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2-dep-Categories

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Ort, Datum	Yannick Ehrhardt

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1 Introduction

Category theory aims to generalize the notion of function. To that end arrows with a domain and a codomain as well as a composition of those arrows are defined. However, many of the structures that we aim to generalize with category theory, like sets or types, also have dependent functions. These are not captured by their respective categories. Hence, if we wish to talk about dependency from a categorical perspective, we need to introduce new structures to the notion of category. One approach for just that, due to Petrakis [3], giving additional structure to categories, works roughly as follows: First, you add family arrows, which have a domain but no codomain. Then you can add dependent arrows, which have a regular object as domain and a family arrow as codomain. Along the way, it also makes sense to introduce Σ-objects, which generalize notions like the Σ-type. These structures give rise to what are called $\mathtt{fam}$-, $(\mathtt{fam}, \Sigma)$-, $\mathtt{dep}$ and $(\mathtt{dep}, \Sigma)$-categories. However, these can still be extended by introducing 2-$\mathtt{fam}$-arrows, arrows between family arrows. To motivate this, consider Martin-Loef Type Theory (MLTT). For the notions from MLTT that we will use here, refer to the HoTT Book [5]. Type-families are given by

- a function $f : A \to U$ from
- a type $A : U$ to
- the universe U.

Now considering a second such family $h : A \to U$, we can define a type of functions, which, in some sense, maps from f to h.
This type is given by

$$\prod_{x:A}(f(x) \to h(x)).$$

What we will do in this thesis, is to consider a generalized notion of this kind of function, the 2-$\mathtt{fam}$-arrows, which extend the previously mention approach to dependency in category theory. We will extend many of the already established examples and we will also extend the theorems on how $(\mathtt{fam}, \Sigma)$-categories induce $\mathtt{dep}$- and $(\mathtt{dep}, \Sigma)$-categories to work with this additional structure. We will also show, that the example of a $\mathtt{fam}$- and $(\mathtt{fam}, \Sigma)$-category given by a topos can be extended to also include 2-$\mathtt{fam}$-arrows in a sensible way and investigate the structure from applying the aforementioned theorems to the topos examples.
For that, first, in the remainder of the introduction (Section 1), we will give definitions of categories and functors, to make the notation we will be using clear. These definitions are largely adopted from the respective definitions in [2].
Then, in Section 2, we will lay down the definitions for $\mathtt{fam}$-, $(\mathtt{fam}, \Sigma)$-, $\mathtt{dep}$ and $(\mathtt{dep}, \Sigma)$-categories as well as prove some statements about them that will be useful later. This section is largely adopted form [3].
In Section 3, we will then introduce 2-$\mathtt{fam}$-arrows and add them to the previously defined structures. We will define 2-$\mathtt{fam}$-, (2-$\mathtt{fam}, \Sigma$)-, 2-$\mathtt{dep}$ and (2-$\mathtt{dep}, \Sigma$)-categories and we see that all (2-$\mathtt{fam}, \Sigma$)-categories induce 2-$\mathtt{dep}$- and (2-$\mathtt{dep}, \Sigma$)-categories.
After that, there will be some concluding comments (Section 4).

Now, as mentioned, we will spend the rest of this section with setting the stage for the later discussions. First we define the notion of a "category":

Definition 1.0.1 (Category)**.**
C consisting of

- a collection C_0 of objects,
- a collection $\mathrm{Hom}(a, b)$ of arrows or morphisms from a to b, $\forall a, b \in C_0$,
- an operation $\mathrm{Hom}(b, c) \times \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, c), (f, h) \mapsto f \circ h, \forall a, b, c \in C_0$ (called composition) and
- an arrow $\mathrm{id}_a \in \mathrm{Hom}(a, a), \forall a \in C_0$ (called the identity)

is called a category, iff

1. $\forall a, b \in C_0 \forall f \in \mathrm{Hom}(b, a)$:
$$\mathrm{id}_a \circ f = f = f \circ \mathrm{id}_b$$
and
2. $\forall a, b, c, d \in C_0 \forall f \in \mathrm{Hom}(b, a) \forall h \in \mathrm{Hom}(c, b) \forall p \in \mathrm{Hom}(d, c)$:
$$f \circ (h \circ p) = (f \circ h) \circ p.$$

We also define C_1 to be the collections of all arrows in C.

Recall earlier we used types as a motivation. So we better make sure that types actually fulfill all the definitions we present. And, as we will see, they do. With one caveat, we will need function extensionality and we will be using it explicitly many times. Hence, we give ourselves function extensionality within type universes as an axiom:

Axiom 1.0.2 (Function Extensionality for types)**.**
We assume for any type universe that we will be working with, that function extensionality holds within it.

Then, since types are our main example and motivation, we will now show, that small types form a category. This is of course pretty easy, but we shall do still it for the sake of completeness.

Proposition 1.0.3 (Small types form a category)**.**
Suppose U is a type universe.
Then, $\mathrm{Type}(U)$ consisting of

- $\mathrm{Type}(U)_0 := U$,
- $\forall A, B \in \mathrm{Type}(U)_0 : \ \mathrm{Hom}(A, B) := A \to B$,
- $\forall A, B, C \in \mathrm{Type}(U)_0 : \ \circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ with computation rule $f \circ h := \lambda x.f(h(x))$ and
- $\forall A \in \mathrm{Type}(U)_0 : \mathrm{id}_A := \lambda x.x : A \to A$

is a category.

Proof.
To see the statement, we have to show the two conditions in Definition 1.0.1.

1. Let $A, B \in \mathrm{Type}(U)_0$ and $f \in \mathrm{Hom}(B, A)$. Hence $A, B : U$ and $f : B \to A$.
As expected, we will be using function extensionality for this, so let $x : B$.
Using the definition for the composition and identities, we thus get

$$(id_A \circ f)(x) = id_A(f(x)) = f(x)$$

and

$$(f \circ id_B)(x) = f(id_B(x)) = f(x).$$

Hence $id_A \circ f = f = f \circ id_B$.

2. Let $A, B, C, D \in \mathrm{Type}(U)_0,\ f \in \mathrm{Hom}(B, A),\ h \in \mathrm{Hom}(C, B)$ and $p \in \mathrm{Hom}(D, C)$.
Hence $A, B, C, D : U,\ f : B \to A,\ h : C \to B$ and $p : D \to C$.
We have to show

$$f \circ (h \circ p) = (f \circ h) \circ p.$$

This time we will not need function extensionality, just the definition of the composition:

$$f \circ (h \circ p) = \lambda x.f((h \circ p)(x)) = \lambda x.f(h(p(x))) = \lambda x.\big(\lambda y.f(h(y))\big)(p(x))$$

$$= \lambda x.(f \circ h)(p(x)) = (f \circ h) \circ p. \quad \square$$

Later, when we present the additional structures to categories we will also want to define maps between them for some of them. These will of course be based on functors. So it is practical to also quickly give a definition for functors:

Definition 1.0.4 (Functor)**.**
Let C and D be categories. Then, F consisting of

- an assignment $F_0 : C_0 \to D_0$ and
- an assignment $F_1^{a,b} : \mathrm{Hom}(a, b) \to \mathrm{Hom}(F_0(a), F_0(b)), \forall a, b \in C_0$

is called a functor from C to D, iff

1. $\forall a \in C_0 : F_1^{a,a}(id_a) = id_{F_0(a)}$ and
2. $\forall a, b, c \in C_0 \forall f \in \mathrm{Hom}(a, b) \forall h \in \mathrm{Hom}(b, c) : F_1^{a,c}(h \circ f) = F_1^{b,c}(h) \circ F_1^{a,b}(b)$.

Example 1.0.5 (A functor from $\mathrm{Type}(U)$ to $\mathrm{Type}(U)$)**.**
Let U be a type universe and $A : U$. Let $\mathrm{Type}(U)$ as in Proposition 1.0.3. Then, $\mathrm{Hom}(A, -)$ consisting of

- $\lambda B.\mathrm{Hom}(A, B) : U \to U$ and
- $\forall B, C \in \mathrm{Type}(U)_0 :\ \lambda f.\lambda y.(f \circ y) : (A \to B) \to (B \to C) \to A \to C$

is a functor from $\mathrm{Type}(U)$ to $\mathrm{Type}(U)$.

Proof.
We have to show the two conditions from Definition 1.0.4.

- Let $B : \mathcal{U}$. We have to show that

$$(\lambda f.\lambda y.(f \circ y))(\mathrm{id}_B) = \mathrm{id}_{A \to B}.$$

For this we fist simplify the right side of this equation.

$$(\lambda f.\lambda y.(f \circ y))(\mathrm{id}_B) = \lambda y.(\mathrm{id}_B \circ y)$$

Now, recall that we showed $\mathrm{id}_B \circ y = y$ in the proof for Proposition 1.0.3. Hence we can say

$$\lambda y.(\mathrm{id}_B \circ y) = \lambda y.y = \mathrm{id}_{A \to B}.$$

Thus we get $(\lambda f.\lambda y.(f \circ y))(\mathrm{id}_B) = \mathrm{id}_{A \to B}$.

- Let $B, C, D : \mathcal{U},\ f : B \to C$ and $h : C \to D$. We have to show that

$$\mathrm{Hom}(A, h \circ f) = \mathrm{Hom}(A, h) \circ \mathrm{Hom}(A, f).$$

For this we just simplify both sides of this equation, which yields:

$$\mathrm{Hom}(A, h \circ f) = (\lambda f.\lambda y.(f \circ y))(h \circ f) = \lambda y.((h \circ f) \circ y) = \lambda y.(h \circ (f \circ y))$$

and

$$\begin{aligned} \mathrm{Hom}(A, h) \circ \mathrm{Hom}(A, f) &= (\lambda f.\lambda y.(f \circ y))(h) \circ (\lambda f.\lambda y.(f \circ y))(f) \\ &= (\lambda y.(h \circ y)) \circ (\lambda y.(f \circ y)) = \lambda x.(\lambda y.(h \circ y))((\lambda y.(f \circ y))(x)) \\ &= \lambda x.(\lambda y.(h \circ y))(f \circ x) = \lambda x.(h \circ (f \circ x)). \quad \square \end{aligned}$$

2 Categories with dependent arrows

In this section, we will set the groundwork for our work in Section 3. We will give definitions for fam-, (fam, Σ)-, dep- and (dep, Σ)-categories. After each of these definitions, we will discuss a couple of examples and prove some useful and interesting statements. Also, just as the definitions build on each other, so will the examples rely on examples from previous definitions. Of particular importances will be the proofs for the statements that (fam, Σ)-categories naturally induce dep- and (dep, Σ)-categories, as we will rely on these heavily in the 3rd section, when we discuss the analogous statements for $(2\text{-}\mathrm{fam}, \Sigma)$-categories. As mentioned in the introduction, the definitions and most of the statements are due to Petrakis [3].

2.1 fam-Categories

fam-categories are the foundation of the everything we will do in this thesis. All subsequent definitions will make use of the structure of a fam-category. This makes sense, consider again types. To define the Σ-type, you first need type-families, to define dependent functions, you also first need type-families to give the codomain.

Definition 2.1.1 (fam-Category)**.**
$\mathrm{fam}C$ consisting of

- a Category C,
- a collection $\mathrm{fHom}(a)$ of fam-arrows, $\forall a \in C_0$ and
- an operation $\mathrm{fHom}(a) \times \mathrm{Hom}(b, a) \to \mathrm{fHom}(b)$, $\forall a, b \in C_0$, as shown in the diagram:

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[rr, bend right, "\lambda\circ f"'] & a \arrow[r, "\lambda"] & \cdot \end{tikzcd},$$

  (called composition of fam-arrows)

is called a fam-category, iff

(fam_1) $\forall a \in C_0 \forall \lambda \in \mathrm{fHom}(a)$:

$$\lambda \circ \mathrm{id}_a = \lambda$$

$$\begin{tikzcd} a \arrow[r, "\mathrm{id}_a"] \arrow[rr, bend right, "\lambda"'] & a \arrow[r, "\lambda"] & \cdot \end{tikzcd}.$$

and

(fam_2) $\forall a, b, c \in C_0 \forall \lambda \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b, a) \forall h \in \mathrm{Hom}(c, b)$:

$$(\lambda \circ f) \circ h = \lambda \circ (f \circ h)$$

$$\begin{tikzcd} c \arrow[r, "h"] \arrow[rr, bend left, "f\circ h"] \arrow[rrr, bend left=50, "\lambda\circ(f\circ h)"] \arrow[rrr, bend right=50, "(\lambda\circ f)\circ h"'] & b \arrow[r, "f"] \arrow[rr, bend right, "\lambda\circ f"'] & a \arrow[r, "\lambda"] & \cdot \end{tikzcd}.$$

We will refer to the collection of all $\mathtt{fam}$-arrows in a $\mathtt{fam}$-category $\mathtt{famC}$ by $\mathtt{famC}_2$. Furthermore, for $\mathtt{fam}$-categories $\mathtt{famC}$, we also say $\mathtt{famC}_0 := C_0$ and $\mathtt{famC}_1 := C_1$.

So, to summarize, in $\mathtt{fam}$-categories, we have

- $\mathtt{fam}$-arrows, which have a domain but no codomain and
- a way to compose $\mathtt{fam}$-arrows with regular arrows.

And again, considering types, this makes sense. Type-families have the universe as codomain. But the universe is not an object in the category given by its small types. Only the domain is an object. Similarly, we know, that type-families can be composed with functions of appropriate codomain, as type-families are ultimately just functions.
Next, we will have a look at a very simple example of a $\mathtt{fam}$-category. This is going to be a continuous theme, after each of the major definitions, we will discuss a trivial example.

Example 2.1.2 (A $\mathtt{fam}$-category with trivial $\mathtt{fam}$-arrows)**.**
Let C be any category and A any collection. Then, $\mathtt{famC}$ consisting of

- C,
- $\forall a \in C_0 : \mathrm{fHom}(a) := A$ and
- $\forall a, b \in C_0 : (\lambda, f) \mapsto \lambda$

is a $\mathtt{fam}$-category.

Proof.
To see this, we have to show the conditions in Definition 2.1.1.

($\mathtt{fam}_1$) For ($\mathtt{fam}_1$), let $a \in C_0$ and $\lambda \in \mathrm{fHom}(a)$.
By the definition of the $\mathtt{fam}$-arrow composition here,

$$\lambda \circ \mathrm{id}_a = \lambda,$$

which is precisely ($\mathtt{fam}_1$).

($\mathtt{fam}_2$) Now let $a, b, c \in C_0$, $\lambda \in \mathrm{fHom}(a)$, $f \in \mathrm{Hom}(b, a)$ and $h \in \mathrm{Hom}(c, b)$.
Again, by the definition of the $\mathtt{fam}$-arrow composition, we have

$$(\lambda \circ f) \circ h = \lambda \circ h = \lambda$$

and

$$\lambda \circ (f \circ h) = \lambda.$$

Hence $(\lambda \circ f) \circ h = \lambda \circ (f \circ h)$. □

Now, just because this example was easy, does not mean it is useless. The following is a special case of this. What we will see, is that, for every category, there is a $\mathtt{fam}$-category, where the $\mathtt{fam}$-arrows are simply the objects of the category. This category with constant $\mathtt{fam}$-arrows will accompany us on our entire journey.

Corollary 2.1.3 (A $\mathtt{fam}$-category with constant $\mathtt{fam}$-arrows)**.**
Let C be a category.
Then, $\mathtt{famC}$ consisting of

- C,

- $\forall a \in C_0 : \mathsf{fHom}(a) := C_0$ and
- $\forall a, b \in C_0 : (\lambda, f) \mapsto \lambda$

is a fam-category.

