^{\times n}$.*

Before we can prove this we need the generalization of [Pet21, p. 2] to an arbitrary but finite number of factors in the product. We state this as the following lemma.

Lemma 7.30. *Let $\mathscr{C}$ be a cartesian closed category and $a_1, \dots, a_n, a_1', \dots, a_n', a_1'', \dots, a_n'' \in \mathscr{C}_0$. Let $f_i\colon a_i \to a_i'$ and $g_i\colon a_i' \to a_i''$ for every $i = 1, \dots, n$. Then*

$$\prod_{i=1}^n (g_i \circ f_i) = \Big(\prod_{i=1}^n g_i\Big) \circ \Big(\prod_{i=1}^n f_i\Big).$$

Proof: Let g_i, f_i be given as in the lemma. Then $\prod_{i=1}^n (g_i \circ f_i)$ is the unique arrow $\prod_{i=1}^n a_i \to \prod_{i=1}^n a_i''$ such that for each $i = 1, \dots, n$ the rectangle

$$\begin{tikzcd} \prod_{i=1}^n a_i \arrow[r, "\mathtt{pr}_{a_i}"] \arrow[d, "\prod_{i=1}^n (g_i\circ f_i)"'] & a_i \arrow[d, "(g_i\circ f_i)"] \\ \prod_{i=1}^n a_i'' \arrow[r, "\mathtt{pr}_{a_i''}"] & a_i'' \end{tikzcd}$$

commutes. It suffices to check that the rectangle

$$\begin{tikzcd} \prod_{i=1}^n a_i \arrow[r, "\mathtt{pr}_{a_i}"] \arrow[d, "\left(\prod_{i=1}^n g_i\right)\circ\left(\prod_{i=1}^n f_i\right)"'] & a_i \arrow[d, "(g_i\circ f_i)"] \\ \prod_{i=1}^n a_i'' \arrow[r, "\mathtt{pr}_{a_i''}"] & a_i'' \end{tikzcd} \tag{30}$$

commutes for every $i = 1, \dots, n$, as then the equality follows by the universal property of the product. So let i be given. By the universal property of the product, the rectangles

$$\begin{tikzcd} \prod_{i=1}^n a_i \arrow[r, "\mathtt{pr}_{a_i}"] \arrow[d, "\prod_{i=1}^n f_i"'] & a_i \arrow[d, "f_i"] \\ \prod_{i=1}^n a_i' \arrow[r, "\mathtt{pr}_{a_i'}"] & a_i' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \prod_{i=1}^n a_i' \arrow[r, "\mathtt{pr}_{a_i'}"] \arrow[d, "\prod_{i=1}^n g_i"'] & a_i' \arrow[d, "g_i"] \\ \prod_{i=1}^n a_i'' \arrow[r, "\mathtt{pr}_{a_i''}"] & a_i'' \end{tikzcd}$$

commute. So the commutativity of the diagram (30) follows immediately and we obtain the desired equality. Q.E.D.

Proof of Lemma 7.29: We define the isomorphism $I\colon \mathscr{C}^n \to \mathscr{C}^{\times n}$ in the following way:
- Let $(c_1, \dots, c_n) \in \mathscr{C}^n$. Then $I_0(c_1, \dots, c_n) = \prod_{i=1}^n c_i$.

- Let $(f_1, \dots, f_n)\colon (c_1, \dots, c_n) \to (d_1, \dots, d_n)$. Then

$$I_1(f_1, \dots, f_n) = f_1 \times \cdots \times f_n \colon \prod_{i=1}^{n} c_i \to \prod_{i=1}^{n} d_i.$$

To see that this is a functor is immediate. We first check the compatibility with composition. For this let

$$\begin{aligned}(f_1, \dots, f_n)&\colon (c_1, \dots, c_n) \to (d_1, \dots, d_n),\\ (g_1, \dots, g_n)&\colon (d_1, \dots, d_n) \to (e_1, \dots, e_n).\end{aligned}$$

Then $(g_1, \dots, g_n) \circ (f_1, \dots, f_n) = (g_1 \circ f_1, \dots, g_n \circ f_n)$. But by lemma 7.30 we have

$$\begin{aligned}I_1\big((g_1, \dots, g_n) \circ (f_1, \dots, f_n)\big) &= I_1(g_1 \circ f_1, \dots, g_n \circ f_n) = \prod_{i=1}^{n}(g_i \circ f_i)\\ &= \Big(\prod_{i=1}^{n} g_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big) && \text{(by lemma 7.30)}\\ &= I_1(g_1, \dots, g_n) \circ I_1(f_1, \dots, f_n).\end{aligned}$$

The compatibility with arrows is also immediate, as

$$I_1(\mathbf{1}_{c_1}, \dots, \mathbf{1}_{c_n}) = \mathbf{1}_{c_1} \times \dots \mathbf{1}_{c_n} = \mathbf{1}_{c_1 \times \cdots \times c_n}.$$

The inverse functor is given by

$$\begin{aligned}\forall \Big(\prod_{i=1}^{n} c_i\Big) \in \mathscr{C}_0^{\times n} &: I_0^{-1}\Big(\prod_{i=1}^{n} c_i\Big) = (c_1, \dots, c_n),\\ \forall \Big(\prod_{i=1}^{n} f_i\Big) \in \mathscr{C}_1^{\times n} &: I_1^{-1}\Big(\prod_{i=1}^{n} f_i\Big) = (f_1, \dots, f_n).\end{aligned}$$

Q.E.D.

This allows us to identify a functor $\mathcal{F}\colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$ with a functor $F\colon (\mathscr{C}^{\times n})^{\mathrm{op}} \to \mathsf{Set}$ by precomposing with the obtained isomorphism. This allows us to make the following identifications:

- We identify the objects $(A_1, \dots, A_n, a)$ of $\mathsf{Groth}(\mathscr{I}, \mathscr{C}, \mathcal{F})$ with $(A_1, \dots, A_n, a)$, where $a \in F_0(A_1 \times \cdots \times A_n)$.
- For arrows $(\phi^1, \dots, \phi^n)\colon (A_1, \dots, A_n, a) \to (B_1, \dots, B_n, b)$ we replace the condition $\mathcal{F}_1(\Phi^{+,1}, \dots, \Phi^{+,n})(b) = \mathcal{F}_1(\Phi^{-,1}, \dots, \Phi^{-,n})$ by $F_1(\Phi^{+,1} \times \cdots \times \Phi^{+,n})(b) = F_1(\Phi^{+,1} \times \cdots \times \Phi^{+,n})$.

From now on we shall no longer distinguish between functors $\mathcal{F}\colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$ and the associated functor $(\mathscr{C}^{\times n})^{\mathrm{op}} \to \mathsf{Set}$.

Theorem 7.31. *Let $n \in \mathbb{N}_{>0}, \mathscr{C}$ be a cartesian closed category, $\mathcal{F}\colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$ be a n-product preserving functor and $\mathscr{T}, \mathscr{T}' \in n\text{-}\mathsf{Tuples}$. Then $\mathsf{Groth}(\mathscr{T}, \mathscr{C}, \mathcal{F}) \cong \mathsf{Groth}(\mathscr{T}', \mathscr{C}, \mathcal{F})$ if there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma \cdot \mathscr{T}' = \mathscr{T}$.*

Example 7.32. Before we prove this theorem, we shall examine it in the case of a concrete example. For this, let $\mathsf{Groth}(\text{-} \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \text{-}, \mathscr{C}, \mathcal{F})$ be given for a cartesian closed category $\mathscr{C}$ and $\mathcal{F}\colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$. As we have seen there exists $\sigma = (123) \in \mathfrak{S}_3$ such that $\sigma \cdot (\text{-} \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \text{-}) = \text{-} \substack{\rightarrow\\ \rightarrow\\ \leftarrow} \text{-}$. We want to see that this translates into an isomorphism

$$\mathsf{Groth}(\text{-} \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \text{-}, \mathscr{C}, \mathcal{F}) \cong \mathsf{Groth}(\text{-} \substack{\rightarrow\\ \rightarrow\\ \leftarrow} \text{-}, \mathscr{C}, \mathcal{F}).$$

To this end, we start by examining the arrows. An arrow in $\mathsf{Groth}(\text{-} \substack{\rightarrow\\\leftarrow\\\rightarrow} \text{-}, \mathscr{C}, \mathcal{F})$ consists of two quadruplets $(A_1, A_2, A_3, a) \in \mathscr{C}_0^3 \times \mathcal{F}_0(A_1, A_2, A_3)$ and $(B_1, B_2, B_3, b) \in \mathscr{C}_0^3 \times \mathcal{F}_0(B_1, B_2, B_3, b)$ as well as arrows

$$\phi^1 \colon A_1 \to B_1, \qquad \phi_2 \colon B_2 \to A_2, \qquad \phi^3 \colon A_3 \to B_3.$$

On the other hand, an arrow $(\psi^1, \psi^2, \psi^3) \colon (A_1, A_2, A_3, a) \to (B_1, B_2, B_3, b)$ in $\mathsf{Groth}(\text{-} \substack{\rightarrow\\\rightarrow\\\leftarrow} \text{-}$ consists of arrows

$$\psi^1 \colon A_1 \to B_1, \qquad \psi^2 \colon A_2 \to B_2, \qquad \psi^3 \colon B_3 \to A_3.$$

The key observation to the equivalence of categories is the following:

Lemma 7.33. *Let $\mathscr{C}$ be a cartesian closed category and $\mathcal{F} \colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$ and $\mathscr{I}_1, \mathscr{I}_2 \in n\text{-}\mathsf{Tuples}$. Then*

$$\mathsf{Groth}(\mathscr{I}_1, \mathscr{C}, \mathcal{F})_0 = \mathsf{Groth}(\mathscr{I}_2, \mathscr{C}, \mathcal{F})_0.$$

Furthermore, $\mathsf{Groth}(\mathscr{I}_1, \mathscr{C}, \mathcal{F})_0$ is stable under $\mathfrak{S}_n$ in the following sense: Let $\sigma \in \mathfrak{S}_n$. Then the following are equivalent:

- $(A_1, \ldots, A_n, a) \in \mathsf{Groth}(\mathscr{I}_1, \mathscr{C}, \mathcal{F})$,
- $(A_{\sigma(1)}, \ldots, A_{\sigma(n)}, a) \in \mathsf{Groth}(\mathscr{I}_1, \mathscr{C}, \mathcal{F})$.

Proof: The first equality is immediate from the definitions. As if $(A_1, \ldots, A_n, a)$ is an object of $\mathsf{Groth}(\mathscr{I}_1, \mathscr{C}, \mathcal{F})$, then $(A_1, \ldots, A_n) \in \mathscr{C}^n$ and $a \in \mathcal{F}_0(A_1, \ldots, A_n)$, but this is sufficient that $(A_1, \ldots, A_n, a) \in \mathsf{Groth}(\mathscr{I}_2, \mathscr{C}, \mathcal{F})$.

To see the equivalence, we simply remark that as $\mathscr{C}$ is a cartesian closed category, we have that for any $(A_1, \ldots, A_n) \in \mathscr{C}^n$ and $\sigma \in \mathfrak{S}_n$ the equalities

$$(A_1, \ldots, A_n) = A_1 \times \cdots \times A_n = A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} = (A_{\sigma(1)}, \ldots, A_{\sigma(n)})$$

hold. Ergo if $a \in \mathcal{F}_0(A_1, \ldots, A_n)$, then also $a \in \mathcal{F}_0(A_{\sigma(1)}, \ldots, A_{\sigma(n)})$, which proves the desired equivalence. Q.E.D.

In our concrete example this allows us to do the following: We shuffle the entries of $(A_1, \ldots, A_n, a)$ according to σ, so we can map the arrows accordingly. In a diagram this would look like the following:

$$\begin{array}{ccc} (A_1\ , & A_2, & A_3) \\ \downarrow \phi^1 & \uparrow \phi^2 & \downarrow \phi^3 \\ (B_1\ , & B_2, & B_3) \end{array} \quad \mapsto \quad \begin{array}{ccc} (A_3\ , & A_1, & A_2) \\ \downarrow \phi^3 & \downarrow \phi^1 & \uparrow \phi^2 \\ (B_3\ , & B_1, & B_2). \end{array}$$

As $(A_1, A_2, A_3, a) \in \mathsf{Groth}(\text{-} \substack{\rightarrow\\\leftarrow\\\rightarrow} \text{-}, \mathscr{C}, \mathcal{F})_0$, we have $(A_3, A_1, A_2, a) \in \mathsf{Groth}(\text{-} \substack{\rightarrow\\\rightarrow\\\leftarrow} \text{-}, \mathscr{C}, \mathcal{F})$. Hence we have that $(\phi^3, \phi^2, \phi^1) \colon (A_3, A_1, A_2, a) \to (B_3, B_1, B_2, b)$ is an arrow of $\mathsf{Groth}(\text{-} \substack{\rightarrow\\\rightarrow\\\leftarrow} \text{-}, \mathscr{C}, \mathcal{F})$ if we can prove that

$$\mathcal{F}_1(\Phi^{+,3} \times \Phi^{+,1} \times \Phi^{+,2})(b) = \mathcal{F}_1(\Phi^{-,3} \times \Phi^{-,1} \times \Phi^{-,2})(a).$$

But this is immediate as $(\phi^1, \phi^2, \phi^3) \in \mathsf{Groth}(\text{-} \substack{\rightarrow\\\leftarrow\\\rightarrow} \text{-}, \mathscr{C}, \mathcal{F})_1$. Furthermore it is easy to see that this rule is indeed bijective on arrows and objects.

Now, motivated from this example we can prove the general theorem.

Proof of Theorem 7.31: Let $\mathscr{T}, \mathscr{T}' \in n\text{-}\mathsf{Tuples}$ and $\sigma \in \mathfrak{S}_n$ such that $\sigma \cdot \mathscr{T}' = \mathscr{T}$. This is equivalent to $\tau \cdot \mathscr{I} = \mathscr{I}$, if we set $\tau := \sigma^{-1}$. We then define a functor $\mathcal{E}^{\sigma, \mathscr{T}} \colon \mathsf{Groth}(\mathscr{T}, \mathscr{C}, \mathcal{F}) \to \mathsf{Groth}(\mathscr{T}', \mathscr{C}, \mathcal{F})$ in the following way:

- For $(A_1, \dots, A_n, a) \in \mathsf{Groth}(\mathscr{T}, \mathscr{C}, \mathcal{F})_0$ we set

$$\mathcal{E}_0^{\sigma,\mathscr{T}}(A_1, \dots, A_n, a) = (A_{\sigma(1)}, \dots, A_{\sigma(n)}, a).$$

- For $(\phi^1, \dots, \phi^n)\colon (A_1, \dots, A_n, a) \to (B_1, \dots, B_n, b)$ we set

$$\mathcal{E}_1^{\sigma,\mathscr{T}}(\phi^1, \dots, \phi^n) = (\phi^{\sigma(1)}, \dots, \phi^{\sigma(n)}).$$

As $A_{\sigma(1)}, \dots, A_{\sigma(n)} \in \mathscr{C}_0$, and $A_1 \times \dots \times A_n \cong A_{\sigma(1)} \times \dots \times A_{\sigma(n)}$, we have $a \in \mathcal{F}_0(A_{\sigma(1)} \times \dots \times A_{\sigma(n)})$. To see that the rule is well-defined on arrows, we show that

$$(\phi^{\sigma(1)}, \dots, \phi^{\sigma(n)})\colon (A_{\sigma(1)}, \dots, A_{\sigma(n)}, a) \to (B_{\sigma(1)}, \dots, B_{\sigma(n)}, b).$$