Proof.
The statement is clear, as it is simply a special case of Example 2.1.2. □

Referring to these fam-arrows as constant is very natural. For this, again consider a type universe U. If we apply the above construction to $\mathsf{Type}(\mathsf{U})$, there is clearly a correspondence between the selected fam-arrows and the constant type-families.
This now marks the 3rd time in this subsection, that we have talked about types in relation to fam-categories. We still have to show, that we can actually give a fam-structure for $\mathsf{Type}(\mathsf{U})$ that captures all type-families. This we do now:

Proposition 2.1.4 (Small types form a fam-category)**.**
Let U be a type universe.
Let $\mathsf{Type}(\mathsf{U})$ be the category as per Example 1.0.3.
Then $\mathsf{Type}^{\mathsf{f}}(\mathsf{U})$ consisting of

- $\mathsf{Type}(\mathsf{U})$,
- $\forall A \in \mathsf{Type}(\mathsf{U})_0 : \ \mathsf{fHom}(A) := A \to \mathsf{U}$ and
- $\forall A, B \in \mathsf{Type}(\mathsf{U})_0 : (\mu, f) \mapsto \lambda x.\mu(f(x))$

is a fam-category.

Proof.
Like in Example 2.1.2, to see that $\mathsf{Type}^{\mathsf{f}}(\mathsf{U})$ is a fam-category, we have to show the conditions (fam_1) and (fam_2).

(fam_1) Let $A : \mathsf{U}$ and $\mu : A \to \mathsf{U}$.
Let $x : A$. We can see that $\mu(\mathrm{id}_A(x)) = \mu(x)$.
Thus by function extensionality, $\mu \circ \mathrm{id}_A = \mu$.

(fam_2) Let $A, B, C : \mathsf{U},\ \mu : A \to \mathsf{U},\ f : B \to A$ and $h : C \to B$.
We have to show that

$$(\mu \circ f) \circ h = \mu \circ (f \circ h).$$

Thus, using the above definitions, we get

$$(\mu \circ f) \circ h = \lambda z.(\mu \circ f)(h(z)) = \lambda z.\mu(f(h(z))) = \lambda z.\mu((f \circ h)(z)) = \mu \circ (f \circ h).$$ □

With this we can see, that the initial motivation is founded, at least for this definition. Small types indeed give a fam-category. Later, as we introduce more definitions, we will always show, that small types form an example of them. Eventually, we will show, that type universes give $(2\text{-}\mathsf{dep}, \Sigma)$-categories, which are, in a sense, the combination of all of the structures we are going to discuss.

Let us then go to our second major example: Toposes. One of our aims is to show that all toposes can naturally be given the structure of a $(2\text{-}\mathsf{dep}, \Sigma)$-category. Naturally, for this we have to show that they can be given the structure of a fam-category. To do this, we will use a construction which is due to Pitts [4].

The full construction presented by Pitts is an example of what he called a type-category. The definition for type-categories and the definition for (fam, Σ)-categories (which we will give later in Definition 2.2.1) are actually almost the same. The difference is, that type-categories are required to have a terminal object, while (fam, Σ)-categories are not. Hence the full construction actually satisfies the definition for (fam, Σ)-categories. Here however we will only focus on the part relevant to fam-categories.
We will revisit this in the subsection on (fam, Σ)-categories, where we will show that toposes can be given a (fam, Σ)-structure. For the definition and other notions of toposes that we will use here, please refer to [1]. Now we are going to explicitly spell out the notation for the bits of the topos-structure, that we are going to use:

Notation 2.1.5 (Notation for toposes)**.**
If C is a topos, we will use

- 1 for a terminal object,
- $\forall a \in C_0$: 1_a for the unique arrow in $\mathrm{Hom}(a, 1)$,
- Ω for a subobject classifier and
- $\top$ for the truth arrow in $\mathrm{Hom}(1, \Omega)$.

Using this notation, we will now see how a topos becomes a fam-category:

Lemma 2.1.6 (Toposes as fam-categories)**.**
Let C be a Topos.
Then $\mathsf{fam}C$ consisting of

- C,
- $\forall a \in C_0 : \mathrm{fHom}(a) := \coprod_{b \in cC_0} \mathrm{Hom}(a \times b, \Omega)$ and
- $\forall a, k \in C_0 : ((b, e), f) \mapsto (b, e \circ (f \times \mathrm{id}_b))$

is a fam-category.

Proof.
As before, to see that $\mathsf{fam}C$ is a fam-category we have to show the conditions in the definition.

(fam_1) To show (fam_1), we have to show

$$\forall a \in C_0 \forall (k, e) \in \mathrm{fHom}(a) : (k, e \circ \mathrm{id}_a \times \mathrm{id}_k) = (k, e).$$

But this is immediately clear, as regular composition respects identities and $\forall a, b \in C_0 : \mathrm{id}_a \times \mathrm{id}_b = \mathrm{id}_{a \times b}$.

(fam_2) Let $a, b, c \in C_0$, $(k, e) \in \mathrm{fHom}(a)$, $f \in \mathrm{Hom}(b, a)$ and $h \in \mathrm{Hom}(c, b)$.
We have to show

$$((k, e) \circ f) \circ h = (k, e) \circ (f \circ h).$$

Using the definition of the fam-arrow composition and basic properties of products, we have

$$((k, e) \circ f) \circ h = (k, e \circ (f \times \mathrm{id}_k)) \circ h = (k, e \circ (f \times \mathrm{id}_k) \circ (h \times \mathrm{id}_k))$$

$$= \big(k, e \circ ((f \circ h) \times \mathrm{id}_k)\big) = (k, e) \circ (f \circ h). \quad \square$$

As mentioned in the introduction, we will show theorems on that every $(2\text{-}\mathsf{fam}, \Sigma)$-category is a 2-$\mathsf{dep}$- and $(2\text{-}\mathsf{dep}, \Sigma)$-category, as well as comparable theorems for the 1-versions. Given such theorems it is of course reasonable to ask, if all (dep, Σ)- and $(2\text{-}\mathsf{dep}, \Sigma)$-categories are given by (fam, Σ)- and $(2\text{-}\mathsf{fam}, \Sigma)$-categories respectively. The answer to that question will be no and the following example of commutative rings will help us get to that answer.

Example 2.1.7 (A fam-structure for a commutative ring)**.**
Let R be a commutative ring.
Let BR be the category given by the additive group of R, with object $*$ and morphisms the ring elements.
Then, fR consisting of

- BR,
- $f\mathrm{Hom}(*) = R \times R$ and
- $\forall r \in R \forall (a, b) \in R \times R : (a, b) \circ r := (a + r, b + r)$

is a fam-category.

Proof.
To see this, again, we just have to show (fam_1) and (fam_2).

(fam_1) Let $(a, b) \in f\mathrm{Hom}(*)$.
We simply have to show that $(a + 0, b + 0) = (a, b)$.
Since 0 is a neutral element, this clearly holds.

(fam_2) Let $(a, b) \in f\mathrm{Hom}(*)$ and $r, q \in R$.
We have to show that $((a + r) + q, (b + r) + q) = (a + (r + q), b + (r + q))$.
This though follows immediately from the associativity of addition in rings. □

2.2 (fam, Σ)-Categories

Now, we will add Σ-objects to fam-categories, turning them into (fam, Σ)-categories. Σ-objects aim to generalize notions like the Σ-type into a categorical framework. As such, in addition to Σ-objects, (fam, Σ)-categories will also contain first projection arrows and Σ-arrows, which we use to capture certain parts of the behaviour of the Σ-type. It is also possible to define second projection arrows, but at the moment we lack the necessary prerequisites for this, so will will address these later.

Definition 2.2.1 ((fam, Σ)-Categories)**.**
ΣC consisting of

- a fam-category $\mathsf{fam}C$,
- an object $\sum_a \lambda \in \mathsf{fam}C_0$, $\forall a \in \mathsf{fam}C_0 \forall \lambda \in f\mathrm{Hom}(a)$,
(called the Σ-object of a and λ),
- an arrow $pr_1^{a,\lambda} \in \mathrm{Hom}\big(\sum_a \lambda, a\big)$, $\forall a \in \mathsf{fam}C_0 \forall \lambda \in f\mathrm{Hom}(a)$,
(called the first projection arrow for a and λ) and
- an arrow $\Sigma_\lambda f \in \mathrm{Hom}\big(\sum_b (\lambda \circ f), \sum_a \lambda\big)$, $\forall a, b \in \mathsf{fam}C_0 \forall \lambda \in f\mathrm{Hom}(a) \forall f \in \mathrm{Hom}(b, a)$,
(called the Σ-arrow for λ and f)

is called a (fam, Σ)-category, iff (pull) $\forall a, b \in \mathrm{famC}_0 \forall \lambda \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b, a)$: the following diagram is a pullback

$$\begin{tikzcd}
\sum_b(\lambda \circ f) \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{b,\lambda\circ f}"'] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
b \arrow[r, "f"'] & a
\end{tikzcd},$$

(s_1) $\forall a \in \mathrm{famC}_0 \forall \lambda \in \mathrm{fHom}(a)$:

$$\Sigma_\lambda \mathrm{id}_a = \mathrm{id}_{\sum_a \lambda}$$

and

(s_2) $\forall a, b, c \in \mathrm{famC}_0 \forall \lambda \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b, a) \forall h \in \mathrm{Hom}(c, b)$:

$$\Sigma_\lambda(f \circ h) = \Sigma_\lambda f \circ \Sigma_{\lambda \circ f} h.$$

Furthermore, we define $C_0 := \mathrm{famC}_0$, $C_1 := \mathrm{famC}_1$ and $C_2 := \mathrm{famC}_2$.

Remark 2.2.2 (Well-definedness of conditions in Definition 2.2.1).
We can see that the conditions (s_1) and (s_2) are well defined. To do so, we want to see, that both sides of the respective equations have equal domains and codomains. To see this, use (fam_1) for (s_1) and (fam_2) for (s_2).

With the definition in hand, let us now consider some examples. Note though, that, as mentioned before, the definition is almost the same as the definition of type-categories presented by Pitts [4]. It is only missing the requirement of a terminal object. So all examples of type-categories are also examples of (fam, Σ)-categories.

Example 2.2.3 (A (fam, Σ)-category with trivial Σ-objects).
Let famC be a fam-category. Then, ΣC consisting of

- famC,
- $\forall a \in \mathrm{famC}_0 \forall \lambda \in \mathrm{fHom}(a) : \sum_a \lambda := a$,
- $\forall a \in \mathrm{famC}_0 \forall \lambda \in \mathrm{fHom}(a) : \mathrm{pr}_1^{a,\lambda} := \mathrm{id}_a$ and
- $\forall a, b \in \mathrm{famC}_0 \forall \lambda \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b, a) : \Sigma_\lambda f := f$,

is a (fam, Σ)-category.

Proof.
To see this, we have to show the conditions (pull), (s_1) and (s_2) from Definition 2.2.1.

(pull) For this, Let $a, b \in \mathrm{famC}_0$, $\lambda \in \mathrm{famC}_0$ and $f \in \mathrm{Hom}(b, a)$.
We have to show, that

$$\begin{tikzcd}
b \arrow[r, "f"] \arrow[d, "\mathrm{id}_b"'] & a \arrow[d, "\mathrm{id}_a"] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

is a pullback.
This, however, is clearly the case.

(s_1) Let again $a \in \mathtt{famC}_0$ and $\lambda \in \mathtt{fHom}(a)$. We have to show that $\mathrm{id}_a = \mathrm{id}_a$.
Hence, by reflexivity of equality, we see that (s_1) holds.

(s_2) That (s_2) holds we can see analogously to (s_1).
Again, just unravel the definitions and use reflexivity. □

Next we will look again at the constant $\mathtt{fam}$-arrows from Corollary 2.1.3. We will see, that, under the condition that we have binary products, we can find a Σ-structure, turning these $\mathtt{fam}$-categories into $(\mathtt{fam}, \Sigma)$-categories.

Example 2.2.4 (A $(\mathtt{fam}, \Sigma)$-category with constant $\mathtt{fam}$-arrows)**.**
Let C be a category with binary products. Let $\mathtt{famC}$ be the $\mathtt{fam}$-category given by C and Corollary 2.1.3. Let there be an assignment

$$\mathtt{famC}_0 \times \mathtt{famC}_0 \to \mathtt{famC}_0, (a, b) \mapsto a \times b,$$

where $a \times b$ is a product of a and b. Then, ΣC consisting of

- $\mathtt{famC}$,
- $\forall a \in \mathtt{famC}_0 \forall \lambda \in \mathtt{fHom}(a) : \sum_a \lambda := a \times \lambda$,
- $\forall a \in \mathtt{famC}_0 \forall \lambda \in \mathtt{fHom}(a) : \mathrm{pr}_1^{a,\lambda} \in \mathrm{Hom}(a \times \lambda, a)$ the canonical projection and
- $\forall a, b \in \mathtt{famC}_0 \forall \lambda \in \mathtt{fHom}(a) \forall f \in \mathrm{Hom}(b, a) : \Sigma_\lambda f := f \times \mathrm{id}_\lambda$,

is a $(\mathtt{fam}, \Sigma)$-category.

Proof.
We again have to show the conditions $(\mathtt{pull})$, (s_1) and (s_2).

$(\mathtt{pull})$ Let $a, b \in \mathtt{famC}_0$, $\lambda \in \mathtt{famC}_0$ and $f \in \mathrm{Hom}(b, a)$.
We have to show that the square

$$\begin{tikzcd} b \times \lambda \arrow[r, "f \times \mathrm{id}_\lambda"] \arrow[d, "\mathrm{pr}_1^{b,\lambda}"'] & a \times \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\ b \arrow[r, "f"'] & a \end{tikzcd},$$

is a pullback.
Since the diagram is clearly commutative, we only need to have a look at whether it fulfills the universal pullback property.
For this, let $P \in \mathtt{famC}_0$, $p \in \mathrm{Hom}(P, a \times \lambda)$ and $q \in \mathrm{Hom}(P, b)$ a competitor to the diagram:

$$\begin{tikzcd} P \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\ & b \times \lambda \arrow[r, "f \times \mathrm{id}_\lambda"] \arrow[d, "\mathrm{pr}_1^{b,\lambda}"] & a \times \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\ & b \arrow[r, "f"'] & a \end{tikzcd}.$$

We now have to find $r \in \mathrm{Hom}(P, b \times \lambda)$ making the diagram

$$\begin{tikzcd}
P \arrow[dr, dashed, red, "r"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& b \times \lambda \arrow[r, "f \times id_\lambda"] \arrow[d, "pr_1^{b,\lambda}"] & a \times \lambda \arrow[d, "pr_1^{a,\lambda}"] \\
& b \arrow[r, "f"'] & a
\end{tikzcd}.$$

commutative and show that r is unique.

existence

For this, let $pr_2^a \in \mathrm{Hom}(a \times \lambda, \lambda)$ and $pr_2^b \in \mathrm{Hom}(b \times \lambda, \lambda)$ be the canonical second projections. For the red arrow in the diagram, we take

$$r := (q, pr_2^a \circ p).$$

We now have to show that the diagram actually becomes commutative with this. Hence we have to show commutativity of the triangles, then we are done with this part. For this we have to show

(i) $pr_1^{b,\lambda} \circ (q, pr_2^a \circ p) = q$ and

(ii) $f \times id_\lambda \circ (q, pr_2^a \circ p)$.

(i) follows from basic properties of products.
For (ii), consider

$$f \times id_\lambda \circ (q, p_2^a \circ p) = (f \circ q, pr_2^a \circ p) = (pr_1^{a,\lambda} \circ p, pr_2^a \circ p)$$
$$= (pr_1^{a,\lambda}, pr_2^a) \circ p = p.$$

uniqueness

For this let $w \in \mathrm{Hom}(P, b \times \lambda)$ also making the diagram commutative.
We have to show

$$r = w.$$

Now, since both are arrows pointing to a product, it suffices to show, that they make the same product-competitor diagram for said product commutative:

$$\begin{tikzcd}
& & b \\
P \arrow[urr, bend left, "q"] \arrow[drr, bend right, "pr_2^a \circ p"'] & b \times \lambda \arrow[ur] \arrow[dr] & \\
& & \lambda
\end{tikzcd}$$

Hence it suffices to show

(i) $pr_1^{b,\lambda} \circ w = pr_1^{b,\lambda} \circ r$ and

(ii) $pr_2^b \circ w = pr_2^b \circ r$.