This means we have to prove that there exists $\mathcal{T}' \in \mathsf{Fun}(\mathscr{T}', \mathscr{C})$ such that $\mathcal{T}'_0(x'_i) = A_{\sigma(i)}$ as well as $\mathcal{T}'_0(y'_i) = B_{\sigma(i)}$ and $\mathcal{T}'_1(\mathtt{ar}'_i) = \phi^{\sigma(i)}$ for all $i = 1, \dots, n$. We define $\mathcal{T}'$ by $\mathcal{T}'_0(x'_i) = A_{\sigma(i)}$ and $\mathcal{T}'_0(y'_i) = B_{\sigma(i)}$ and $\mathcal{T}'_1(\mathtt{ar}'_i) = \phi^{\sigma(i)}$. It remains to check that this is well-defined, i.e. that $\mathcal{F}_1(\mathtt{ar}'_i)\colon A_{\sigma(i)} \to B_{\sigma(i)}$. We know that $\sigma \cdot \mathscr{T}' = \mathscr{T}$. Therefore we have $\mathtt{ar}_i\colon x_i \to y_i$ if $\mathtt{ar}'_{\sigma^{-1}(i)}\colon x_{\sigma^{-1}(i)} \to y_{\sigma^{-1}(i)}$. As we have $\mathcal{T} \in \mathsf{Fun}(\mathscr{T}, \mathscr{C})$ such that

$$\mathcal{T}_0(x_i) = A_i, \mathcal{T}_0(y_i) = B_i, \mathcal{T}_1(\mathtt{ar}_i) = \phi^i,$$

we know that $\mathcal{T}'_1(\mathtt{ar}'_{\sigma^{-1}(i)})\colon \mathcal{T}'_0(x'_{\sigma^{-1}(i)} \to \mathcal{T}'_0(y'_{\sigma^{-1}(i)}$, hence $\mathcal{T}'_1(\mathtt{ar}'_{\sigma^{-1}(i)})\colon A_i \to B_i$, which means $\mathcal{T}'_1(\mathtt{ar}'_{\sigma^{-1}(i)}) = \phi^i$ for every $i = 1, \dots, n$, which yields the desired $\mathcal{T}'_1(\mathtt{ar}'_i)\colon A_{\sigma(i)} \to B_{\sigma(i)}$ by replacing i by $\sigma(i)$, as desired.

For the functoriality we have to check

$$\mathcal{E}_1^{\sigma,\mathscr{T}}\big((\psi^1, \dots, \psi^n) \circ (\phi^1, \dots, \phi^n)\big) = \mathcal{E}_1^{\sigma,\mathscr{T}}(\psi^1, \dots, \psi^n) \circ \mathcal{E}_1^{\sigma,\mathscr{T}}(\phi^1, \dots, \phi^n).$$

But this is immediate from the definition, as we have $\mathcal{E}_1^{\sigma,\mathscr{T}}\big((\psi^1, \dots, \psi^n) \circ (\phi^, \dots, \phi^n)\big) = \mathcal{E}_1^{\sigma,\mathscr{T}}(\theta^1, \dots, \theta^n)$, where θ^i is either $\psi^i \circ \phi^i$ or $\phi^i \circ \psi^i$, so $\theta^{\sigma(i)} = \psi^{\sigma(i)} \circ \phi^{\sigma(i)}$ or $\theta^{\sigma(i)} = \phi^{\sigma(i)} \circ \psi^{\sigma(i)}$. This allows us to compute

$$\begin{aligned}
\mathcal{E}_1^{\sigma,\mathscr{T}}\big((\psi^1, \dots, \psi^n) \circ (\phi^1, \dots, \phi^n)\big) &= \mathcal{E}_1^{\sigma,\mathscr{T}}(\theta^1, \dots, \theta^n) \\
&= (\theta^{\phi(1)}, \dots, \theta^{\phi(n)}) \\
&= (\psi^{\sigma(1)}, \dots, \psi^{\sigma(n)}) \circ (\phi^{\sigma(1)}, \dots, \phi^{\sigma(n)}) \\
&= \mathcal{E}_1^{\sigma,\mathscr{T}}(\psi^1, \dots, \psi^n) \circ \mathcal{E}_1^{\sigma,\mathscr{T}}(\phi^1, \dots, \phi^n).
\end{aligned}$$

To see that this functor is a isomorphism, we simply remark that for every $\sigma \in \mathfrak{S}_n$ we have $\sigma^{-1} \in \mathfrak{S}_n$ and

$$\mathcal{E}^{\sigma,\mathscr{T}} \circ \mathcal{E}^{\sigma^{-1},\mathscr{T}} = \mathcal{E}^{\sigma^{-1},\mathscr{T}} \circ \mathcal{E}^{\sigma,\mathscr{T}} = \mathrm{id}^{\mathsf{Groth}(\mathscr{T},\mathscr{C},\mathcal{F})}.$$

Q.E.D.

This allows us to discard the additional information in which order the arrows arise and simply count the instances of arrows in each direction. So we simplify in the following manner:

Definition 7.34. Let $n \in \mathbb{N}_{>0}, \mathscr{C}$ be a cartesian closed category as well as $\mathcal{F}\colon (\mathscr{C}^n)^{\mathrm{op}} \to \mathsf{Set}$. Then we call $\mathsf{Groth}(i, n-i, \mathscr{C}, \mathcal{F})$ to be the category $\mathsf{Groth}(\mathscr{I}, \mathscr{C}, \mathcal{F})$ where $\mathscr{I} \in n\text{-}\mathsf{Tuples}$ is the category defined by the following condition:

For all $j = 1, \dots, i$, we have $\mathtt{ar}_i\colon x_i \to y_i$ and for all $j = i+1, \dots, n$ we have $\mathtt{ar}_i\colon y_i \to x_i$.

One can visualize $\mathscr{I}$ as

$$\begin{tikzcd}[row sep=small]
x_1 & x_2 & \cdots & x_i & x_{i+1} & \cdots & x_n \\
\bullet \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[d] & \bullet & \cdots & \bullet \\
\bullet & \bullet & \cdots & \bullet & \bullet \arrow[u] & \cdots & \bullet \arrow[u] \\
y_1 & y_2 & \cdots & y_i & y_{i+1} & \cdots & y_n.
\end{tikzcd}$$

7.7 "Generalization" of the generalized Chu category

We have already found a Grothendieck construction equivalent to the standard Chu category. This begs the question whether there exists a Grothendieck construction equivalent to the generalized Chu category. The following Grothendieck construction will answer this question in a positive way.

Definition 7.35. Let $\mathscr{C}$ be a category, $\Gamma\colon \mathscr{C} \to \mathsf{Fun}((\mathscr{C}^2)^{\mathrm{op}}, \mathsf{Set})$ be a functor. We define $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)$ to be the category given by the following data:

- The *objects* of $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)$ are quadruples $(C, D, c; d)$ such that $C, D \in \mathscr{C}_0$ and $c \in [\Gamma_0(d)]_0(C, D)$.
- The *arrows* of $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)$ are triplets $(\phi^0, \phi^+, \phi^-)\colon (C, D, c, d) \to (C', D', c', d')$ such that $\phi^+\colon C \to C', \phi^-\colon D' \to D, \phi^0\colon d \to d'$ and the equality

$$\Big(\Gamma_1(\phi^0)_{(C,D')} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \phi^-)\Big)(c) = \big(\Gamma_0(d')\big)_1(\phi^+, \mathbf{1}_{D'})(c')$$

  holds.
- The *composition* of two arrows

$$\begin{aligned}(\phi^0, \phi^+, \phi^-)\colon (C, D, c, d) &\to (C', D', c', d'),\\ (\psi^0, \psi^+, \psi^-)\colon (C', D', c', d') &\to (C'', D'', c'', d'')\end{aligned}$$

  is given by $(\psi^0, \psi^+, \psi^-) \circ (\phi^0, \phi^+, \phi^-) = (\psi^0 \circ \phi^0, \psi^+ \circ \phi^+, \phi^- \circ \psi^-)$.

We visualize this as in figure 7.2.

Lemma 7.36. *Let $\mathscr{C}$ be a category, $\Gamma\colon \mathscr{C} \to \mathsf{Fun}((\mathscr{C}^2)^{\mathrm{op}}, \mathsf{Set})$ be a functor. Then $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)$ as defined above is a category.*

Proof: The well-definedness of objects and arrows is immediate. It remains to show the well-definedness of the composition. For this let two arrows

$$\begin{aligned}(\phi^0, \phi^+, \phi^-)\colon (C, D, c, d) &\to (C', D', c', d'),\\ (\psi^0, \psi^+, \psi^-)\colon (C', D', c', d') &\to (C'', D'', c'', d'')\end{aligned}$$

in $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, F, \Gamma)$ be given. By definition we obtain $\psi^+ \circ \phi^+\colon C \to C'', \phi^- \circ \psi^-\colon D'' \to D, \psi^0\colon d \to d''$. It remains to check the condition

$$\Big(\Gamma_1(\psi^0 \circ \phi^0)_{(C,D'')} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, (\phi^- \circ \psi^-))\Big)(c) = \big(\Gamma_0(d'')\big)_1(((\psi^+ \circ \phi^+), \mathbf{1}_{D'})(c'').$$

We compute (with explanations at the end)

$$\Big(\Gamma_1(\psi^0 \circ \phi^0)_{(C,D'')} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, (\phi^- \circ \psi^-))\Big)(c) \tag{31}$$

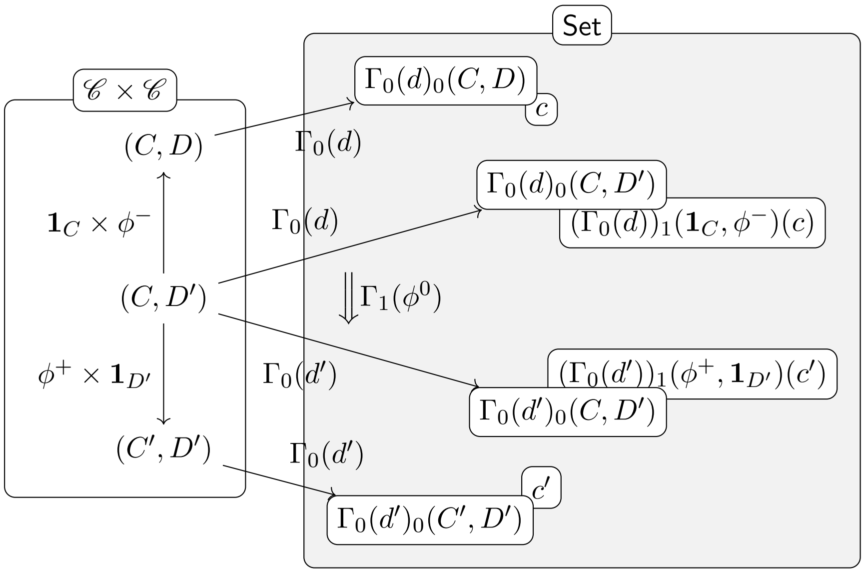


Figure 7.2: An illustration of the arrows in $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)$

$$= \Big(\Gamma_1(\psi^0 \circ \phi^0)_{(C,D'')} \circ \big(\Gamma_0(d)\big)_1\big((\mathbf{1}_C, \phi^-) \circ (\mathbf{1}_C, \psi^-)\big)\Big)(c) \tag{32}$$
$$= \Big(\Gamma_1(\psi^0 \circ \phi^0)_{(C,D'')} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \psi^-) \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \phi^-)\Big)(c) \tag{33}$$
$$= \Big(\Gamma_1(\psi^0)_{(C,D'')} \circ \Gamma_1(\phi^0)_{(C,D'')} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \psi^-) \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \phi^-)\Big)(c) \tag{34}$$
$$= \Big(\Gamma_1(\psi^0)_{(C,D'')} \circ \big(\Gamma_0(d')\big)_1(\mathbf{1}_C, \psi^-) \circ \Gamma_1(\phi^0)_{(C,D')} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \phi^-)\Big)(c) \tag{35}$$
$$= \Big(\Gamma_1(\psi^0)_{(C,D'')} \circ \big(\Gamma_0(d')\big)_1(\mathbf{1}_C, \psi^-) \circ \big(\Gamma_0(d')\big)_1(\phi^+, \mathbf{1}_{D'})\Big)(c') \tag{36}$$
$$= \Big(\Gamma_1(\psi^0)_{(C,D'')} \circ \big(\Gamma_0(d')\big)_1\big((\phi^+, \mathbf{1}_{D'}) \circ (\mathbf{1}_C, \psi^-)\big)\Big)(c') \tag{37}$$
$$= \Big(\Gamma_1(\psi^0)_{(C,D'')} \circ \big(\Gamma_0(d')\big)_1\big((\mathbf{1}_{C'}, \psi^-) \circ (\phi^+, \mathbf{1}_{D''})\big)\Big)(c') \tag{38}$$
$$= \Big(\Gamma_1(\psi^0)_{(C,D'')} \circ \big(\Gamma_0(d')\big)_1(\phi^+, \mathbf{1}_{D''}) \circ \big(\Gamma_0(d')\big)_1(\mathbf{1}_{C'}, \psi^-)\Big)(c') \tag{39}$$
$$= \Big(\big(\Gamma_0(d'')\big)_1(\phi^+, \mathbf{1}_{D''}) \circ \Gamma_1(\psi^0)_{(C',D'')} \circ \big(\Gamma_0(d')\big)_1(\mathbf{1}_{C'}, \psi^-)\Big)(c') \tag{40}$$
$$= \Big(\big(\Gamma_0(d'')\big)_1(\phi^+, \mathbf{1}_{D''}) \circ \big(\Gamma_0(d'')\big)_1(\psi^+, \mathbf{1}_{D''})\Big)(c'') \tag{41}$$
$$= \Big(\big(\Gamma_0(d'')\big)_1\big((\psi^+, \mathbf{1}_{D''}) \circ (\phi^+, \mathbf{1}_{D''})\big)\Big)(c'') \tag{42}$$
$$= \Big(\big(\Gamma_0(d'')\big)_1\big((\psi^+ \circ \phi^+), \mathbf{1}_{D''})\big)\Big)(c''). \tag{43}$$

It remains to justify the equations.

(31)=(32): This follows from lemma 7.30.

(32)=(33): This follows as $\Gamma_0(d)$ is a contravariant functor $\mathscr{C}^2 \to \mathsf{Set}$.

(33)=(34): This follows from the fact that for any three functors $F, G, H \colon \mathscr{C} \to \mathscr{D}$ and two natural transformations, $\eta \colon F \Rightarrow G, \mu \colon G \to H$ we have $(\mu \circ \eta)_x = \mu_x \circ \eta_x$ for all $x \in \mathscr{C}$. This fact is proven in [Pet20b, Definition 1.1.5.3].

(34)= (35): As $\Gamma_1(\phi^0)$ is a natural transformation $\Gamma_1(\phi^0) \colon \Gamma_0(d) \to \Gamma_0(d')$ we have a commutative diagram

$$\begin{tikzcd}
{[\Gamma_0(d)]_0(C,D')} \arrow[r, "{[\Gamma_0(d)]_1(\mathbf{1}_C,\psi^-)}"] \arrow[d, "{\Gamma_1(\phi^0)_{(C,D')}}"'] & {[\Gamma_0(d)]_0(C,D'')} \arrow[d, "{\Gamma_1(\phi^0)_{(C,D'')}}"] \\
{[\Gamma_0(d')]_0(C,D')} \arrow[r, "{[\Gamma_0(d')]_1(\mathbf{1}_C,\psi^-)}"] & {[\Gamma_0(d')]_0(C,D'')}.
\end{tikzcd}$$

This yields the equality used.

(35)=(36)**:** Here we used the equality

$$\Big(\Gamma_1(\phi^0)_{(C,D')} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C,\phi^-)\Big)(c) = \big(\Gamma_0(d')\big)_1(\phi^+,\mathbf{1}_{D'})(c')$$

that stems from the arrow (ϕ^0,ϕ^+,ϕ^-).

(36)=(37)**:** Here we used that $(\Gamma_0(d')$ is a contravariant functor.

(37)=(38)**:** Here we used the fact that

$$\begin{aligned}(\phi^+,\mathbf{1}_{D'})\circ(\mathbf{1}_C,\psi^-) &= \phi^+,\psi^- = (\mathbf{1}_{C'}\circ\phi^+),(\psi^-\circ\mathbf{1}_{D''})\\ &= (\mathbf{1}_{C'},\psi^-)\circ(\phi^+,\mathbf{1}_{D''}).\end{aligned}$$

(38)=(39)**:** Here we used again that $\Gamma_0(d')$ is a contravariant functor.

(39)=(40)**:** Here we used that $\Gamma_1(\psi^0)\colon \Gamma_0(d') \Rightarrow \Gamma_0(d'')$ is a natural transformation, so we obtain a commutative square

$$\begin{tikzcd}
{[\Gamma_0(d')]_0(C',D'')} \arrow[r, "{[\Gamma_0(d')]_1(\phi^+,\mathbf{1}_{D''})}"] \arrow[d, "{\Gamma_1(\psi^0)_{(C',D'')}}"'] & {[\Gamma_0(d')]_0(C,D'')} \arrow[d, "{\Gamma_1(\psi^0)_{(C,D'')}}"] \\
{[\Gamma_0(d'')]_0(C',D'')} \arrow[r, "{[\Gamma_0(d'')]_1(\phi^+,\mathbf{1}_{D''})}"] & {[\Gamma_0(d'')]_0(C,D'')},
\end{tikzcd}$$

which yields the desired equality.