(i) is clear, as both sides of the equation are assumed to be equal to q.
To show (ii), consider

$$\mathrm{pr}_2^b \circ w = \mathrm{id}_\lambda \circ \mathrm{pr}_2^b \circ w = \mathrm{pr}_2^a \circ f \times \mathrm{id}_\lambda \circ w = \mathrm{pr}_2^a \circ p.$$

Analogously we also get

$$\mathrm{pr}_2^b \circ r = \mathrm{pr}_2^a \circ p.$$

Hence, $\mathrm{pr}_2^b \circ w = \mathrm{pr}_2^b \circ r$, demonstrating uniqueness.

To summarize, with this we can see, that we indeed have a pullback.

(s_1) Let $a \in \mathrm{fam}C_0$ and $\lambda \in \mathrm{fHom}(a)$. We have to show

$$\mathrm{id}_a \times \mathrm{id}_\lambda = \mathrm{id}_{a\times\lambda}.$$

However, this is clear.

(s_2) Let $a, b, c \in \mathrm{fam}C_0,\ \lambda \in \mathrm{fHom}(a),\ f \in \mathrm{Hom}(b, a)$ and $h \in \mathrm{Hom}(c, b)$.
We have to show

$$(f \circ h) \times \mathrm{id}_\lambda = (f \times \mathrm{id}_\lambda) \circ (h \times \mathrm{id}_\lambda).$$

Again, this is just a basic property of products. □

After the constant families, we will now return to types and we will see that small types indeed form (fam, Σ)-categories. Observe that much of the proof below is similar to the proof above. This makes sense, after all Σ-types are simply dependent products.

Proposition 2.2.5 (Small types form a (fam, Σ)-category).
Let U be a type universe. Let $\mathrm{Type}^f(U)$ be the fam-category from Proposition 2.1.4 using U.
Then, $\mathrm{Type}^\Sigma(U)$ consisting of

- $\mathrm{Type}^f(U)$,
- $\forall A \in \mathrm{Type}^f(U)_0 \forall \mu \in \mathrm{fHom}(A) : \sum_A \mu := \sum_{x:A} \mu(x)$ the inductively defined Σ-type,
- $\forall A \in \mathrm{Type}^f(U)_0 \forall \mu \in \mathrm{fHom}(A) :\ \mathrm{pr}_1^{A,\mu} : \sum_A \mu \to A$ the first projection and
- $\forall A, B \in \mathrm{Type}^f(U)_0 \forall \mu \in \mathrm{fHom}(A) \forall f \in \mathrm{Hom}(B, A) :\ \Sigma_\mu f : \sum_B (\mu \circ f) \to \sum_A \mu$,
  with computation rule $\Sigma_\mu f(x, y) = (f(x), y)$,

is a (fam, Σ)-category.

Proof.
Again, we simply have to show the conditions (pull), (s_1) and (s_2).

(pull) Let $A, B \in \mathrm{Type}^f(U)_0,\ \mu \in \mathrm{fHom}(A)$ and $f \in \mathrm{Hom}(B, A)$.
We have to show that the square

$$\begin{tikzcd}
\sum_B(\mu \circ f) \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{B,\mu\circ f}"'] & \sum_A \mu \arrow[d, "\mathrm{pr}_1^{A,\mu}"] \\
B \arrow[r, "f"'] & A
\end{tikzcd},$$

is a pullback.
It is easy to see that the square is commutative using function extensionality.
So we just have to show the universal pullback property.
For this, let $P \in \mathcal{Type}^f(\mathcal{U})_0,\ p \in \mathrm{Hom}(P, \sum_A \mu)$ and $q \in \mathrm{Hom}(P, B)$ a competitor to the diagram:

$$\begin{tikzcd}
P \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& \sum_B(\mu \circ f) \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{B,\mu\circ f}"] & \sum_A \mu \arrow[d, "\mathrm{pr}_1^{A,\mu}"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}.$$

We now have to find $r : P \to \sum_A \mu$, making the pullback-competitor diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] \arrow[dr, dashed, "r"] & & \\
& \sum_B(\mu \circ f) \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{B,\mu\circ f}"] & \sum_A \mu \arrow[d, "\mathrm{pr}_1^{A,\mu}"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}.$$

commutative and show that r is unique.

existence

For this, let $\mathrm{pr}_2^A : \prod_{x:\sum_A \mu} \mu(\mathrm{pr}_1^{A,\mu}(x))$ and $\mathrm{pr}_2^B : \prod_{x:\sum_B(\mu\circ f)} \mu(f(\mathrm{pr}_1^{B,\mu\circ f}(x)))$ the second projections.
Now let

$$r := \lambda x.(q(x), \mathrm{pr}_2^A(p(x))) : P \to \sum_B (\mu \circ f).$$

To see that r makes the pullback-competitor diagram commutative, we need to show

(i) $\mathrm{pr}_1^{B,\mu\circ f} \circ r = q$ and
(ii) $\Sigma_\mu f \circ r = p$.

This can be done using function extensionality, let $x : P$.
Thus we can see

$$(\mathrm{pr}_1^{B,\mu\circ f} \circ r)(x) = \mathrm{pr}_1^{B,\mu\circ f}(r(x)) = \mathrm{pr}_1^{B,\mu\circ f}\big(q(x), \mathrm{pr}_2^A(p(x))\big) = q(x)$$

and

$$(\Sigma_\mu f \circ r)(x) = \Sigma_\mu f(r(x)) = \Sigma_\mu f\big(q(x), \mathrm{pr}_2^A(p(x))\big) = \big(f(q(x)), \mathrm{pr}_2^A(p(x))\big)$$
$$= \big(\mathrm{pr}_1^{A,\mu}(p(x)), \mathrm{pr}_2^A(p(x))\big) = p(x).$$

uniqueness

Let $w : P \to \sum_B (\mu \circ f)$ such that it makes the pullback-competitor diagram commutative. We have to show, that

$$r = w.$$

To do so, we use function extensionality again, let $x : P$. Thus

$$\begin{aligned} w(x) &= \left(\mathrm{pr}_1^{B,\mu}(w(x)), \mathrm{pr}_2^B(w(x))\right) = \left(q(x), \mathrm{pr}_2^B(w(x))\right) \\ &= \left(q(x), \mathrm{pr}_2^A(\Sigma_\mu f(w(x)))\right) = \left(q(x), \mathrm{pr}_2^A(p(x))\right) = r(x). \end{aligned}$$

Thus clearly, $r = w$.

With this we can see, that we have a pullback.

(s_1) Let $A \in \mathrm{Type}^f(U)_0$ and $\mu \in \mathrm{fHom}(A)$. We have to show

$$\Sigma_\mu \mathrm{id}_A = \mathrm{id}_{\sum_A \mu}.$$

This we can see easily, by applying function extensionality:
Let $(x, y) : \sum_A \mu$, then

$$\Sigma_\mu \mathrm{id}_A(x, y) = (\mathrm{id}_A(x), y) = (x, y) = \mathrm{id}_{\sum_A \mu}(x, y).$$

(s_2) Let $A, B, C \in \mathrm{Type}^f(U)_0,\ \mu \in \mathrm{fHom}(A),\ f \in \mathrm{Hom}(B, A)$ and $h \in \mathrm{Hom}(C, B)$.
We have to show

$$\Sigma_\mu (f \circ h) = \Sigma_\mu f \circ \Sigma_{\mu \circ f} h.$$

We are going to use function extensionality again, let $(x, y) : \sum_b (\mu \circ f \circ h)$.
Thus

$$\Sigma_\mu (f \circ h)(x, y) = (f(h(x)), y) = \Sigma_\mu f(h(x), y) = (\Sigma_\mu f \circ \Sigma_{\mu \circ f} h)(x, y). \quad \square$$

Next, we want to discuss the example of a topos again. As mentioned before, the structure we used for toposes in Lemma 2.1.6, is actually part of a type-category-structure on toposes, introduced by Pitts [4]. Now we will verify, that this structure does fulfill the definition of a (fam, Σ)-category.

Lemma 2.2.6 (Toposes as (fam, Σ)-categories)**.**
Let C be a topos. Let $\mathrm{fam}C$ be the fam-category by Lemma 2.1.6 using C.
Then, ΣC consisting of

- $\mathrm{fam}C$,
- $\forall a \in \mathrm{fam}C_0 \forall (k, e) \in \mathrm{fHom}(a) : \sum_a (k, e)$ the pullback of

$$\begin{tikzcd} & a \times k \arrow[d, "e"] \\ 1 \arrow[r, "\top"] & \Omega \end{tikzcd}$$

- $\forall a \in \mathrm{fam}C_0 \forall (k, e) \in \mathrm{fHom}(a) :\ \mathrm{pr}_1^{a,\lambda} := \mathrm{pr}_1^{a,k} \circ p$,
where $\mathrm{pr}_1^{a,k} \in \mathrm{Hom}(a \times k, a)$ is the canonical first projection
and $p \in \mathrm{Hom}\left(\sum_a (k, e), a \times k\right)$ the arrow given by the pullback, and

- $\forall a, b \in \mathsf{famC}_0 \forall (k, e) \in \mathsf{fHom}(a) \forall f \in \mathrm{Hom}(b, a)$: $\Sigma_{(k,e)}f$ the unique arrow making the diagram

$$\begin{tikzcd}
\sum_b(k,e)\circ f \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(k,e)}f"] \arrow[ddr, bend right] & & b\times k \arrow[dr, "f\times \mathrm{id}_k"] & \\
& \sum_a(k,e) \arrow[rr] \arrow[d] & & a\times k \arrow[d, "e"] \\
& 1 \arrow[rr, "\top"] & & \Omega
\end{tikzcd}$$

  commutative

is a (fam, Σ)-category.

Proof.
As always, we need to prove the 3 conditions in Definition 2.2.1.

(pull) Let $a, b \in \mathsf{famC}_0$, $(k, e) \in \mathsf{fHom}(a)$ and $f \in \mathrm{Hom}(b, a)$.
We have to show that the following square, which we call $(**)$

$$\begin{tikzcd}
\sum_b(k,e)\circ f \arrow[r, "\Sigma_{(k,e)}f"] \arrow[d, "\mathrm{pr}_1^{b,(k,e)\circ f}"'] & \sum_a(k,e) \arrow[d, "\mathrm{pr}_1^{a,(k,e)}"] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

is a pullback.
For this, let

- $p_a \in \mathrm{Hom}\big(\sum_a(k,e), a\times k\big)$ and
- $p_b \in \mathrm{Hom}\big(\sum_b(k,e)\circ f, b\times k\big)$

be the arrows from the defining pullback of the Σ-objects.
Furthermore let

- $\mathrm{pr}_1^{a,k} \in \mathrm{Hom}(a\times k, a)$ and
- $\mathrm{pr}_1^{b,k} \in \mathrm{Hom}(b\times k, b)$

be the canonical projections.
Since $\mathrm{pr}_1^{a,(k,e)} = \mathrm{pr}_1^{a,k}\circ p_a$ and $\mathrm{pr}_1^{b,(k,e)\circ f} = \mathrm{pr}_1^{b,k}\circ p_b$, $(**)$ is the outer square in the following diagram, which we call $(*)$:

$$\begin{tikzcd}
\sum_b(k,e)\circ f \arrow[r, "\Sigma_{(k,e)}f"] \arrow[d, "p_b"] \arrow[dd, bend right=60, "\mathrm{pr}_1^{b,(k,e)\circ f}"'] & \sum_a(k,e) \arrow[d, "p_a"'] \arrow[dd, bend left=60, "\mathrm{pr}_1^{a,(k,e)}"] \\
b\times k \arrow[r, "f\times \mathrm{id}_k"] \arrow[d, "\mathrm{pr}_1^{b,k}"] & a\times k \arrow[d, "\mathrm{pr}_1^{a,k}"'] \\
b \arrow[r, "f"'] & a
\end{tikzcd}.$$

Notice that the left and right triangle of $(*)$ are commutative. Furthermore, using the definitions for $\Sigma_\lambda f$ and $f \times \mathrm{id}_k$, we immediately see, that the upper and lower square in $(*)$ are commutative. Thus $(*)$ is entirely commutative and thus, $(**)$ also is.
By Example 2.2.4 we already know, that the lower square in $(*)$ is a pullback.
Hence it suffices to show, that the upper square is a pullback, then by the pullback lemma, the outer square, which is $(**)$, also is.
For this, combine the upper square with the defining diagram for $\sum_a(k,e)$, which yields the commutative diagram:

$$\begin{tikzcd}
\sum_b(k,e) \circ f \arrow[r, "\Sigma_{(k,e)} f"] \arrow[d, "p_b"'] & \sum_a(k,e) \arrow[r] \arrow[d, "p_a"] & 1 \arrow[d, "\top"] \\
b \times k \arrow[r, "f \times \mathrm{id}_k"'] & a \times k \arrow[r, "e"] & \Omega
\end{tikzcd}$$

By the definitions for $\sum_b(k,e) \circ f$ and $\sum_a(k,e)$, both the right and outer square of this diagram are pullbacks. By the pullback lemma, thus the left square, which is the upper square in $(*)$, is as well. Hence $(**)$ is a pullback.

(s_1) Let $a \in \mathrm{fam}C_0$ and $(k,e) \in \mathrm{fHom}(a)$. We have to show, that

$$\Sigma_{(k,e)} \mathrm{id}_a = \mathrm{id}_{\sum_a(k,e)}.$$

To do so, since $\Sigma_{(k,e)} \mathrm{id}_a$ is defined via pullback, we can simply show that $\mathrm{id}_{\sum_a(k,e)}$ also satisfies the universal pullback property.

(s_2) Let $a, b, c \in \mathrm{fam}C_0$, $(k,e) \in \mathrm{fHom}(a)$, $f \in \mathrm{Hom}(b,a)$ and $h \in \mathrm{Hom}(c,b)$.
We have to show

$$\Sigma_{(k,e)}(f \circ h) = \Sigma_{(k,e)} f \circ \Sigma_{(k,e) \circ f} h. \tag{1}$$

To do so, we will show, that both sides of this equation make the same pullback-competitor diagram commutative. First consider $\Sigma_{(k,e)} f \circ \Sigma_{(k,e) \circ f} h$.
The arrow $\Sigma_{(k,e)} f$ is defined by

$$\begin{tikzcd}
\sum_b(k,e) \circ f \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(k,e)} f"] \arrow[ddr, bend right] & & b \times k \arrow[dr, "f \times \mathrm{id}_k"] & \\
& \sum_a(k,e) \arrow[rr] \arrow[d] & & a \times k \arrow[d, "e"] \\
& 1 \arrow[rr, "\top"'] & & \Omega
\end{tikzcd}$$

and $\Sigma_{(k,e) \circ f} h$ is defined by

$$\begin{tikzcd}
\sum_c(k,e) \circ f \circ h \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(k,e) \circ f} h"] \arrow[ddr, bend right] & & c \times k \arrow[dr, "h \times \mathrm{id}_k"] & \\
& \sum_b(k,e) \circ f \arrow[rr] \arrow[d] & & b \times k \arrow[d, "e \circ f \times \mathrm{id}_k"] \\
& 1 \arrow[rr, "\top"'] & & \Omega
\end{tikzcd}.$$

Putting these diagrams together, we get the following diagram, which we call $(*)$:

$$\begin{tikzcd}
\sum_c(k,e)\circ f\circ h \arrow[rrr] \arrow[dr, dashed, red, "\Sigma_{(k,e)\circ f}h"] \arrow[dddrr, bend right] & & & c\times k \arrow[dr, "h\times id_k"] & & \\
& \sum_b(k,e)\circ f \arrow[rrr] \arrow[dr, dashed, red, "\Sigma_{(k,e)}f"] \arrow[ddr, bend right] & & & b\times k \arrow[dr, "f\times id_k"] \arrow[ddr, bend right, "e\circ f\times id_k"'] & \\
& & \sum_a(k,e) \arrow[rrr] \arrow[d] & & & a\times k \arrow[d, "e"] \\
& & 1 \arrow[rrr, "\top"'] & & & \Omega
\end{tikzcd}.$$

Now consider the other side of Equation 1.
The arrow $\Sigma_{(k,e)}(f\circ h)$ is defined by

$$\begin{tikzcd}
\sum_c(k,e)\circ f\circ h \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(k,e)}(f\circ h)"] \arrow[ddr, bend right] & & c\times k \arrow[dr, "(f\circ h)\times id_k"] & \\
& \sum_a(k,e) \arrow[rr] \arrow[d] & & a\times k \arrow[d, "e"] \\
& 1 \arrow[rr, "\top"'] & & \Omega
\end{tikzcd}.$$

We can clearly see, that (except for the red arrow) this diagram is a subdiagram of $(*)$. Thus, since we are working with a pullback, whatever in $(*)$ has the place of the red arrow, is equal to that red arrow.
Thus Equation 1 follows. $\square$

Example 2.2.7 (A (fam,Σ)-structure for a commutative ring)**.**
Let R be a commutative ring. Let fR be the fam-category given by Example 2.1.7 and R.
Then, ΣR consisting of

- fR,
- $\forall (a,b)\in f\mathrm{Hom}(*):\ \sum_*(a,b) := *$,
- $\forall (a,b)\in f\mathrm{Hom}(*):\ pr_1^{*,(a,b)} := a\cdot b$ and
- $\forall (a,b)\in f\mathrm{Hom}(*)\forall r\in \mathrm{Hom}(*,*) : \Sigma_{(a,b)}r := r(1+r+a+b)$

is a (fam,Σ)-category.