(40)=(41)**:** Here we used the equality

$$\Big(\Gamma_1(\psi^0)_{(C',D'')} \circ \big(\Gamma_0(d')\big)_1(\mathbf{1}_{C'},\psi^-)\Big)(c') = \big(\Gamma_0(d'')\big)_1(\psi^+,\mathbf{1}_{D''})(c'')$$

that stems from the arrow (ψ^0,ψ^+,ψ^-).

(41)=(42)**:** Here we used that $(\Gamma_0(d''))_1$ is a contravariant functor.

(42)=(43)**:** At last we used the equality

$$(\psi^+,\mathbf{1}_{D''})\circ(\phi^+,\mathbf{1}_{D''}) = \big((\psi^+\circ\phi^+),(\mathbf{1}_{D''}\circ\mathbf{1}_{D''})\big) = \big((\psi^+\circ\phi^+),\mathbf{1}_{D''}\big).$$

So we have proven that the composition of two arrows is well-defined. It remains to check the associativity and the existence of identities.

Associativity: Let three arrows

$$\begin{aligned}(\phi^0,\phi^+,\phi^-)&\colon (C_1,D_1,c_1,d_1)\to(C_2,D_2,c_2,d_2),\\ (\psi^0,\psi^+,\psi^-)&\colon (C_2,D_2,c_2,d_2)\to(C_3,D_3,c_3,d_3),\\ (\theta^+,\theta^+,\theta^-)&\colon (C_3,D_3,c_3,d_3)\to(C_4,D_4,c_4,d_4)\end{aligned}$$

be given. Then

$$\begin{aligned}(\theta^0,\theta^+,\theta^-)\circ\big((\psi^0,\psi^+,\psi^-)\circ(\phi^0,\phi^+,\phi^-)\big) &= (\theta^0,\theta^+,\theta^-)\circ(\psi^0\circ\phi^0,\psi^+\circ\phi^+,\phi^-\circ\psi^-)\\ &= \big(\theta^0\circ(\psi^0\circ\phi^0),\theta^+\circ(\psi^+\circ\phi^+),(\phi^-\circ\psi^-)\circ\theta^-\big)\end{aligned}$$

$$
\begin{aligned}
&= \big((\theta^0 \circ \psi^0) \circ \phi^0, (\theta^+ \circ \psi^+) \circ \phi^+, \phi^- \circ (\psi^- \circ \theta^-)\big) \\
&= (\theta^0 \circ \psi^0, \theta^+ \circ \psi^+, \psi^- \circ \theta^-) \circ (\phi^0, \phi^+, \phi^-) \\
&= \big((\theta^0, \theta^+, \theta^-) \circ (\psi^0, \psi^+, \psi^-)\big) \circ (\phi^0, \phi^+, \phi^-).
\end{aligned}
$$

Identities: We want to show that $\mathbf{1}_{(C,D,c,d)} = (\mathbf{1}_d, \mathbf{1}_C, \mathbf{1}_D)$ for all $(C, D, c, d) \in \overleftarrow{\overrightarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)_0$. It is immediate that $\mathbf{1}_d \colon d \to d, \mathbf{1}_C \colon C \to C, \mathbf{1}_D \colon D \to D$, so it remains to check the condition

$$\Big(\Gamma_1(\mathbf{1}_d)_{(C,D)} \circ \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \mathbf{1}_D)\Big)(c) = \big(\Gamma_0(d)\big)_1(\mathbf{1}_C, \mathbf{1}_D)(c).$$

For this we simply remark that $\Gamma_1(\mathbf{1}_d)_{(C,D)} = \mathbf{1}_{(C,D)}$, so the equality holds. To see that $(\mathbf{1}_d, \mathbf{1}_C, \mathbf{1}_D)$ fulfils the axioms of an identity is immediate. So $\overleftarrow{\overrightarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)$ is a category. Q.E.D.

Now we want to see that this construction actually generalizes the generalized Grothendieck construction. The first problem at hand is that the generalized Chu construction uses a functor $\mathscr{C} \to \mathscr{C}$, not a functor $\mathscr{C} \to \mathsf{Fun}((\mathscr{C}^2)^{\mathrm{op}}, \mathsf{Set})$. This we can remedy by the following construction.

Definition 7.37. Let $\mathscr{C}$ be a category and $\Gamma \colon \mathscr{C} \to \mathscr{C}$ be an endofunctor. We then construct a functor $F^\Gamma \colon \mathscr{C} \to \mathsf{Fun}((\mathscr{C}^2)^{\mathrm{op}}, \mathsf{Set})$ in the following way.

- For all $c \in \mathscr{C}_0$ we set $F_0^\Gamma(c) := \mathrm{Hom}_{\mathscr{C}}(\text{-} \times \text{-}, \Gamma_0(c))$.
- For all arrows $f \colon c \to c'$ in $\mathscr{C}$ we set $F_1^\Gamma(f)$ to be the natural transformation

  $$F_1^\Gamma(f) \colon \mathrm{Hom}_{\mathscr{C}}(\text{-} \times \text{-}, \Gamma_0(c)) \to \mathrm{Hom}_{\mathscr{C}}(\text{-} \times \text{-}, \Gamma_0(c'))$$

  defined by

  $$
  \begin{aligned}
  F_1^\Gamma(f)_{(C,D)} \colon \mathrm{Hom}_{\mathscr{C}}(C \times D, \Gamma_0(c)) &\to \mathrm{Hom}(C \times D, \Gamma_0(c')), \\
  h &\mapsto \Gamma_1(f) \circ h.
  \end{aligned}
  $$

Lemma 7.38. *Let $\mathscr{C}$ be a category and $\Gamma \colon \mathscr{C} \to \mathscr{C}$ be an endofunctor. Then F^Γ defined as above is a functor.*

Proof: The well-definedness is immediate. It remains to check the axioms of a functor. *Compatibility with composition:* Let $f \colon c \to c'$ and $g \colon c' \to c''$ be arrows in $\mathscr{C}$. It suffices to check the equality of the natural transformations on the arrows. So let $r \in \mathrm{Hom}_{\mathscr{C}}(C \times D, \Gamma_0(c))$ be given. Then

$$
\begin{aligned}
F_1^\Gamma(g \circ f)_{(C,D)}(r) &= \Gamma_1(g \circ f) \circ r = \Gamma_1(g) \circ \Gamma_1(f) \circ r \\
&= \Gamma_1(g) \circ F_1^\Gamma(f)_{(C,D)}(r) = \big(F_1^\Gamma(g)_{(C,D)} \circ F_1^\Gamma(f)_{(C,D)}\big)(r).
\end{aligned}
$$

Compatibility with identities: We have to show that $\Gamma_1(\mathbf{1}_c)$ is the identity natural transformation. It again suffices to check this on the arrows. So let $f \in \mathrm{Hom}_{\mathscr{C}}(C \times D, \Gamma_0(c))$ be given. Then

$$F^\Gamma(\mathbf{1}_c)_{(C,D)}(f) = \Gamma_1(\mathbf{1}_c) \circ f = \mathbf{1}_{\Gamma_0(c)} \circ f = f.$$

So we have shown that $F^\Gamma \colon \mathscr{C} \to \mathsf{Fun}((\mathscr{C}^2)^{\mathrm{op}}, \mathsf{Set})$ is a functor. Q.E.D.

Remark 7.39. Before we identify the generalized Chu category with our Grothendieck construction, we will make a few identifications in case $\mathscr{C}$ is a cartesian closed category.

- We will identify the objects of $\overleftarrow{\overrightarrow{\mathsf{Groth}}}(\mathscr{C}, \Gamma)$ with quadruples (C, D, c, d) such that $c \in (\Gamma_0(d))_0(C \times D)$.

- We will identify the arrows $(\mathbf{1}_C, \phi^-)$ and $(\phi^+, \mathbf{1}_{D'})$ with the arrows $\mathbf{1}_C \times \phi^-$ and $\phi^+ \times \mathbf{1}_{D'}$.

Theorem 7.40. *Let $\mathscr{C}$ be a cartesian closed category and $\Gamma\colon \mathscr{C} \to \mathscr{C}$ be an endofunctor. Then*

$$\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, F^{\Gamma}) = \mathsf{Chu}(\mathscr{C}, \Gamma).$$

Proof: We start by identifying the objects. So let $(C, D, c, d) \in \overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, F^{\Gamma})$. Then $C, D, d \in \mathscr{C}_0$ and $c \in (F_0^{\Gamma}(d))_0(C \times D) = \mathrm{Hom}_{\mathscr{C}}(C \times D, \Gamma_0(d)$. So $c\colon C \times D \to \Gamma_0(d)$, ergo $(d, C, c, D) \in \mathsf{Chu}(\mathscr{C}, \Gamma)$. One can immediately verify that this identification is one-to-one.

Next we identify the arrows. So let $(\phi^0, \phi^+, \phi^-)\colon (C, D, c, d) \to (C', D', c', d')$ be an arrow in $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, F^{\Gamma})$. Then

$$\phi^0\colon d \to d', \qquad \phi^+\colon C \to C', \qquad \phi^-\colon D' \to D.$$

The condition on the arrows translates to

$$\Gamma_1(\phi^0) \circ c \circ (\mathbf{1}_C \times \phi^+) = c' \circ (\phi^+ \times \mathbf{1}_{D'}),$$

which is exactly the desired commutativity of the rectangle

$$\begin{tikzcd}
C \times D' \arrow[rr, "\phi^+ \times \mathbf{1}_{D'}"] \arrow[d, "\mathbf{1}_C \times \phi^+"'] & & C' \times D' \arrow[d, "c'"] \\
C \times D \arrow[r, "c"'] & \Gamma_0(d) \arrow[r, "\Gamma_1(\phi^0)"'] & \Gamma_0(d')
\end{tikzcd}$$

once the objects of $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, F^{\Gamma})$ are identified with the objects of $\mathsf{Chu}(\mathscr{C}, \Gamma)$. So we have the desired equality $\overrightarrow{\underleftarrow{\mathsf{Groth}}}(\mathscr{C}, F^{\Gamma}) = \mathsf{Chu}(\mathscr{C}, \Gamma)$. Q.E.D.

Chapter 8

The Chu construction and topoi

Chu spaces, which are objects of Chu categories $\mathsf{Chu}(\mathsf{Set}, X)$ for a set X, have found their way into many domains of mathematics, theoretical informatics and theoretical physics. Now as Set is the archetypical topos, it is sensible to ask whether the Chu category $\mathsf{Chu}(\mathscr{T}, \gamma)$ over a topos $\mathscr{T}$ and an object $\gamma \in \mathscr{T}$ is again a topos.

8.1 The definition of a topos

We start by recalling the categorical notion of a topos, which we take from [Awo10, Definition 8.16].

Definition 8.1 (Topoi). A *topos* $\mathscr{T}$ is a category such that the following two conditions hold:

(Tps$_1$) The category $\mathscr{T}$ is complete (has all finite limits).

(Tps$_2$) The category $\mathscr{T}$ has all exponentials.

(Tps$_3$) The category $\mathscr{T}$ has a *subobject classifier*, that is an object Ω together with an arrow $t\colon \top \to \Omega$ such that for every object $E \in \mathscr{T}_0$ and any subobject $U \hookrightarrow E$ there exists a unique arrow making the diagram

$$\begin{tikzcd} U \arrow[r, "!_U"] \arrow[d, hook] & \top \arrow[d, "t"] \\ E \arrow[r, "u"] & \Omega \end{tikzcd}$$

commute and U a pullback. The arrow u is the *classifying arrow* of $U \hookrightarrow E$.

Remark 8.2. We do not have to demand that $\mathscr{T}$ is a cartesian closed category, as $\mathscr{T}$ has all finite limits if and only if it has finite products and equalizers, which again can only be if it has pullbacks and a terminal object. So $\mathscr{T}$ already has finite products and a terminal object $\top$ as it has all finite limits. The proofs for the equivalences can be found in [Awo10, Section 5.4].

8.2 The Chu construction over Set

The category Set is, as we have already mentioned, the archetypical topos. Our next goal is to prove that $\mathsf{Chu}(\mathsf{Set}, K)$ is generally not a topos. To this end we first make the following observation.

Proposition 8.3. *Let K be an arbitrary set. Then $\mathsf{Chu}(\mathsf{Set}, K)$ is bicomplete.*

Proof: As Set is bicomplete, we can use theorem 6.14 to deduce that $\mathsf{Chu}(\mathsf{Set}, K)$ is bicomplete.

Q.E.D.

This proves that the first condition of a topos is fulfilled. So we direct our attention to the subobject classifier. How does such an object look in the Chu construction? A subobject classifier $(\Omega, w, \mho) \in \mathsf{Chu}(\mathsf{Set}, K)$ comes with an arrow $(\phi^+, \phi^-)\colon T \to (\Omega, w, \mho)$, where T is the terminal object of $\mathsf{Chu}(\mathsf{Set}, K)$. So we have to determine this terminal object.

Lemma 8.4. *The terminal object of the category* $\mathsf{Chu}(\mathsf{Set}, K)$ *for arbitrary* X *is given by* $(\mathbb{1}, ¡_K, \emptyset)$*, where* $\mathbb{1}$ *is the set* $\mathbb{1} := \{\emptyset\}$ *and* $¡_K$ *is the unique arrow* $¡_K\colon \emptyset \to K$.[7]

Proof: Let K be an arbitrary set. Suppose we are given $(A, f, X) \in \mathsf{Chu}(\mathsf{Set}, X)$. We have to find a unique arrow $!_{(A,f,X)}\colon (A, f, X) \to (\mathbb{1}, ¡_X, \emptyset)$. By definition we find unique arrows $!_A\colon A \to \mathbb{1}$ and $¡_X\colon \emptyset \to X$. For $(!_A, ¡_X)\colon (A, f, X) \to (\mathbb{1}, ¡_K, \emptyset)$ to be an arrow in $\mathsf{Chu}(\mathsf{Set}, K)$, we need to show the commutativity of

$$\begin{tikzcd} A \times \emptyset \arrow[r, "\mathbf{1}_A \times ¡_X"] \arrow[d, "!_A \times \mathbf{1}_\emptyset"'] & A \times X \arrow[d, "f"] \\ \mathbb{1} \times \emptyset \arrow[r, "¡_K"] & K. \end{tikzcd}$$

But by lemma 6.16 we know that $A \times \emptyset \cong \emptyset$. Using this isomorphism we obtain that both arrows $f \circ (\mathbf{1}_A \times ¡_X)$ and $¡_K \circ (!_A \times \mathbf{1}_\emptyset)$ are identical to the arrow $¡_K$, so the diagram commutes. The arrows $!_A$ and $¡_X$ are unique, hence is $(!_A, ¡_X)$. Q.E.D.

So we have identified the terminal object of $\mathsf{Chu}(\mathsf{Set}, K)$. Our next goal is to find an arrow $(\phi^+, \phi^-)\colon (\mathbb{1}, ¡_K, \emptyset) \to (\Omega, w, \mho)$ in $\mathsf{Chu}(\mathsf{Set}, K)$. This arrows is given by the arrows $\phi^+\colon \mathbb{1} \to \Omega$ and $\phi^-\colon \mho \to \emptyset$. But as there exists only one arrow in Set with codomain $\emptyset$, the arrow $\mathbf{1}_\emptyset$, we know that $\mho = \emptyset$. It remains to show that this equality leads to a contradiction. For this we examine the subobjects in $\mathsf{Chu}(\mathsf{Set}, K)$ a little further.

A subobject $(\theta^+, \theta^-)\colon (B, g, Y) \hookrightarrow (A, f, X)$ is given by an object and a monomorphism. But how do monomorphisms in $\mathsf{Chu}(\mathsf{Set}, K)$ look like? If we are given the situation

$$\begin{tikzcd} (C, h, Z) \arrow[r, shift left, "(\alpha^+, \alpha^-)"] \arrow[r, shift right, "(\beta^+, \beta^-)"'] & (B, g, Y) \arrow[r, "(\theta^+, \theta^-)"] & (A, f, X), \end{tikzcd}$$

then $(\theta^+, \theta^-) \circ (\alpha^+, \alpha^-) = (\theta^+, \theta^-) \circ (\beta^+, \beta^-)$ has to imply $(\alpha^+, \alpha^-) = (\beta^+, \beta^-)$. To dissect this further, if $\theta^+ \circ \alpha^+ = \theta^+ \circ \beta^+$ and $\alpha^- \circ \theta^- = \beta^- \circ \theta^-$, then $\alpha^+ = \beta^+$, $\alpha^- = \beta^-$. So we see that if θ^+ is a monomorphism and θ^- is an epimorphism, then (θ^+, θ^-) is a monomorphism in $\mathsf{Chu}(\mathsf{Set}, K)$. One should note that these conditions on θ^+, θ^- are sufficient, but not necessary.