Proof.
We simply have to show (pull), (s_1) and (s_2) again.

(pull) Let $(a,b)\in f\mathrm{Hom}(*)$ and $f\in \mathrm{Hom}(*,*)$.
We have to show that the square

$$\begin{tikzcd}
* \arrow[r, "f(1+f+a+b)"] \arrow[d, "(a+f)(b+f)"'] & * \arrow[d, "ab"] \\
* \arrow[r, "f"'] & *
\end{tikzcd}$$

is a pullback.
For this we first have to show that it is commutative.
We can see commutativity, by showing

$$ab + f(1 + f + a + b) = f + (a + f)(b + f).$$

For this, we can just simplify both sides of this equation, which get us:

$$ab + f(1 + f + a + b) = ab + f + f^2 + fa + fb$$

and

$$f + (a + f)(b + f) = f + ab + fb + fa + f^2.$$

By commutativity of addition in rings, we can thus see that the square is commutative.
Now we show the universal pullpack property holds here. So let $*$ and $p, q \in \mathrm{Hom}(*, *)$ be a competitor to the square. Hence we have the following commutative diagram, which we call (d):

$$\begin{tikzcd}
* \arrow[rrd, bend left, "p"] \arrow[rdd, bend right, "q"'] & & \\
& * \arrow[r, "f(1+f+a+b)"] \arrow[d, "(a+f)(b+f)"] & * \arrow[d, "ab"] \\
& * \arrow[r, "f"'] & *
\end{tikzcd},$$

We now have to show existence and uniqueness of an arrow as in the universal pullback property for this diagram.

existence

Let $r := -f(1 + f + a + b) + p$. We have to show that

$$\begin{tikzcd}
* \arrow[rrd, bend left, "p"] \arrow[rdd, bend right, "q"'] \arrow[rd, dashed, "r"] & & \\
& * \arrow[r, "f(1+f+a+b)"] \arrow[d, "(a+f)(b+f)"] & * \arrow[d, "ab"] \\
& * \arrow[r, "f"'] & *
\end{tikzcd}.$$

is commutative. We only have to do so for the triangles though, as we have already seen that the square is commutative.
For the upper triangle this is immediately clear.
For the left triangle, use that $p - f + ab = q$, since (d) is commutative, and verify

$$(a + f)(b + f) \circ r = (a + f)(b + f) \circ (-f(1 + f + a + b) + p)$$

$$= ab + fa + fb + f^2 - f - f^2 - fa - fb + p = ab - f + p = q.$$

Hence existence is clear.

uniqueness

That we have uniqueness is clear, since all morphisms are isomorphism.

(s_1) Let $(a,b) \in \mathrm{fHom}(*)$. We have to show that

$$0(1+0+a+b) = 0.$$

For this we can simply use the general fact, that in rings $\forall r \in R: \ 0 \cdot r = 0$.

(s_2) Let $(a,b) \in \mathrm{fHom}(*)$ and $r, q \in \mathrm{Hom}(*,*)$. We have to show, that

$$(r+q)(1+r+q+a+b) = r(1+r+a+b) + q(1+q+a+b+2r).$$

For this we just simplify both sides of the equation, which yields:

$$\begin{aligned}(r+q)(1+r+q+a+b) &= r + r^2 + rq + ra + rb + q + qr + q^2 + qa + qb \\ &= r^2 + q^2 + 2rq + ra + rb + qa + qb\end{aligned}$$

and

$$\begin{aligned}r(1+r+a+b) + q(1+q+a+b+2r) &= r + r^2 + ra + rb + q + q^2 + qa + qb + 2qr \\ &= r^2 + q^2 + 2rq + ra + rb + qa + qb. \quad \square\end{aligned}$$

2.3 fam- and (fam, Σ)-Functors

Now we will have a short look at fam- and (fam, Σ)-functors, as we want to introduce the 2-fam-versions of these later. We will also discuss some examples of these, so that we will be able to give examples for the 2-fam-versions later. First we will consider fam-functors:

Definition 2.3.1 (fam-Functor).
Let $\mathrm{fam}C$, $\mathrm{fam}D$ be fam-categories and C, D their underlying regular categories respectively. Then, $\mathrm{fam}F$ consisting of

- $F : C \to D$ a functor and
- $\forall a \in C : \mathrm{fam}F_2^a : \mathrm{fHom}(a) \to \mathrm{fHom}(F(a))$,

is a fam-functor from $\mathrm{fam}C$ to $\mathrm{fam}D$, iff
$\forall a, b \in C_0 \forall \lambda \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b,a) :$

$$\mathrm{fam}F_2^b(\lambda \circ f) = \mathrm{fam}F_2^a(\lambda) \circ F(f).$$

Example 2.3.2 (A fam-functor from $\mathrm{Type}^f(U)$ to $\mathrm{Type}^f(U)$).
Let U be a type universe, $A : U$ and $a : A$. Let $\mathrm{Type}^f(U)$ be the fam-category form Proposition 2.1.4. Then, $\mathrm{Hom}^{f,a}(A, -)$ consisting of

- $\mathrm{Hom}(A, -)$ the functor from Example 1.0.5 and
- $\forall B \in \mathrm{Type}^f(U)_0 : \ \mathrm{fam}F_2^B := \lambda\mu.\lambda f.\mu(f(a)) : (B \to U) \to (A \to B) \to U$

is a fam-functor from $\mathrm{Type}^f(U)$ to $\mathrm{Type}^f(U)$.

Proof.
We have to show that the condition in Definition 2.3.1 is fulfilled.
So let $B, C : U, \ \mu : B \to U$ and $h : C \to B$. We have to show that

$$\big(\lambda\mu.\lambda f.\mu(f(a))\big)(\mu \circ h) = \big(\lambda\mu.\lambda f.\mu(f(a))\big)(\mu) \circ \mathrm{Hom}(A, h).$$

To do so we just simplify both sides of the equation, which yields:

$$\big(\lambda\mu.\lambda f.\mu(f(a))\big)(\mu\circ h) = (\lambda f.(\mu\circ h))(f(a)) = \lambda f.\big(\mu(h(f(a)))\big).$$

and

$$\begin{aligned}\big(\lambda\mu.\lambda f.\mu(f(a))\big)(\mu)\circ \mathrm{Hom}(A,h) &= \big(\lambda f.\mu(f(a))\big)\circ \mathrm{Hom}(A,h)\\ = \lambda x.\big((\lambda f.\mu(f(a)))(\mathrm{Hom}(A,h)(x))\big) &= \lambda x.\big(\mu((\mathrm{Hom}(A,f)(x))(a))\big)\\ = \lambda x.\big(\mu(((\lambda y.h\circ y)(x))(a))\big) = \lambda x.\big(\mu((h\circ x)(a))\big) &= \lambda x.\big(\mu(h(x(a)))\big). \quad \square\end{aligned}$$

Example 2.3.3 (A $\mathtt{fam}$-functor induced by a ring homomorphism)**.**
Let R, Q be a commutative ring and $f : R \to Q$ a ring homomorphism.
Let fR and fQ be the $\mathtt{fam}$-categories given by R and Q via Example 2.1.7 respectively.
Then, $\mathtt{fam}F$ consisting of

- $(\mathrm{id}_{\{*\}}, f)$ and
- $F_2^* : (a, b) \mapsto (f(a), f(b))$

is a $\mathtt{fam}$-functor from fR to fQ.

Proof.
That (id_*, f) forms a functor $BR \to BQ$ is easy to see and that F_2^* is well defined is also clear.
Hence we simply have to consider the condition in Definition 2.3.1.
Let $(a, b) \in f\mathrm{Hom}(*)$ and $r \in \mathrm{Hom}(*, *)$.
We have to show

$$F_2^*((a, b)\circ r) = F_2^*(a, b)\circ f(r).$$

For this we can just simplify both sides of this equation, which yields:

$$F_2^*((a, b)\circ r) = F_2^*(a + r, b + r) = (f(a) + f(r), f(b) + f(r))$$

and

$$F_2^*(a, b)\circ f(r) = (f(a), f(b))\circ f(r) = (f(a) + f(r), f(b) + f(r)). \quad \square$$

Next, we are going to look at $(\mathtt{fam}, \Sigma)$-functors. Interestingly, they have the same structure as $\mathtt{fam}$-functors, they are just subject to additional conditions. This is not entirely surprising, after all, $(\mathtt{fam}, \Sigma)$-categories do not have collections of things that $\mathtt{fam}$-categories do not have. $(\mathtt{fam}, \Sigma)$-categories only provide us with additional objects and arrows.

Definition 2.3.4 ($(\mathtt{fam}, \Sigma)$-Functor)**.**
Let ΣC, ΣD be $(\mathtt{fam}, \Sigma)$-categories and $\mathtt{fam}C, \mathtt{fam}D$ their underlying $\mathtt{fam}$-categories respectively. Then, a $\mathtt{fam}$-functor

$$\mathtt{fam}F = (F, \mathtt{fam}F_2) : \mathtt{fam}C \to \mathtt{fam}D$$

is called a $(\mathtt{fam}, \Sigma)$-functor from ΣC to ΣD, iff

1. $\forall a \in \Sigma C_0 \forall \lambda \in f\mathrm{Hom}(a)$:
$$F\Big(\sum_a \lambda\Big) = \sum_{F(a)} \mathtt{fam}F_2^a(\lambda)$$
and
2. $\forall a, b \in \Sigma C_0 \forall \lambda \in f\mathrm{Hom}(a) \forall f \in \mathrm{Hom}(b, a)$:
$$F(\Sigma_\lambda f) = \Sigma_{\mathtt{fam}F_2^a(\lambda)} F(f).$$

Example 2.3.5 (A (fam, Σ)-functor induced by a ring homomorphism)**.**
Let R, Q be a commutative ring and $f : R \to Q$ a ring homomorphism.
Let famF be the fam-functor given by R, Q and f through Example 2.3.3.
Let ΣR and ΣQ be the (fam, Σ)-categories given by Example 2.2.7.
famF is a (fam, Σ)-functor from ΣR to ΣQ.

Proof.
We just have to show the two conditions from Definition 2.3.4.

1. This condition is immediately clear, as both ΣR and ΣQ both only have one object.

2. Let $(a, b) \in \mathsf{fHom}(*)$ and $r \in \mathsf{Hom}(*, *)$.
   We have to show

   $$f(r \cdot (1 + r + a + b)) = f(r) \cdot (1 + f(r) + f(a) + f(b)).$$

   This is clear, as f is a ring homomorphism. □

2.4 dep-Categories

Now we begin to add arrows that point from an object to a fam-arrow of that object, which we call dep-arrows. We also want to be able to compose these arrows with regular arrows of appropriate codomain. Later, when we get to 2-dep-categories, we will also give ourselves a composition of a dep-arrow with the 2-fam-arrows mentioned in the introduction. A minor complication that we will have when composing dep-arrows and regular arrows, is that the resulting dep-arrow will have both a different domain and a different codomain from the original one. It is easy to see why this is necessary by considering the typing of the dep-arrows, which we will see in the definition below. The reader is also encouraged to compare how the codomain changes for these arrows with how the codomain behaves in type theory, when composing a dependent function with a regular function.

Definition 2.4.1 (dep-Category)**.**
dC consisting of

- a fam-category famC,
- a collection $\mathsf{dHom}(a, \lambda)$ of dep-arrows, $\forall a \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a)$ and
- an application of dep-arrows,$\forall a, b \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a)$, as shown in the diagram:

$$\begin{tikzcd}
b \arrow[r, "f"] \arrow[rr, bend right, "\lambda \circ f"] \arrow[rr, Rightarrow, bend right=50, "\phi \circ f"'] & a \arrow[r, "\lambda"] \arrow[r, Rightarrow, bend left=60, "\phi"] & \cdot
\end{tikzcd}$$

is called a dep-category, iff

(dep_1) $\forall a \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a) \forall \phi \in \mathsf{dHom}(a, \lambda)$:

$$\phi \circ id_a = \phi$$

and

(dep_2) $\forall a, b, c \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a) \forall \varphi \in \mathsf{dHom}(a, \lambda) \forall f \in \mathsf{Hom}(b, a) \forall h \in \mathsf{Hom}(c, b):$

$$\varphi \circ (f \circ h) = (\varphi \circ f) \circ h.$$

We also define dC_3 to be the collection of all dep-arrows of dC.
Furthermore, we say $\mathsf{dC}_0 = \mathsf{famC}_0$, $\mathsf{dC}_1 := \mathsf{famC}_1$ and $\mathsf{dC}_2 := \mathsf{famC}_2$.

With this definition, we will again look at examples as usual. First we will look at a trivial example, then at an example with constant fam-arrows and then show that small types fulfill the definition. After that, we will deviate from the previous pattern and not immediately look at the example of a topos, but rather one of the theorems mentioned in the introduction, which states that every (fam, Σ)-category induces a dep-category.

Example 2.4.2 (A dep-category with trivial dep-arrows)**.**
Let famC be a fam-category and A any collection.
Then, dC consisting of

- famC,
- $\forall a \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a) : \mathsf{dHom}(a, \lambda) := A$ and
- $\forall a, b \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a) : (\varphi, f) \mapsto \varphi$

is a dep-category.

Proof.
To show that dC is a dep-category, we simply need to show (dep_1) and (dep_2).
Both of these follow immediately though, as the application is constant in the second variable. Hence dC is a dep-category. □

Example 2.4.3 (A dep-category with constant fam-arrows)**.**
Let C be a category and famC the fam-category given by Corollary 2.1.3 and C.
Then, dC consisting of

- famC,
- $\forall a \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a) : \mathsf{dHom}(a, \lambda) := \mathsf{Hom}(a, \lambda)$ and
- $\forall a, b \in \mathsf{famC}_0 \forall \lambda \in \mathsf{fHom}(a) :$ the regular composition of arrows

is a dep-category.