So we move on to our contradiction that will prove that $\mathsf{Chu}(\mathsf{Set}, K)$ is not a topos.

Lemma 8.5. *Let* K *be an arbitrary set, such that there exist sets* A, X *and a map* $f\colon A \times X \to K$*, such that the cardinality of* X *is greater or equal 2. Then the category* $\mathsf{Chu}(\mathsf{Set}, K)$ *does not have a subobject classifier.*

Proof: Suppose $\mathsf{Chu}(\mathsf{Set}, K)$ contains a subobject classifier $(\Omega, w, \emptyset)$ with an arrow $(\phi^+, \phi^-)\colon (\mathbb{1}, ¡_K, \emptyset) \to (\Omega, w, \emptyset)$. Consider $(A, f, X) \in \mathsf{Chu}(\mathsf{Set}, K)_0$ where f is not dependent on the input of X, i.e. given an $a \in A$ and arbitrary $x, x' \in X$ we have the equality $f(a, x) = f(a, x')$. Suppose we have a subobject $(B, g, \mathbb{1}) \in \mathsf{Chu}(\mathsf{Set}, X)_0$, where $B \hookrightarrow A$ is a subset and g is defined by restriction, i.e. $g = f\big|_{B \times \mathbb{1}}$. Consider $(i_B, !_X)\colon (B, g, \mathbb{1}) \to (A, f, X)$, where i_B

[7] In this notation we suppress the isomorphism $\mathbb{1} \times \emptyset \xrightarrow{\cong} \emptyset$ that would need to be precomposed with $¡_K$. But as we always identify isomorphic objects this does not change our results.

is the inclusion $B \hookrightarrow A$. To see that this is an arrow in $\mathsf{Chu}(\mathsf{Set}, K)$, we need to show the commutativity of

$$\begin{tikzcd} B \times X \arrow[r, "\mathbf{1}_B \times !_X"] \arrow[d, "i_A \times \mathbf{1}_X"'] & B \times \mathbb{1} \arrow[d, "g"] \\ A \times X \arrow[r, "f"] & K. \end{tikzcd}$$

But this is immediate as g is the restriction of f to $B \times \mathbb{1}$ and f is independent of the input in X. So given $b \in B, x \in X$ we can compute

$$\begin{aligned} \big(f \circ (i_B \times \mathbf{1}_X)\big)(b, x) &= f(b, x) \\ &= f(b, 0) && (f \text{ independent of } X) \\ &= f\big|_{B \times \mathbb{1}}(b, 0) && (\text{as } (b, 0) \in B \times \mathbb{1}) \\ &= g(b, 0) \\ &= \big(g \circ (\mathbf{1}_B \times !_X)\big)(b, x). \end{aligned}$$

So $(i_B, !_X)$ is an arrow in $\mathsf{Chu}(\mathsf{Set}, K)$ and a monomorphism, as i_B is a monomorphism and $\mathbf{1}_K$ is an epimorphism. As $(\Omega, w, \emptyset)$ is a subobject classifier, we find a unique $(\xi^+, \xi^-)\colon (A, f, X) \to (\Omega, w, \emptyset)$ making the diagram

$$\begin{tikzcd} (B, g, \mathbb{1}) \arrow[r, "(!_B, ¡_{\mathbb{1}})"] \arrow[d, hook, "(i_B !_X)"'] & (\mathbb{1}, ¡_K, \emptyset) \arrow[d, "(\phi^+, \mathbf{1}_\emptyset)"] \\ (A, f, X) \arrow[r, "(\xi^+, \xi^-)"] & (\Omega, w, \emptyset) \end{tikzcd} \tag{44}$$

commute and $(B, g, \mathbb{1})$ a pullback. As $\emptyset$ is the initial object of Set we immediately obtain $\xi^- = ¡_X$. Now consider the situation given by the diagram

$$\begin{tikzcd} (B, h, X) \arrow[drr, bend left, "(!_B, ¡_X)"] \arrow[ddr, bend right, "(i_B, \mathbf{1}_X)"'] & & \\ & (B, g, \mathbb{1}) \arrow[r, "(!_B, ¡_{\mathbb{1}})"] \arrow[d, hook, "(i_B, !_X)"'] & (\mathbb{1}, ¡_K, \emptyset) \arrow[d, "(\phi^+, \mathbf{1}_\emptyset)"] \\ & (A, f, X) \arrow[r, "(\xi^+, \xi^-)"] & (\Omega, w, \emptyset), \end{tikzcd} \tag{45}$$

where h is the restriction $f\big|_{B \times X}$. We want to show that $(i_B, \mathbf{1}_X)$ is an arrow in $\mathsf{Chu}(\mathsf{Set}, K)$. For this we need to show the commutativity of

$$\begin{tikzcd} B \times X \arrow[r, "\mathbf{1}_B \times \mathbf{1}_X"] \arrow[d, "i_B \times \mathbf{1}_X"'] & B \times X \arrow[d, "h"] \\ A \times X \arrow[r, "f"] & K. \end{tikzcd}$$

But this is immediate, as h is the restriction of f to $B \times X$. Next we show that

$$\begin{tikzcd} (B, h, X) \arrow[r, "(!_B, ¡_X)"] \arrow[d, "(i_B, \mathbf{1}_X)"'] & (\mathbb{1}, ¡_K, \emptyset) \arrow[d, "(\phi^+, \mathbf{1}_\emptyset)"] \\ (A, f, X) \arrow[r, "(\xi^+, ¡_X)"] & (\Omega, w, \emptyset) \end{tikzcd}$$

commutes. For this we need to show the commutativity of

$$
\begin{tikzcd}
B \arrow[r, "!_B"] \arrow[d, "i_B"'] & \mathbb{1} \arrow[d, "\phi^+"] \\
A \arrow[r, "\xi^+"'] & \Omega
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X & \emptyset \arrow[l, "¡_X"'] \\
X \arrow[u, "\mathbf{1}_X"] & \emptyset. \arrow[l, "¡_X"] \arrow[u, "\mathbf{1}_\emptyset"']
\end{tikzcd}
$$

As the commutativity of the right diagram is immediate from the definition of $\emptyset$, we only need to consider the left one. But the left diagram commutes as (44) commutes. So as $(B, g, \mathbb{1}$ is a pullback , we obtain a unique arrow $(\rho^+, \rho^-)\colon (B, h, X) \to (B, g, \mathbb{1})$ making (45) commute. We show that such an arrow can not exist. If such an arrow would exist, we would have the commutative diagram

$$
\begin{tikzcd}
X & & \\
& \mathbb{1} \arrow[ul, dashed, "\rho^-"'] & \emptyset \arrow[l, "¡_{\mathbb{1}}"'] \arrow[ull, bend right, "¡_X"'] \\
& X \arrow[u, "!_X"] \arrow[uul, bend left, "\mathbf{1}_X"] & \emptyset. \arrow[l, "¡_X"] \arrow[u, "\mathbf{1}_\emptyset"']
\end{tikzcd}
$$

But such a ξ^- does not exist if the cardinality of X is greater or equal 2. So $(B, g, \mathbb{1})$ is not a pullback, therefore no subobject classifier in $\mathsf{Chu}(\mathsf{Set}, K)$ exists. Q.E.D.

Now we examine which categories of Chu spaces $\mathsf{Chu}(\mathsf{Set}, K)$ have objects (A, f, X) such that X has cardinality greater than two. Here we arrive at the following conclusion.

Theorem 8.6. *Let K be an arbitrary set. Then the category $\mathsf{Chu}(\mathsf{Set}, K)$ has an object (A, f, X) such that the cardinality of X is greater or equal 2, so $\mathsf{Chu}(\mathsf{Set}, K)$ is not a topos.*

Proof: If K itself has cardinality greater or equal two, we can consider the object $(\mathbb{1}, \mathtt{pr}_K, K) \in \mathsf{Chu}(\mathsf{Set}, K)_0$, where $\mathtt{pr}_K\colon \mathbb{1} \times K \to K$ is the projection. With the preceding lemma we can conclude that $\mathsf{Chu}(\mathsf{Set}, K)$ is not a topos.

Now let K have cardinality 1 or 0. This means that K is isomorphic to either $\mathbb{1}$ or $\emptyset$. We first consider the case $K \cong \mathbb{1}$, but as we identify isomorphic objects we arrive at $K = \mathbb{1}$.

If $K = \mathbb{1}$, then every pair (A, X) of sets can be made a object of $\mathsf{Chu}(\mathsf{Set}, \mathbb{1})$ using $(A, !_{A\times X}, X)$. Hence we can simply choose $\mathbb{2} = \{0, 1\}$ and the conditions of the preceding lemma are satisfied. Hence $\mathsf{Chu}(\mathsf{Set}, \mathbb{1})$ is not a topos.

At last consider $\mathsf{Chu}(\mathsf{Set}, \emptyset)$. As there exists only one arrow with codomain $\emptyset$, we know that all objects $(A, f, X) \in \mathsf{Chu}(\mathsf{Set}, \emptyset)_0$ are given by objects which fulfil $A \times X \cong \emptyset$. Using lemma 6.16 we know that for an arbitrary set X we have $\emptyset \times X \cong \emptyset$. This allows us to consider the object $(\emptyset, \mathbf{1}_\emptyset, \mathbb{2})$ in $\mathsf{Chu}(\mathsf{Set}, \emptyset)$, where we suppress the isomorphism $\emptyset \times \mathbb{2} \cong \emptyset$ in the notation of $\mathbf{1}_\emptyset$. So we can again use the preceding lemma, so $\mathsf{Chu}(\mathsf{Set}, \emptyset)$ is not a topos.

So in all possible cases for K we have that $\mathsf{Chu}(\mathsf{Set}, K)$ is not a topos. Q.E.D.

Chapter 9

Conclusion

9.1 Open questions and future tasks

As this thesis is in no regard an exhausting treatment of the topic of Chu categories, there remain open questions and tasks, of which we name a few.

1. Are there categories which require one of the generalizations of the Chu category we gave for an embedding? For example is there a category $\mathscr{C}$ such that there exists a fully faithful functor $F\colon \mathscr{C} \to \mathsf{Groth}(\mathscr{C}, F)$, where $F\colon (\mathscr{C} \times \mathscr{C})^{\mathrm{op}} \to \mathsf{Set}$ is a functor not given as a Hom-functor $\mathrm{Hom}(\text{-} \times \text{-}, \gamma)$?

2. Can one externalize the Grothendieck construction over finite products given in section 7.6 in the following way:

When one wants to define a projection $\Pi\colon \mathsf{Chu}(\mathscr{C}, \gamma) \to \mathscr{D}$ for a to be determined category $\mathscr{C}$, one immediately arrives at the notion of the *antiparallel product* of categories, that is a category $\mathscr{C} \times^{\leftrightarrows} \mathscr{C}$ which has the same objects as the category $\mathscr{C} \times \mathscr{C}$, but the arrows differ as such. An arrow $f\colon (a, b) \to (c, d)$ in $\mathscr{C} \times^{\leftrightarrows} \mathscr{C}$ is given as a pair $\langle f^+, f^- \rangle\colon (a, b) \to (c, d)$ where $f^+\colon a \to c$ and $f^-\colon d \to b$.

Now the first thing one sees is that unlike with the "standard" product of categories, one obtains ad hoc

$$(\mathscr{C} \times^{\leftrightarrows} \mathscr{C}) \times^{\leftrightarrows} \mathscr{C} \neq \mathscr{C} \times^{\leftrightarrows} (\mathscr{C} \times^{\leftrightarrows} \mathscr{C}),$$

which is good, as it mimics the property of the Grothendieck construction over finite products. Now for a given $\mathscr{I} \in n\text{-}\mathsf{Tuples}$ can one find an ordering of n copies of $\mathscr{C}$ and parenthesis such that

$$\mathsf{Groth}(\mathscr{I}, \mathscr{C}, \mathrm{Hom}(\text{-} \times \cdots \times \text{-}, \gamma)) = (\mathscr{C} \times^{\leftrightarrows} \ldots \times^{\leftrightarrows} \mathscr{C} \times^{\leftrightarrows} (\mathscr{C} \times^{\leftrightarrows} \ldots \times^{\leftrightarrows} \mathscr{C} \times^{\leftrightarrows} (\ldots)))?$$

3. Can one find a functor $\mathsf{F}\colon \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathsf{Fun}(\mathsf{Chu}(\mathscr{C}, \gamma)^{\mathrm{op}}, \mathsf{Set}$ making the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "E^{\mathscr{C},\gamma}"] \arrow[d, "\mathsf{y}"'] & \mathsf{Chu}(\mathscr{C}, \gamma) \arrow[d, "\mathsf{y}"] \\
\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set}) \arrow[r, "\mathsf{F}"'] & \mathsf{Fun}(\mathsf{Chu}(\mathscr{C}, \gamma)^{\mathrm{op}}, \mathsf{Set})
\end{tikzcd}$$

commute?

4. One can employ the Chu construction over a symmetric monoidal category by replacing all occurrences of $\times$ in the definition with $\otimes$. Now if we are given a category with products, coproducts, a terminal object and an initial object, it is remarked in [BW20, Example 16.1.3] that there are two ways to obtain a symmetric monoidal structure on $\mathscr{C}$, on the one hand one takes the products and the terminal object and on the other hand one takes the coproducts and the initial object. Therefore we find two (a priori different) Chu structures on $\mathscr{C}$. Now can one find a generalized construction, that encompasses the information of both these Chu constructions as well as possible interplay between products and coproducts in the underlying category $\mathscr{C}$?

5. We have seen that the bicompleteness of the base category $\mathscr{C}$ implies the bicompletness of the Chu construction $\mathsf{Chu}(\mathscr{C}, \gamma)$ for any choice of $\gamma \in \mathscr{C}_0$. Now the obvious question is whether a reverse implication holds. Unfortunately we can a priori only construct "pseudolimits" using the bicompleteness of the Chu construction, as the uniqueness in the universal property can not be trivially reduced to the uniqueness in the Chu construction. This gives rise to two questions.

   Are these "pseudolimits" interesting in other mathematical research, i.e. can they be used in answers to other categorical questions?

   What restriction have to be put on $\mathscr{C}$ and γ such that the bicompleteness of $\mathsf{Chu}(\mathscr{C}, \gamma)$ implies the bicompleteness of $\mathscr{C}$? One can for example observe that this implication holds if $\gamma = \top$, the terminal object, but this also gives us an isomorphism $\mathsf{Chu}(\mathscr{C}, \top) \cong \mathscr{C} \times \mathscr{C}^{\mathrm{op}}$, so this case is not particularly interesting. So $\top = \gamma$ is sufficient, but is it also necessary?

6. Can theorem 6.14 be generalized to the generalized Chu category?

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Catalogue of categories

Glossary of symbols

This index contains non-alphabetic symbols and special notations used in this paper. An example for a special notation would be the isomorphism $F_{ab}\colon F_0(a) \times F_0(b) \to F_0(a \times b)$ associated to a product preserving functor. Letters will be used in he according manner.

- The letters a and b will always refer to objects of a given category $\mathscr{C}$.
- The letter F will always refer to a functor whose (co-)domain is an arbitrary category.
- The letter f shall always refer to an arrow whose domain is a product.
- The letter u denotes an arrow with arbitrary codomain and domain.
- The letters μ, η are used to denote natural transformations.

Index

The font of the page number denotes the following. If the page number is bold, it refers to a definition. If it is italic, it refers to an equivalent characterisation, and a normal upright font denotes theorems, lemmas and corollaries using this definition. So if we take for example the following entry:

Bicomplete category, **19**,*20*,25

It means that the notion of "bicomplete categories" is introduced in page 19, a condition equivalent to the bicompleteness of a category is given in 20, and a theorem using bicomplete categories can be found in 25.

Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und ohne fremde Hilfsmittel angefertigt habe.

Die Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

Luis Gambarte

München, den