Proof.
Again, we simply have to show the conditions in Definition 2.4.1.
And again, we can easily see that the conditions hold, as the regular composition of arrows respects identities, which gives us (dep_1), and is associative, which gives us (dep_2). □

Proposition 2.4.4 (Small types form a dep-category)**.**
Let U be a type universe and $\mathsf{Type}^f(U)$ the fam-category given by Proposition 2.1.4 and U.
Then, $\mathsf{Type}^d(U)$ consisting of

- $\mathsf{Type}^f(U)$,
- $\forall A \in \mathsf{Type}^f(U)_0 \forall \mu \in \mathsf{fHom}(A) :\ \mathsf{dHom}(a, \mu) := \prod_{x:A} \mu(x)$ and

- $\forall A, B \in \mathrm{Type}^{f}(U)_0 \forall \mu \in \mathrm{fHom}(A)$:

$$\circ := \lambda\phi.\lambda f.\lambda x.\phi(f(x)) : \Big(\prod_{x:A} \mu(x)\Big) \to \prod_{f:B\to A} \prod_{x:A} \mu(f(x))$$

is a dep-category.

Proof.
We again have to show that the conditions hold, this time we will have to put some more effort in though.

(dep_1) Let $A : U$, $\mu : A \to U$ and $\phi : \prod_{x:A} \phi(x)$.
We have to show

$$\phi \circ \mathrm{id}_A = \phi.$$

This we can do using function extensionality, so let $x : A$.
We can see, using the definitions

$$(\phi \circ \mathrm{id}_A)(x) = \phi(\mathrm{id}_A(x)) = \phi(x).$$

Hence clearly $\phi \circ \mathrm{id}_A = \phi$.

(dep_2) Let $A, B, C : U$, $\mu : A \to U$, $f : B \to A$, $h : C \to B$ and $\phi : \prod_{x:A} \phi(x)$.
We have to show

$$\phi \circ (f \circ h) = (\phi \circ f) \circ h.$$

Again, we can use function extensionality, let $x : C$.
Using the definitions we get

$$(\phi \circ (f \circ h))(x) = \phi((f \circ h)(x)) = \phi(f(h(x))) = (\phi \circ f)(h(x)) = ((\phi \circ f) \circ h)(x). \quad \square$$

Theorem 2.4.5 (Every (fam, Σ)-category induces a dep-category)**.**
Let ΣC be a (fam, Σ)-category and $\mathrm{fam}C$ its fam-structure.
Let $\forall a \in \Sigma C_0 \forall \lambda \in \mathrm{fHom}(a)$:

$$\mathrm{dHom}(a, \lambda) := \{\phi \in \mathrm{Hom}(a, \sum_a \lambda) | \mathrm{pr}_1^{a,\lambda} \circ \phi = \mathrm{id}_a\}.$$

Let $\forall a, b \in \Sigma C_0 \forall \lambda \in \mathrm{fHom}(a)$: an operation

$$\circ_d : \mathrm{dHom}(a, \lambda) \to \prod_{f \in \mathrm{Hom}(b,a)} \mathrm{dHom}(b, \lambda \circ f),$$

such that it returns the unique arrow making the following pullback-competitor diagram commutative:

$$\begin{tikzcd}
b \arrow[dr, dashed, red, "\phi \circ_d f"] \arrow[drr, bend left, "\phi \circ f"] \arrow[ddr, bend right, "\mathrm{id}_b"'] & & \\
& \sum_b (\lambda \circ f) \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{b,\lambda\circ f}"] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& b \arrow[r, "f"'] & a
\end{tikzcd}.$$

Then, dC consisting of

- famC,
- $\forall a \in \Sigma C_0 \forall \lambda \in \mathrm{fHom}(a) : \mathrm{dHom}(a, \lambda)$ and
- $\forall a, b \in \Sigma C_0 \forall \lambda \in \mathrm{fHom}(a) :\ \circ_d$

is a dep-category.

The proof of the theorem largely follows the proof of the same statement in [3].

Proof.
First we have to see, that $\circ_d$ actually maps to the right collection of dep-arrows.
For this let $a, b \in \Sigma C_0,\ \lambda \in \mathrm{fHom}(a),\ \phi \in \mathrm{dHom}(a, \lambda)$ and $f \in \mathrm{Hom}(b, a)$.
By considering the defining diagram, it is clear, that $\phi \circ_d f \in \mathrm{Hom}\big(b, \sum_b(\lambda \circ f)\big)$, but we still need to show, that

$$\mathrm{pr}_1^{b,\lambda\circ f} \circ (\phi \circ_d f) = \mathrm{id}_b.$$

This however, we can see by considering the left triangle in the defining diagram.
So what remains to be done, is to show the conditions (dep_1) and (dep_2).

(dep_1) Let $a \in \Sigma C_0,\ \lambda \in \mathrm{fHom}(a)$ and $\phi \in \mathrm{dHom}(a, \lambda)$.
We have to show

$$\phi \circ_d \mathrm{id}_a = \phi.$$

For this it suffices to show that the diagram

$$\begin{tikzcd}
a \arrow[dr, "\phi"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\mathrm{id}_a"'] & & \\
& \sum_a \lambda \arrow[r, "\Sigma_\lambda \mathrm{id}_a"] \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& a \arrow[r, "\mathrm{id}_a"'] & a
\end{tikzcd} .$$

is commutative.
For this we just need to show the commutativity of the triangles.
For the left triangle, this is clear, by the condition on dep-arrows.
So we just have to demonstrate commutativity of the upper triangle.
To see this, we use (s_1) and see that $\Sigma_\lambda \mathrm{id}_a = \mathrm{id}_{\sum_a \lambda}$.
With this we know that the upper triangle is commutative.
Thus we have seen that (dep_1) holds.

(dep_2) Let $a, b, c \in \Sigma C_0,\ \lambda \in \mathrm{fHom}(a),\ \phi \in \mathrm{dHom}(a, \lambda),\ f \in \mathrm{Hom}(b, a)$ and $h \in \mathrm{Hom}(c, b)$.
We have to show

$$\phi \circ_d (f \circ h) = (\phi \circ_d f) \circ_d h.$$

For this, first consider the definition of the left side of this equation:

$$\begin{tikzcd}
c \arrow[drr, bend left, "(\phi \circ_d f) \circ h"] \arrow[dr, dashed, red, "(\phi \circ_d f) \circ_d h"] \arrow[ddr, bend right, "id_c"'] & & \\
& \sum_c (\lambda \circ f) \circ h \arrow[r, "\Sigma_{\lambda \circ f} h"] \arrow[d, "pr_1^{c, \lambda \circ f \circ h}"] & \sum_b (\lambda \circ f) \arrow[d, "pr_1^{b, \lambda \circ f}"] \\
& c \arrow[r, "h"'] & b
\end{tikzcd}.$$

If we can show, that the right side of this equation also makes the above diagram commutative when substituting the red arrow for it, we are done. So we have to show

(i) $id_c = pr_1^{a, \lambda \circ f \circ h} \circ \phi \circ_d (f \circ h)$ and

(ii) $(\phi \circ_d f) \circ h = \Sigma_{\lambda \circ f} h \circ \phi \circ_d (f \circ h)$.

To do so, consider the definition for $\phi \circ_d (f \circ h)$:

$$\begin{tikzcd}
c \arrow[drr, bend left, "\phi \circ f \circ h"] \arrow[dr, dashed, red, "\phi \circ_d (f \circ h)"] \arrow[ddr, bend right, "id_c"'] & & \\
& \sum_c (\lambda \circ f) \circ h \arrow[r, "\Sigma_\lambda (f \circ h)"] \arrow[d, "pr_1^{c, \lambda \circ f \circ h}"] & \sum_a \lambda \arrow[d, "pr_1^{a, \lambda}"] \\
& c \arrow[r, "f \circ h"'] & a
\end{tikzcd}.$$

We call this diagram $(*)$.
With this we can immediately see, from the left triangle, that (i) holds.
For (ii), consider the definition of $\phi \circ_d f$:

$$\begin{tikzcd}
b \arrow[drr, bend left, "\phi \circ f"] \arrow[dr, dashed, red, "\phi \circ_d f"] \arrow[ddr, bend right, "id_b"'] & & \\
& \sum_b (\lambda \circ f) \arrow[r, "\Sigma_\lambda f"] \arrow[d, "pr_1^{b, \lambda \circ f}"] & \sum_a \lambda \arrow[d, "pr_1^{a, \lambda}"] \\
& c \arrow[r, "f"'] & a
\end{tikzcd}.$$

We call this diagram $(**)$.

We can put parts of $(*)$ and $(**)$ together, to form the following commutative diagram:

$$\begin{tikzcd}
c \arrow[rr, bend left, "h"] \arrow[dr, dashed, red, "\phi \circ_d (f \circ h)"] \arrow[ddr, bend right, "id_c"'] & & b \arrow[drr, bend left, "\phi \circ f"] \arrow[dr, dashed, red, "\phi \circ_d f"] & & \\
& \sum_c (\lambda \circ f) \circ h \arrow[rr, "\Sigma_{\lambda \circ f} h"] & & \sum_b (\lambda \circ f) \arrow[r, "\Sigma_\lambda f"] \arrow[d, "pr_1^{b, \lambda \circ f}"] & \sum_a \lambda \arrow[d, "pr_1^{a,\lambda}"] \\
& c \arrow[rr, "h"'] & & b \arrow[r, "f"] & a
\end{tikzcd} .$$

We can clearly see, that the inner square in this diagram is a pullback.
Furthermore, we can also see, that c, $\phi \circ f \circ h$ and h are a competitor to that pullback.
Lastly, we can see, that both $(\phi \circ_d f) \circ h$ and $\Sigma_{\lambda \circ f} h \circ \phi \circ_d (f \circ h)$ fulfill the universal pullback property for the put together diagram. Thus they are equal, showing that (ii) holds. Thus we have

$$\phi \circ_d (f \circ h) = (\phi \circ_d f) \circ_d h,$$

what we wanted to show. □

We will now address the example of a topos again. We have seen in Lemma 2.2.6 that every topos is a (fam, Σ)-category. With the above theorem in hand, we thus know that every topos is a dep-category. But just knowing that is not enough to understand the dep-structure of a topos, so we have to put in some additional work:

Lemma 2.4.6 (Toposes as dep-categories).
Let C be a Topos and let ΣC be the (fam, Σ)-category given by Lemma 2.2.6 and C.
Let dC be the dep-category given by ΣC and Theorem 2.4.5.
Let $a \in dC_0$ and $(k, e) \in fHom(a)$.
Then, there is a bijective correspondence between

- the $f \in Hom(a, k)$ fulfilling $e \circ (id_a, f) = T \circ 1_a$ and
- the dependent arrows from a to (k, e) in dC.

In one direction, this correspondence is given by

$$\chi : dHom(a, (k, e)) \to \{f \in Hom(a, k) | e \circ (id_a, f) = T \circ 1_a\},$$

$$\phi \mapsto pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi,$$

where $p_{a,(k,e)}$ is the arrow from the pullback as in Lemma 2.2.6 and $pr_2^{a,k} : a \times k \to k$ canonical.
In the other direction the correspondence is given by

$$\delta : \{f \in Hom(a, k) | e \circ (id_a, f) = T \circ 1_a\} \to dHom(a, (k, e)),$$

$$f \mapsto \delta(f),$$

where $\delta(f)$ is the unique arrow given by the universal pullback property:

$$\begin{tikzcd}
a \arrow[dr, dashed, red, "\delta(f)"] \arrow[drr, bend left, "(\mathrm{id}_a, f)"] \arrow[ddr, bend right] & & \\
& \sum_a(k,e) \arrow[r] \arrow[d] & a \times k \arrow[d, "e"] \\
& 1 \arrow[r, "\top"] & \Omega
\end{tikzcd}.$$

Furthermore, it can be shown, that

$$\forall b \in dC_0 \forall \phi \in d\mathrm{Hom}(a,(k,e)) \forall h \in \mathrm{Hom}(b,a) : \chi(\phi \circ_d h) = \chi(\phi) \circ h.$$

Proof.
We first have to consider whether χ and δ are well defined. We start with χ.
We have to show, that, given $\phi \in d\mathrm{Hom}(a,(k,e))$, we have

1. $pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi \in \mathrm{Hom}(a,k)$ and
2. $e \circ (\mathrm{id}_a, pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi) = \top \circ 1_a$.

This, we can do in the following way:

1. Recall from the respective definitions, that
   - $\phi \in \mathrm{Hom}(a, \Sigma_a(k,e))$,
   - $p_{a,(k,e)} \in \mathrm{Hom}(\Sigma_a(k,e), a \times k)$ and
   - $pr_2^{a,k} \in \mathrm{Hom}(a \times k, k)$.

   Thus clearly $pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi \in \mathrm{Hom}(a,k)$.
2. For this, we first show that $(\mathrm{id}_a, pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi) = p_{a,(k,e)} \circ \phi$.
   We do this, by showing that the diagram

$$\begin{tikzcd}
& & a \\
a \arrow[urr, bend left, "\mathrm{id}_a"] \arrow[r, "p_{a,(k,e)} \circ \phi"'] \arrow[drr, bend right, "pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi"'] & a \times k \arrow[ur, "pr_1^{a,k}"] \arrow[dr, "pr_2^{a,k}"] & \\
& & k
\end{tikzcd}$$

   is commutative.
   For the upper triangle, use the conditions on ϕ in Theorem 2.4.5 and the definition of $pr_1^{a,(k,e)}$ in Lemma 2.2.6 and verify

$$\mathrm{id}_a = pr_1^{a,(k,e)} \circ \phi = pr_1^{a,k} \circ p_{a,(k,e)} \circ \phi.$$

For the lower triangle, simply use reflexivity.
Since $(\mathrm{id}_a, \mathrm{pr}_2^{a,k} \circ p_{a,(k,e)} \circ \phi)$ is given as by the universal product property:

$$\begin{tikzcd}
 & & a \\
a \arrow[r, dashed, red, "{(\mathrm{id}_a,\, \mathrm{pr}_2^{a,k} \circ p_{a,(k,e)} \circ \phi)}"'] \arrow[urr, bend left, "\mathrm{id}_a"] \arrow[drr, bend right, "\mathrm{pr}_2^{a,k} \circ p_{a,(k,e)} \circ \phi"'] & a \times k \arrow[ur, "\mathrm{pr}_1^{a,k}"] \arrow[dr, "\mathrm{pr}_2^{a,k}"] & \\
 & & k
\end{tikzcd},$$

we thus know that

$$(\mathrm{id}_a, \mathrm{pr}_2^{a,k} \circ p_{a,(k,e)} \circ \phi) = p_{a,(k,e)} \circ \phi.$$

Hence, we only have to show $e \circ p_{a,(k,e)} \circ \phi = T \circ 1_a$.
Furthermore, since $1_a = 1_{\sum_a(k,e)} \circ \phi$, it suffices to show

$$e \circ p = T \circ 1_{\sum_a(k,e)}.$$

However, this holds by the defining diagram of $\sum_a(k,e)$ in Lemma 2.2.6.

Hence we see that χ is well defined.

Now we consider δ.
That the resulting arrows have the correct type is clear by the defining diagram of δ. Hence we need only consider that the condition on dependent arrows from Theorem 2.4.5 is fulfilled by these arrows. So let $f \in \mathrm{Hom}(a, k)$, such that

$$e \circ (\mathrm{id}_a, f) = T \circ 1_a.$$

We have to show that

$$\mathrm{pr}_1^{a,(k,e)} \circ \delta(f) = \mathrm{id}_a.$$

By the defining diagram for $\delta(f)$, we have the following commutative diagram:

$$\begin{tikzcd}
a \arrow[dr, dashed, red, "\delta(f)"] \arrow[drr, bend left, "{(\mathrm{id}_a,\, f)}"] \arrow[ddr, bend right] & & & \\
 & \sum_a(k,e) \arrow[r, "p_{a,(k,e)}"] \arrow[d] & a \times k \arrow[r, "\mathrm{pr}_1^{a,k}"] \arrow[d, "e"] & a \\
 & 1 \arrow[r, "T"] & \Omega &
\end{tikzcd}$$

By the definition of $\mathrm{pr}_1^{a,(k,e)}$, in Lemma 2.2.6, we have $\mathrm{pr}_1^{a,(k,e)} = \mathrm{pr}_1^{a,k} \circ p_{a,(k,e)}$.
Thus, together with the defining diagram for $\delta(f)$, we get

$$\mathrm{pr}_1^{a,(k,e)} \circ \delta(f) = \mathrm{pr}_1^{a,k} \circ p_{a,(k,e)} \circ \delta(f) = \mathrm{pr}_1^{a,k} \circ (\mathrm{id}_a, f) = \mathrm{id}_a.$$

This concludes the part for δ.

Now we want to show, that χ and δ are inverse to each other. For this we simply have to show that they compose to identity in both directions.

(→) Let $\phi \in dHom(a, (k, e))$.
We have to show
$$\delta(\chi(\phi)) = \phi.$$
Using the defining diagram for $\delta(\chi(\phi))$, we only need to show that the diagram

$$\begin{tikzcd}
a \arrow[dr, "\phi"] \arrow[drr, bend left, "{(id_a,\ \chi(\phi))}"] \arrow[ddr, bend right] & & \\
& \sum_a(k,e) \arrow[r, "p_{a,(k,e)}"] \arrow[d] & a \times k \arrow[d, "e"] \\
& 1 \arrow[r, "\top"] & \Omega
\end{tikzcd}.$$

is commutative.
The inner square is commutative since it is a pullback.
The well definedness of χ guarantees that the outer square is commutative.
Since 1 is a terminal object, the left triangle is commutative.
So what remains is to show $p_{a,(k,e)} \circ \phi = (id_a, \chi(\phi))$.
However, we already saw this earlier in this proof.
Thus $\delta(\chi(\phi)) = \phi$.

(←) Let $f \in Hom(a, k)$ with $e \circ (id_a, f) = \top \circ 1_a$. We have to show
$$\chi(\delta(f)) = f.$$
We have that $\chi(\delta(f)) = pr_2^{a,k} \circ p_{a,(k,e)} \circ \delta(f)$ and that the diagram

$$\begin{tikzcd}
a \arrow[dr, dashed, red, "\delta(f)"] \arrow[drr, bend left, "{(id_a,\ f)}"] \arrow[ddr, bend right] & & \\
& \sum_a(k,e) \arrow[r, "p_{a,(k,e)}"] \arrow[d] & a \times k \arrow[d, "e"] \\
& 1 \arrow[r, "\top"] & \Omega
\end{tikzcd}.$$

is commutative.
Thus
$$\chi(\delta(f)) = pr_2^{a,k} \circ p_{a,(k,e)} \circ \delta(f) = pr_2^{a,k} \circ (id_a, f) = f.$$

With this we have seen that χ and δ are indeed inverses to each other.

What remains now, is to show that
$$\forall b \in dC_0 \forall \phi \in dHom(a, (k, e)) \forall h \in Hom(b, a) : \chi(\phi \circ_d h) = \chi(\phi) \circ h.$$
So let $b \in dC_0$, $\phi \in dHom(a, (k, e))$ and $h \in Hom(b, a)$. We have to show, that
$$pr_2^{b,k} \circ p_{b,(k,e)\circ h} \circ (\phi \circ_d h) = pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi \circ h.$$

To do so, recall the defining diagrams of $\phi \circ_d h$ and $\Sigma_{k,e}h$. Putting these together, we get the diagram

$$\begin{tikzcd}
b \arrow[rrrd, bend left, "\phi\circ h"] \arrow[dr, dashed, "\phi\circ_d h"] \arrow[ddddr, bend right, "\mathrm{id}_b"'] & & & \\
& \sum_b(k,e)\circ h \arrow[rr, "\Sigma_{(k,e)}h"] \arrow[d, "p_{b,(k,e)\circ h}"] & & \sum_a(k,e) \arrow[d, "p_{a,(k,e)}"] \\
& b\times k \arrow[rr, "h\times \mathrm{id}_k"] \arrow[dr, "pr_2^{b,k}"] \arrow[dd, "pr_1^{b,k}"] & & a\times k \arrow[dl, "pr_2^{a,k}"'] \arrow[dd, "pr_1^{a,k}"] \\
& & k & \\
& b \arrow[rr, "h"] & & a
\end{tikzcd},$$

which is commutative.
Using the commutativity of this diagram, we get

$$pr_2^{b,k} \circ p_{b,(k,e)\circ h} \circ (\phi \circ_d h) = pr_2^{b,k} \circ h \times id_k \circ p_{b,(k,e)\circ h} \circ (\phi \circ_d h)$$

$$= pr_2^{b,k} \circ p_{a,(k,e)} \circ \Sigma_{(k,e)}h \circ (\phi \circ_d h) = pr_2^{a,k} \circ p_{a,(k,e)} \circ \phi \circ h. \quad \square$$

Remark 2.4.7 (δ also commutes with composition)**.**
Notice, that as a result of Lemma 2.4.6 we also have that

$$\forall b \in dC_0 \forall f \in d\mathrm{Hom}(a,(k,e)) \forall h \in \mathrm{Hom}(b,a) : \delta(f \circ h) = \delta(f) \circ_d h.$$

This follows since δ and χ, as defined in Lemma 2.4.6, are inverse to each other and since we have seen the analogous statement for ϕ.

Next, we want to see, that for the example of a commutative ring, there is a dep-category, which is not given through Theorem 2.4.5 and the $(\mathfrak{fam}, \Sigma)$-category from Example 2.2.7. For this, we have to give a dep-structure for a commutative ring independently of Theorem 2.4.5, which has the $\mathfrak{fam}$-structure from Example 2.1.7. This, we do now:

Example 2.4.8 (A dep-structure for a commutative ring)**.**
Let R be a commutative ring. Let fR be the $\mathfrak{fam}$-category given by 2.1.7 and R.
Then, dR consisting of

- fR,
- $\forall (a,b) \in f\mathrm{Hom}(*) : \; d\mathrm{Hom}(*,(a,b)) := \{I \in \mathrm{Ideal}(R) | a - b \in I\}$ and
- $\forall (a,b) \in f\mathrm{Hom}(*) \forall I \in d\mathrm{Hom}(*,(a,b)) \forall r \in R : \; I \circ r := I$

is a dep-category.

Proof.
First, notice that the here defined application does have the correct typing as given in Definition 2.4.1. Now, we have to show the conditions (dep_1) and (dep_2).

(dep_1) Let $(a,b) \in \mathrm{fHom}(*)$ and $I \in \mathrm{dHom}(*,(a,b))$.
Thus we have to show $I \circ 0 = I$, which holds by definition.

(dep_2) Let $(a,b) \in \mathrm{fHom}(*)$, $I \in \mathrm{dHom}(*,(a,b))$ and $r,q \in \mathrm{Hom}(*,*)$.
Thus we have to show

$$I \circ (r \circ q) = (I \circ r) \circ q.$$

Here both sides of the equation simplify to just I, making them equal. □

Now we can compare the two dep-structures for a commutative ring:

Proposition 2.4.9 (A dep-structure on a commutative ring not given by Theorem 2.4.5**).**
Let R be a commutative ring.
We use

- dR for the dep-category from Example 2.4.8 and
- ΣR for the (fam, Σ)-category from Example 2.2.7.

Furthermore, $\forall (a,b) \in \mathrm{fHom}(*)$, we call the collection of dep arrows from $*$ to (a,b), given by Theorem 2.4.5 and ΣR, $\mathrm{dHom}_l(*,(a,b))$.

Then, there is a commutative ring R, such that,

$$\exists (a,b) \in \mathrm{fHom}(*) : |\mathrm{dHom}(*,(a,b))| > |\mathrm{dHom}_l(*,(a,b))|,$$

where $|\cdot|$ refers to the cardinality of sets.

Proof.
Let $R := \mathbb{Z}$.
Let $(a,b) \in \mathbb{Z}^2$, such that $a - b$ is not a unit.

First, consider $\mathrm{dHom}_l(*,(a,b))$.
Let $r,q \in \mathrm{dHom}_l(*,(a,b))$.
Hence, by Theorem 2.4.5, $r + ab = 0 = q + ab$.
Thus $r = -ab = q$.
Thus $\mathrm{dHom}_l(*,(a,b))$ has at most one element.

Now consider $\mathrm{dHom}(*,(a,b))$.
We need to find at least 2 ideals, such that they both contain $a - b$.
One ideal that does that, is R itself.
Another one is $< a - b >$.
Since $a - b$ is not a unit, $R \neq < a - b >$.
Hence we find at least 2 elements in $\mathrm{dHom}(*,(a,b))$.
Thus $|\mathrm{dHom}(*,(a,b))| > |\mathrm{dHom}_l(*,(a,b))|$. □

2.5 (dep, Σ)-Categories

Now we will combine the notions of (fam, Σ)- and dep-categories into (dep, Σ)-categories. Combining (fam, Σ)- and dep-categories is not just for fun. Of course it is fun, but it will also allow us to add the notion of a second projection arrow to the Σ-objects. Comparing with types again, it makes sense that we couldn't before, as the second projection is dependent. Hence the second projection arrow needs both Σ-objects and dep-arrows to be defined sensibly.

Definition 2.5.1 ($(\mathtt{dep}, \Sigma)$-Category).
$d\Sigma C$ consisting of

- a dep-category dC,
- a $(\mathtt{fam}, \Sigma)$-category ΣC and
- a dep-arrow $\mathtt{pr}_2^{a,\lambda} \in \mathrm{dHom}(\sum_a \lambda, \lambda \circ \mathtt{pr}_1^{a,\lambda}), \forall a \in dC_0 \forall \lambda \in \mathrm{fHom}(a)$, (called the second projection arrow of a and λ)

is called a $(\mathtt{dep}, \Sigma)$-category, iff

$(\mathtt{fam})$ The $\mathtt{fam}$-structure of dC and ΣC are the same and

$(d\Sigma)$ $\forall a, b \in dC_0 \forall \lambda \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b, a)$:

$$\mathtt{pr}_2^{a,\lambda} \circ \Sigma_\lambda f = \mathtt{pr}_2^{b,\lambda\circ f},$$

meaning the diagram

$$\begin{tikzcd}
\sum_b(\lambda \circ f) \arrow[rr, "\Sigma_\lambda f"] \arrow[dr, "\lambda\circ f\circ \mathtt{pr}_1^{b,\lambda\circ f}"] \arrow[dr, Rightarrow, bend right, "\mathtt{pr}_2^{b,\lambda\circ f}"'] & & \sum_a \lambda \arrow[dl, "\lambda\circ \mathtt{pr}_1^{a,\lambda}"'] \arrow[dl, Rightarrow, bend left, "\mathtt{pr}_2^{a,\lambda}"] \\
& \cdot &
\end{tikzcd}$$

is commutative.

We also define $C_0 := dC_0 = \Sigma C_0,\ C_1 := dC_1 = \Sigma C_1,\ C_2 := dC_2 = \Sigma C_2$ and $C_3 := dC_3$.

Remark 2.5.2 (Well-definedness in Definition 2.5.1).
To see that the equality in $(d\Sigma)$ is well defined, we have to show

1. that the codomains are equal, so $\lambda \circ \mathtt{p}_1^{a,\lambda} \circ \Sigma_\lambda f = \lambda \circ f \circ \mathtt{pr}_1^{b,\lambda\circ f}$ and
2. that the domains are equal, so $\sum_b(\lambda \circ f) = \sum_b(\lambda \circ f)$.

2. is easy and we use $(\mathtt{pull})$ from Definition 2.2.1 for 1.

Now we want to look at another trivial example. Previously, for the dep-and $(\mathtt{fam}, \Sigma)$-categories, we always took a general $\mathtt{fam}$-structure and then added trivial additional structure to it. So what we might want to do here, is to take general dep-and $(\mathtt{fam}, \Sigma)$-structures and add to them trivial second projection arrows. But there is an obvious problem with that. Not every dep-structure can have second projection arrows. Say, for instance, you pick every dHom collection to be empty. This would clearly yield a dep-structure for every $\mathtt{fam}$-category. However, it would clearly be impossible to select second projection arrows from such a dep-structure. Hence we have to do something else. And we will do the next best thing, our $(\mathtt{fam}, \Sigma)$-structure will be general and the dep-structure will be given by Example 2.4.2.

Example 2.5.3 (A $(\mathtt{dep}, \Sigma)$-category with trivial dependent arrows).
Let A be a collection. Let $k \in A$. Let ΣC be a $(\mathtt{fam}, \Sigma)$-Category. Let dC be the dep-Category given by Example 2.4.2 and the $\mathtt{fam}$-structure of ΣC.
Then, $d\Sigma C$ consisting of

- dC,

- ΣC and
- $\forall a \in dC_0 \forall \lambda \in f\mathrm{Hom}(a) : k$

is a (dep, Σ)-category.

Proof.
To show this, we must show the conditions (fam) and $(d\Sigma)$ from Definition 2.5.1.

(fam) That dC and ΣC have the same fam-structure is clear by how they where defined.

$(d\Sigma)$ Let $a, b \in dC_0$, $\lambda \in f\mathrm{Hom}(a)$ and $f \in \mathrm{Hom}(b, a)$.
We have to show

$$k \circ \Sigma_\lambda f = k.$$

Recall the definition of this composition in Example 2.4.2. We can see the equation holds immediately by definition. □

Example 2.5.4 (A (dep, Σ)-category with constant family arrows).
Let C be a category with binary products and an assignment

$$C_0 \times C_0 \longrightarrow C_0, (a, b) \mapsto a \times b,$$

where $a \times b$ is a product of a and b.
Let dC be the dep-category given by Example 2.4.3 and C.
Let ΣC be the (fam, Σ)-category given by Example 2.2.4.
Then, $d\Sigma C$ consisting of

- dC,
- ΣC and
- $\forall a \in dC_0 \forall \lambda \in f\mathrm{Hom}(a) : \mathrm{pr}_2^{a,\lambda} \in \mathrm{Hom}(a \times \lambda, \lambda)$ canonical

is a (dep, Σ)-category.

Proof.
Again we have to show the conditions in the definition.

(fam) It is clear from the construction in the Examples 2.4.3 and 2.2.4, that dC and ΣC have the same fam-structure.

$(d\Sigma)$ Let $a, b \in dC_0$, $\lambda \in f\mathrm{Hom}(a)$ and $f \in \mathrm{Hom}(b, a)$.
We have to show

$$\mathrm{pr}_2^{a,\lambda} \circ (f \times \mathrm{id}_\lambda) = \mathrm{pr}_2^{b,\lambda}.$$

This however follows immediately from the definition of the second projection arrows, as it is a basic property of products. □

Proposition 2.5.5 (Small types form a (dep, Σ)-category).
Let U be a type universe.
Let $\mathrm{Type}^d(U)$ be the dep-category given by Proposition 2.4.4 and U.
Let $\mathrm{Type}^\Sigma(U)$ be the (fam, Σ)-category given by Proposition 2.2.5 and U.
Then, $\mathrm{Type}^{d\Sigma}(U)$ consisting of

- $\mathrm{Type}^d(U)$,
- $\mathrm{Type}^\Sigma(U)$ and

- $\forall A \in \mathsf{Type}^d(U)_0 \forall \mu \in \mathsf{fHom}(A): \; \mathrm{pr}_2^{A,\mu} : \prod_{y:\sum_{x:A}\mu(x)} \mu(\mathrm{pr}_1^{A,\mu}(y))$ the second projection

is a (dep, Σ)-category.

Proof.
For this we again simply have to show the conditions.

(fam) By considering the constructions in the Propositions 2.4.4 and 2.2.5, we can see that $\mathsf{Type}^d(U)$ and ΣC have the same fam-structure.

$(\mathsf{d}\Sigma)$ Let $A, B \in \mathsf{Type}^d(U)_0$, $\mu \in \mathsf{fHom}(A)$ and $f \in \mathrm{Hom}(B, A)$.
Thus $A, B : U$, $\mu : A \to U$ and $f : B \to A$.
We now have to show, that

$$\mathrm{pr}_2^{A,\mu} \circ \Sigma_\mu f = \mathrm{pr}_2^{B,\mu\circ f}.$$

We will show this utilizing function extensionality, so let $(x, y) : \sum_B (\mu \circ f)$.
Hence, using the computation rules, we can see

$$(\mathrm{pr}_2^{A,\mu} \circ \Sigma_\mu f)(x, y) = \mathrm{pr}_2^{A,\mu}(\Sigma_\mu f(x, y)) = \mathrm{pr}_2^{A,\mu}(f(x), y) = y$$

and

$$\mathrm{pr}_2^{B,\mu\circ f}(x, y) = y. \quad \square$$

Next we will extend Theorem 2.4.5. We will see, that (fam, Σ)-categories not only induce dep-categories, but also (dep, Σ)-categories.

Theorem 2.5.6 (Every (fam, Σ)-category induces a (dep, Σ)-category)**.**
Let ΣC be a (fam, Σ)-category.
Let $\mathsf{d}C$ be the dep-category given by Theorem 2.4.5 and C.
Let $\forall a \in \mathsf{d}C_0 \forall \lambda \in \mathsf{fHom}(a) : \mathrm{pr}_2^{a,\lambda}$ the unique arrow fulfilling the pullback-competitor diagram:

$$\begin{tikzcd}
\sum_a \lambda \arrow[dr, dashed, "\mathrm{pr}_2^{a,\lambda}"] \arrow[drr, bend left, "\mathrm{id}_{\sum_a \lambda}"] \arrow[ddr, bend right, "\mathrm{id}_{\sum_a \lambda}"'] & & \\
& \sum_{\sum_a \lambda}(\lambda \circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\lambda \mathrm{pr}_1^{a,\lambda}"] \arrow[d, "\mathrm{pr}_1^{\sum_a \lambda, \lambda\circ \mathrm{pr}_1^{a,\lambda}}"] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_a \lambda \arrow[r, "\mathrm{pr}_1^{a,\lambda}"'] & a
\end{tikzcd}.$$

Then, $\mathsf{d}\Sigma C$ consisting of

- $\mathsf{d}C$,
- ΣC and
- $\forall a \in \mathsf{d}C_0 \forall \lambda \in \mathsf{fHom}(a): \; \mathrm{pr}_2^{a,\lambda}$

is a (dep, Σ)-category.

Proof.
By the defining diagram of the second projection arrows, we can immediately see, that they do fulfill the conditions for being dep-arrows of the appropriate type by Theorem 2.4.5.
So what we have to do, is simply show that the conditions from Definition 2.5.1 hold.

(fam) By the construction in Theorem 2.4.5, we know that the fam-structures of C and dC are the same.

($d\Sigma$) Let $a, b \in dC_0$, $\lambda \in f\mathrm{Hom}(a)$ and $f \in \mathrm{Hom}(b, a)$.
We have to show

$$\mathrm{pr}_2^{a,\lambda} \circ_d \Sigma_\lambda f = \mathrm{pr}_2^{b,\lambda\circ f}.$$

Putting together the defining diagrams for $\mathrm{pr}_2^{b,\lambda\circ f}$ and $\Sigma_\lambda f$, we obtain the following commutative diagram, which we call $(*)$:

$$\begin{tikzcd}
\sum_b(\lambda\circ f) \arrow[drr, bend left, "\mathrm{id}_{\Sigma_b(\lambda\circ f)}"] \arrow[dr, dashed, red, "\mathrm{pr}_2^{b,\lambda\circ f}"] \arrow[ddr, bend right, "\mathrm{id}_{\Sigma_b(\lambda\circ f)}"'] & & & \\
& \sum_{\Sigma_b(\lambda\circ f)}(\lambda\circ f\circ \mathrm{pr}_1^{b,\lambda\circ f}) \arrow[r, "\Sigma_{\lambda\circ f}\mathrm{pr}_1^{b,\lambda\circ f}"] \arrow[d, "\mathrm{pr}_1^{\Sigma_b(\lambda\circ f),\lambda\circ f\circ \mathrm{pr}_1^{b,\lambda\circ f}}"] & \sum_b(\lambda\circ f) \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{b,\lambda\circ f}"] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_b(\lambda\circ f) \arrow[r, "\mathrm{pr}_1^{b,\lambda\circ f}"'] & b \arrow[r, "f"] & a
\end{tikzcd},$$

Now for $\mathrm{pr}_2^{a,\lambda} \circ_d \Sigma_\lambda f$:
Putting together

- the defining diagram of $\mathrm{pr}_2^{a,\lambda} \circ_d \Sigma_\lambda f$, from Theorem 2.4.5,
- the defining diagram for $\Sigma_\lambda \mathrm{pr}_1^{a,\lambda}$ from Definition 2.2.1 and
- the arrow $\Sigma_\lambda f$,

we obtain the following commutative diagram, which we call $(**)$:

$$\begin{tikzcd}
\sum_b(\lambda\circ f) \arrow[drrr, bend left, "\Sigma_\lambda f"] \arrow[drr, bend left, "\mathrm{pr}_2^{a,\lambda}\circ\Sigma_\lambda f"] \arrow[dr, dashed, red, "\mathrm{pr}_2^{a,\lambda}\circ_d\Sigma_\lambda f"] \arrow[ddr, bend right, "\mathrm{id}_{\Sigma_b(\lambda\circ f)}"'] & & & \\
& \sum_{\Sigma_b(\lambda\circ f)}(\lambda\circ \mathrm{pr}_1^{a,\lambda}\circ\Sigma_\lambda f) \arrow[r, "\Sigma_{(\lambda\circ \mathrm{pr}_1^{a,\lambda})}\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{\Sigma_b(\lambda\circ f),\lambda\circ \mathrm{pr}_1^{b,\lambda\circ f}\circ\Sigma_\lambda f}"] & \sum_{\Sigma_a\lambda}(\lambda\circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\lambda \mathrm{pr}_1^{a,\lambda}"] \arrow[d, "\mathrm{pr}_1^{\Sigma_a\lambda,\lambda\circ \mathrm{pr}_1^{a,\lambda}}"] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_b(\lambda\circ f) \arrow[r, "\Sigma_\lambda f"'] & \sum_a\lambda \arrow[r, "\mathrm{pr}_1^{a,\lambda}"] & a
\end{tikzcd}$$

Now, to show that the red arrows in $(*)$ and $(**)$ are equal, we first want to show that, the arrows actually have the same codomain. Hence we need to show

$$\mathrm{pr}_1^{a,\lambda} \circ \Sigma_\lambda f = f \circ \mathrm{pr}_1^{b,\lambda\circ f}$$

This follows from the commutativity of the right square in $(*)$.
To see that the arrows are also equal, we will show that they fulfill the same pullback-competitor diagram. For this we pick appropriate subdiagrams of $(*)$ and $(**)$ (called $(*')$ and $(**')$ respectively) as follows:

$$\begin{tikzcd}
\sum_b(\lambda\circ f) \arrow[rrr, bend left, "\Sigma_\lambda f"] \arrow[dr, dashed, red, "\mathrm{pr}_2^{b,\lambda\circ f}"] \arrow[ddr, bend right, "\mathrm{id}_{\Sigma_b(\lambda\circ f)}"'] & & & \sum_a\lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_{\Sigma_b(\lambda\circ f)}(\lambda\circ f\circ \mathrm{pr}_1^{b,\lambda\circ f}) \arrow[urr, "\Sigma_\lambda f\circ\Sigma_{\lambda\circ f}\mathrm{pr}_1^{b,\lambda\circ f}"] \arrow[d, "\mathrm{pr}_1^{\Sigma_b(\lambda\circ f),\lambda\circ f\circ\mathrm{pr}_1^{b,\lambda\circ f}}"] & & a \\
& \sum_b(\lambda\circ f) \arrow[urr, "f\circ\mathrm{pr}_1^{b,\lambda\circ f}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\sum_b(\lambda\circ f) \arrow[rrr, bend left, "\Sigma_\lambda f"] \arrow[dr, dashed, red, "\mathrm{pr}_2^{a,\lambda}\circ_d\Sigma_\lambda f"] \arrow[ddr, bend right, "\mathrm{id}_{\Sigma_b(\lambda\circ f)}"'] & & & \sum_a\lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_{\Sigma_b(\lambda\circ f)}(\lambda\circ \mathrm{pr}_1^{a,\lambda}\circ\Sigma_\lambda f) \arrow[urr, "\Sigma_\lambda\mathrm{pr}_1^{a,\lambda}\circ\Sigma_{(\lambda\circ\mathrm{pr}_1^{a,\lambda})}\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{\Sigma_b(\lambda\circ f),\lambda\circ\mathrm{pr}_1^{a,\lambda}\circ\Sigma_\lambda f}"] & & a \\
& \sum_b(\lambda\circ f) \arrow[urr, "\mathrm{pr}_1^{a,\lambda}\circ\Sigma_\lambda f"'] & &
\end{tikzcd}.$$

By (pull) and the pullback lemma, the inner squares of both $(*')$ and $(**')$ are pullbacks.
So what remains to be seen, is that $(*')$ and $(**')$ are the same. For this we have to show

(i) $\mathrm{pr}_1^{a,\lambda}\circ\Sigma_\lambda f = f\circ\mathrm{pr}_1^{b,\lambda\circ f}$,

(ii) $\mathrm{pr}_1^{\Sigma_b(\lambda\circ f),\lambda\circ f\circ\mathrm{pr}_1^{b,\lambda\circ f}} = \mathrm{pr}_1^{\Sigma_b(\lambda\circ f),\lambda\circ\mathrm{pr}_1^{a,\lambda}\circ\Sigma_\lambda f}$ and

(iii) $\Sigma_\lambda f\circ\Sigma_{\lambda\circ f}\mathrm{pr}_1^{b,\lambda\circ f} = \Sigma_\lambda\mathrm{pr}_1^{a,\lambda}\circ\Sigma_{(\lambda\circ\mathrm{pr}_1^{a,\lambda})}\Sigma_\lambda f$.

We already have seen (i) earlier.
(ii) then immediately follows from (i).
For (iii), we have to use (s_2):
Using (i) and (s_2), we can see

$$(\Sigma_\lambda f)\circ\Sigma_{(\lambda\circ f)}\mathrm{pr}_1^{b,\lambda\circ f} = \Sigma_\lambda(f\circ\mathrm{pr}_1^{b,\lambda\circ f})$$

and

$$\Sigma_\lambda\mathrm{pr}_1^{a,\lambda}\circ\Sigma_{\left(\lambda\circ\mathrm{pr}_1^{a,\lambda}\right)}(\Sigma_\lambda f) = \Sigma_\lambda(\mathrm{pr}_1^{a,\lambda}\circ\Sigma_\lambda f) = \Sigma_\lambda(f\circ\mathrm{pr}_1^{b,\lambda\circ f}).$$

With this we can see, that (iii) holds as well.
Thus both $pr_2^{b,\lambda\circ f}$ and $pr_2^{a,\lambda} \circ_d \Sigma_\lambda f$ fulfill the same pullback-competitor diagram.
Hence they are equal. $\square$

Remark 2.5.7 (Toposes as (dep, Σ)-categories)**.**
As an extension of Lemma 2.4.6, we can, given an object a and fam-arrow (k, e), find the arrows in $\mathrm{Hom}(a, k)$ which correspond to the second projection arrows given by Theorem 2.5.6. We can do this, by simply applying χ.

Recall the work we did on commutative rings (2.1.7,2.2.7,2.4.8,2.4.9). Now we want to actually put these together, to see that not every (dep, Σ)-category is induced by its (fam, Σ)-structure. First we combine the (fam, Σ)-category from Example 2.2.7 and the dep-category from Example 2.4.8 into a (dep, Σ)-category. Afterwards, by using Proposition 2.4.9, seeing that statement will be easy.

Example 2.5.8 (A (dep, Σ)-structure for a commutative ring)**.**
Let R be a commutative ring. Let $\forall r \in R : \; I(r)$ be the ideal generated by r.
Let dR be the dep-category given be Example 2.4.8.
Let ΣR be the (fam, Σ)-category given by Example 2.2.7.
Then, $d\Sigma R$ consisting of

- dR,
- ΣR and
- $\forall (a, b) \in f\mathrm{Hom}(*) : \; pr_2^{*,(a,b)} := \; I(a - b)$

is a $(\mathrm{dep}, \Sigma) - \mathrm{category}$.

Proof.
To see this, we have to show the conditions in Definition 2.5.1.

(fam) Using the constructions from the Examples 2.4.8 and 2.2.7 it becomes immediately clear that (fam) holds.

$(d\Sigma)$ Let $(a, b) \in f\mathrm{Hom}(*)$ and $r \in \mathrm{Hom}(*, *)$.
We have to show

$$I(a - b) \circ r \cdot (1 + r + a + b) = I(a + r - b - r).$$

Using the definition of the application, we get

$$I(a - b) \circ r \cdot (1 + r + a + b) = I(a - b).$$

Then, using basic arithmetic, we get

$$I(a + r - b - r) = I(a - b).$$

Thus the desired equation indeed holds. $\square$

Remark 2.5.9 (Not all (dep, Σ)-categories are given by their (fam, Σ)-structure)**.**
Because of Lemma 2.4.9 we can immediately see, that the (dep, Σ)-category defined above, is not given through Theorem 2.5.6.
This is since Theorem 2.5.6 uses the dep-structure generated by Theorem 2.4.5 and we already saw, that the thusly generated dep-structure is not compatible with the one for Example 2.4.8, which we used above.

This concludes both this subsection and the 2nd section as a whole. We now have all the definitions and tools we need, in order to tackle the goals of the 3rd section. For further reading on the concepts mentioned in this section, see [3].

3 Categories with dep-arrows and 2-fam-arrows

In this section, as already stated in the introduction, we will add arrows between the fam-arrows to each of the structures of the previous section. To that end, we will define 2-fam, (2-fam, Σ), 2-dep- and $(2\text{-}\mathrm{dep}, \Sigma)$-categories. Similarly to last section, we will give examples for each of these definitions. in particular, we will see, that small types form examples for all of these definitions. We will also see analogous versions of Theorems 2.4.5 and 2.5.6 in the Theorems 3.4.6 and 3.5.8. We will then, again use these on the topos examples. Furthermore, we will see that there are $(2\text{-}\mathrm{dep}, \Sigma)$-categories which are not given by their $(2\text{-}\mathrm{fam}, \Sigma)$-structure. For this we will again use the example of commutative rings.

3.1 2-fam-Categories

In this section we will add 2-fam-arrows to fam-categories. However, we do not simply require arbitrary collections for every pair of fam-arrows over a particular object, we will impose additional conditions:

- Firstly, we will require, that the fam-arrows over a particular object and the 2-fam-arrows between them form a category.
- Secondly, we will also require a composition of 2-fam-arrows with regular arrows as well as certain conditions on this composition.

Definition 3.1.1 (2-fam-Category)**.**
2C consisting of

- a fam-category famC
- a collection $\mathrm{Hom}(\lambda, \mu)$ of 2-fam-arrows, $\forall a \in \mathrm{famC}_0 \forall \lambda, \mu \in \mathrm{fHom}(a)$,
- an operation, $\forall a \in \mathrm{famC}_0 \forall \lambda, \mu, \nu \in \mathrm{fHom}(a)$, as shown in the following diagram:

$$\begin{tikzcd}[column sep=huge] a \arrow[r, bend left=60, "\lambda", name=L] \arrow[r, "\mu" description, name=M] \arrow[r, bend right=60, "\nu"', name=N] & \cdot \arrow[from=L, to=M, Rightarrow, "\eta"] \arrow[from=M, to=N, Rightarrow, "\theta"] \arrow[from=L, to=N, Rightarrow, bend right, "\theta\circ\eta"] \end{tikzcd},$$

- a 2-fam-arrow $\mathrm{id}_\lambda \in \mathrm{Hom}(\lambda, \lambda)$, $\forall a \in \mathrm{famC}_0 \forall \lambda \in \mathrm{fHom}(a)$, and
- an operation, $\forall a, b \in \mathrm{famC}_0 \forall \lambda, \mu \in \mathrm{fHom}(a)$, as shown in the diagram

$$\begin{tikzcd}[column sep=huge] b \arrow[r, "f"'] \arrow[rr, bend left=60, "\lambda\circ f", name=LF] \arrow[rr, bend right=60, "\mu\circ f"', name=MF] & a \arrow[r, bend left, "\lambda", name=L] \arrow[r, bend right, "\mu"', name=M] & \cdot \arrow[from=L, to=M, Rightarrow, "\eta"] \arrow[from=LF, to=MF, Rightarrow, bend right, "\eta\circ f"'] \end{tikzcd}.$$

is called a 2-fam-category, iff (fcat) $\forall a \in \mathtt{famC}_0 : \mathtt{CfHom}(a)$ consisting of

- $\mathtt{fHom}(a)$,
- $\forall \lambda, \mu \in \mathtt{fHom}(a): \ \mathtt{Hom}(\lambda, \mu)$,
- $\forall \lambda, \mu, \nu \in \mathtt{fHom}(a)$: the composition $\mathtt{Hom}(\mu, \nu) \times \mathtt{Hom}(\lambda, \mu) \mapsto \mathtt{Hom}(\lambda, \nu)$ and
- $\forall \lambda \in \mathtt{fHom}(a): \ \mathtt{id}_\lambda$

is a category,

$(2\mathtt{fh}_1)$ $\forall a \in \mathtt{famC}_0 \forall \lambda, \mu \in \mathtt{fHom}(a) \forall \eta \in \mathtt{Hom}(\lambda, \mu)$:

$$\eta \circ \mathtt{id}_a = \eta,$$

$(2\mathtt{fh}_2)$ $\forall a, b, c \in \mathtt{famC}_0 \forall \lambda, \mu \in \mathtt{fHom}(a) \forall \eta \in \mathtt{Hom}(\lambda, \mu) \forall f \in \mathtt{Hom}(b, a) \forall h \in \mathtt{Hom}(c, b)$:

$$\eta \circ (f \circ h) = (\eta \circ f) \circ h.$$

The condition $(2\mathtt{fh}_2)$ can be illustrated using the following two diagrams:

$$\begin{tikzcd} c \arrow[r, "h"] & b \arrow[r, "f"] & a \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd} .$$

$$\begin{tikzcd} c \arrow[rr, bend left=60, "\lambda\circ f\circ h"] \arrow[rr, bend right=60, "\mu\circ f\circ h"'] \arrow[r, phantom, "(\eta\circ f)\circ h \ \Downarrow"] & {} \arrow[r, phantom, "\Downarrow \ \eta\circ(f\circ h)"] & {} \end{tikzcd} .$$

(2fv) $\forall a, b \in \mathtt{famC}_0 \forall \lambda, \mu, \nu \in \mathtt{fHom}(a) \forall \eta \in \mathtt{Hom}(\lambda, \mu) \forall \theta \in \mathtt{Hom}(\mu, \nu) \forall f \in \mathtt{Hom}(b, a)$:

$$(\theta \circ \eta) \circ f = (\theta \circ f) \circ (\eta \circ f).$$

The condition (2fv) can be illustrated using the following two diagrams:

$$\begin{tikzcd} b \arrow[r, "f"] & a \arrow[r, bend left=60, "\lambda"] \arrow[r] \arrow[r, bend right=60, "\nu"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\theta"] & {} \end{tikzcd} .$$

$$\begin{tikzcd} b \arrow[rr, bend left=60, "\lambda\circ f"] \arrow[rr, bend right=60, "\nu\circ f"'] \arrow[r, phantom, "(\theta\circ f)\circ(\eta\circ f) \ \Downarrow"] & {} \arrow[r, phantom, "\Downarrow \ (\theta\circ\eta)\circ f"] & {} \end{tikzcd} .$$

We call the collection of all 2-$\mathfrak{fam}$-arrows in C by C_4.
We also define

- $C_0 := \mathfrak{fam}C_0$,
- $C_1 := \mathfrak{fam}C_1$,
- $C_2 := \mathfrak{fam}C_2$,

Given this definition, we will now look a examples again. As before, we will start with a simple example. Similarly to the Examples 2.2.3 and 2.4.2 in the subsections on $(\mathfrak{fam}, \Sigma)$- and dep-categories respectively, we will take an arbitrary $\mathfrak{fam}$-structure and then add to it a trivial 2-$\mathfrak{fam}$-structure:

Example 3.1.2 (A 2-$\mathfrak{fam}$-category with trivial 2-$\mathfrak{fam}$-arrows)**.**
Let M be any monoid. Let $\mathfrak{fam}C$ be a $\mathfrak{fam}$-Category.
Then, $2C$ consisting of

- $\mathfrak{fam}C$
- $\forall a \in \mathfrak{fam}C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) : \mathrm{Hom}(k, p) := M$
- $\forall a \in \mathfrak{fam}C_0 \forall \lambda, \mu, \nu \in \mathrm{fHom}(a) :\ \circ : M \times M \to M$ the operation in M,
- $\forall a \in \mathfrak{fam}C_0 \forall \lambda \in \mathrm{fHom}(a) :\ \mathrm{id}_\lambda$ the neutral element of M and
- $\forall a, b \in \mathfrak{fam}C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) :\ \circ : (\eta, f) \mapsto \eta$

is a 2-$\mathfrak{fam}$-category.

Proof.
We just have to show the conditions in Definition 3.1.1.

(fcat) We show that $\forall a \in \mathfrak{fam}C_0 : C\mathrm{fHom}(a)$ is a category with these definitions.
For this we have to show that the conditions in Definition 1.0.1 hold.
Since the composition comes from a monoid though, this is immediately clear.

$(2\mathrm{fh}_1)$ That the equation for $(2\mathrm{fh}_1)$ holds is clear, immediately by definition, as the composition of 2-$\mathfrak{fam}$-arrows with regular arrows is constant in the second variable.

$(2\mathrm{fh}_2)$ Now let $a, b, c \in \mathfrak{fam}C_0,\ \lambda, \mu \in \mathrm{fHom}(a),\ f \in \mathrm{Hom}(b, a)$ and $h \in \mathrm{Hom}(c, b)$.
Using the definition of the composition, we obtain the following equations:

$$\eta \circ (f \circ h) = \eta$$

and

$$(\eta \circ f) \circ h = \eta \circ h = \eta.$$

Putting them together, we thus get

$$\eta \circ (f \circ h) = (\eta \circ f) \circ h.$$

$(2\mathrm{fv})$ Let $a, b \in \mathfrak{fam}C_0,\ \lambda, \mu \in \mathrm{fHom}(a),\ \nu \in \mathrm{fHom}(a),\ \theta \in \mathrm{Hom}(\mu, \nu)$ and $f \in \mathrm{Hom}(b, a)$.
We have to show

$$(\theta \circ \eta) \circ f = (\theta \circ f) \circ (\eta \circ f)$$

We can simplify both sides of the equation, which yields

$$(\theta \circ \eta) \circ f = \theta \circ \eta$$

and

$$(\theta \circ f) \circ (\eta \circ f) = \theta \circ \eta$$

Thus we get the desired equation, by reflexivity. $\square$

Example 3.1.3 (A 2-fam-category with constant fam-arrows)**.**
Let C be a category.
Let $\mathrm{fam}C$ be the fam-category from Corollary 2.1.3 using C.
Then, 2C consisting of

- $\mathrm{fam}C$
- $\forall a \in \mathrm{fam}C_0 \forall \lambda, \mu \in \mathrm{fHom}(a)$: the collection of regular arrows $\mathrm{Hom}(\lambda, \mu)$,
- $\forall a \in \mathrm{fam}C_0 \forall \lambda, \mu, \nu \in \mathrm{fHom}(a)$: $\circ$ the composition of regular arrows,
- $\forall a \in \mathrm{fam}C_0 \forall \lambda \in \mathrm{fHom}(a)$: id_λ the regular identity and
- $\forall a, b \in \mathrm{fam}C_0 \forall \lambda, \mu \in \mathrm{fHom}(a)$: $\circ : (\eta, f) \mapsto \eta$

is a 2-fam-category.

Proof.
Again, we have to show the conditions (fcat), $(2\mathrm{fh}_1)$, $(2\mathrm{fh}_2)$ and $(2\mathrm{f}\nu)$.

(fcat) First we consider why $\forall a \in \mathrm{fam}C_0 : C\mathrm{fHom}(a)$ is a category.
This is easy though, as equipping $\mathrm{fHom}(a)$ with the above defined arrows and composition makes it equal to C.
Then, since C is a category, so is $C\mathrm{fHom}(a)$.

$(2\mathrm{fh}_1)$ We can see $(2\mathrm{fh}_1)$ similarly to last example, as the composition of 2-fam- and regular arrows is constant in the second variable.

$(2\mathrm{fh}_2)$ For $(2\mathrm{fh}_2)$ we can use, that both sides of the relevant equation simplify to just the 2-fam-arrow, as the composition is constant in the second variable.

$(2\mathrm{f}\nu)$ Let $a, b \in \mathrm{fam}C_0$, $\lambda, \mu, \nu \in \mathrm{fHom}(a)$, $\eta \in \mathrm{Hom}(\lambda, \mu)$, $\theta \in \mathrm{Hom}(\mu, \nu)$, $f \in \mathrm{Hom}(b, a)$.
We have to show

$$(\theta \circ \eta) \circ f = (\theta \circ f) \circ (\eta \circ f)$$

Both sides of the equation simplify in the following ways:

$$(\theta \circ \eta) \circ f = \theta \circ \eta$$

and

$$(\theta \circ f) \circ (\eta \circ f) = \theta \circ \eta$$

Thus we get the desired equation, by reflexivity. $\square$

Proposition 3.1.4 (Small types form a 2-fam-category)**.**
Let U be a type universe.
Let $\mathrm{Type}^f(U)$ be the fam-category from Proposition 2.1.4 using U.
Then, $\mathrm{Type}^2(U)$ consisting of

- $\mathsf{Type}^f(\mathsf{U})$
- $\forall A \in \mathsf{Type}^f(\mathsf{U})_0 \forall \mu, \nu \in \mathsf{fHom}(A) : \mathsf{Hom}(\mu, \nu) := \prod_{x:A} \big(\mu(x) \to \nu(x)\big)$,
- $\forall A \in \mathsf{Type}^f(\mathsf{U})_0 \forall \mu, \nu, \xi \in \mathsf{fHom}(A)$: a function $\circ$ with computation rule
$$(\eta \circ \theta)(x) \equiv \eta(x) \circ \theta(x)$$
- $\forall A \in \mathsf{Type}^f(\mathsf{U})_0 \forall \mu \in \mathsf{fHom}(A) :\ \mathrm{id}_\mu := \lambda x.\lambda y.y : \prod_{x:A}(\mu(x) \to \mu(x))$ and
- $\forall A, B \in \mathsf{Type}^f(\mathsf{U})_0 \forall \mu, \nu \in \mathsf{fHom}(A)$: a dependent function $\circ$ with computation rule
$$(\eta \circ f)(y) := \eta(f(y))$$

is a 2-fam-category.

Proof.
We again want to show that the conditions from the definition hold.

(fcat) We have to show, that $\forall A \in \mathsf{Type}^f(\mathsf{U})_0 : \mathsf{fHom}(a)$ becomes a category, so let $A \in \mathsf{Type}^f(\mathsf{U})_0$. We now have to show the conditions from Definition 1.0.1:

1. Let $\nu, \mu \in \mathsf{fHom}(A)$ and $\eta \in \mathsf{Hom}(\nu, \mu)$. We need to show
$$\mathrm{id}_\mu \circ \eta = \eta = \eta \circ \mathrm{id}_\nu.$$
For this let $x : A$ and $y : \nu(x)$.
Thus
$$(\mathrm{id}_\mu \circ \eta)(x)(y) = \big(\mathrm{id}_\mu(x) \circ \eta(x)\big)(y) = \mathrm{id}_\mu(x)\big(\eta(x)(y)\big) = \eta(x)(y)$$
and
$$\eta(x)(y) = \eta(x)\big(\mathrm{id}_\nu(x)(y)\big) = \big(\eta(x) \circ \mathrm{id}_\nu(x)\big)(y) = (\eta \circ \nu)(x)(y).$$
Then, using function extensionality, we get the desired equations.
2. Let $\nu, \mu, \xi, o \in \mathsf{fHom}(A)$, $\eta \in \mathsf{Hom}(\nu, \mu)$, $\theta \in \mathsf{Hom}(\mu, o)$ and $\iota \in \mathsf{Hom}(o, \xi)$ as well. Now we have to show
$$(\iota \circ \theta) \circ \eta = \iota \circ (\theta \circ \eta).$$
For this let again $x : A$.
Applying x to both sides of the equation yields:
$$((\iota \circ \theta) \circ \eta)(x) = (\iota \circ \theta)(x) \circ \eta(x) = (\iota(x) \circ \theta(x)) \circ \eta(x)$$
and
$$(\iota \circ (\theta \circ \eta))(x) = \iota(x) \circ (\theta \circ \eta)(x) = \iota(x) \circ (\theta(x) \circ \eta(x)).$$
Now we have to show that the rightmost terms are equal.
For this, notice, that these are just compositions of regular functions.
Hence, by associativity:
$$(\iota(x) \circ \theta(x)) \circ \eta(x) = \iota(x) \circ (\theta(x) \circ \eta(x)).$$
Thus ultimately:
$$((\iota \circ \theta) \circ \eta)(x) = (\iota \circ (\theta \circ \eta))(x).$$
Using function extensionality, we then get the desired equation.

Thus $\mathrm{CfHom}(A)$ indeed is a category.

($2fh_1$) Let $A \in \mathrm{Type}^f(U)_0,\ \nu, \mu \in \mathrm{fHom}(A)$ and $\eta \in \mathrm{Hom}(\nu, \mu)$.
We have to show
$$\eta \circ id_A = \eta.$$
For this let $x : A$.
Thus we have
$$(\eta \circ id_A)(x) = \eta(id_A(x)) = \eta(x).$$
Using function extensionality we thus get the desired equality.

($2fh_2$) Let $A, B, D \in \mathrm{Type}^f(U)_0,\ \eta \in \mathrm{Hom}(\nu, \mu),\ f \in \mathrm{Hom}(B, A)$ and $h \in \mathrm{Hom}(D, B)$.
Again, let $x : D$. We get
$$\big(\eta \circ (f \circ h)\big)(x) = \eta\big(f(h(x))\big) = (\eta \circ f)(h(x)) = \big((\eta \circ f) \circ h\big)(x).$$
Using function extensionality we thus get $\eta \circ (f \circ h) = (\eta \circ f) \circ h$.

($2fv$) Let $A, B \in \mathrm{Type}^f(U)_0,\ \nu, \mu, \xi \in \mathrm{fHom}(A),\ \eta \in \mathrm{Hom}(\nu, \mu)$ and $\theta \in \mathrm{Hom}(\mu, \xi)$.
We now have to show
$$(\theta \circ \eta) \circ f = (\theta \circ f) \circ (\eta \circ f)$$
To do so, let $x : B$. We thus get
$$\big((\theta \circ \eta) \circ f\big)(x) = (\theta \circ \eta)(f(x)) = \theta(f(x)) \circ \eta(f(x))$$
$$= (\theta \circ f)(x) \circ (\eta \circ f)(x) = \big((\theta \circ f) \circ (\eta \circ f)\big)(x)$$
Using function extensionality, we thus get the desired equation. □

Lemma 3.1.5 (Toposes as 2-fam-categories)**.**
Let C be a topos. Let $\mathrm{fam}C$ be the fam-category from Lemma 2.1.6 using C.
Then, $2C$ consisting of

- $\mathrm{fam}C$
- $\forall a \in \mathrm{fam}C_0 \forall (k,t), (p,e) \in \mathrm{fHom}(a)$:
$$\mathrm{Hom}((k,t),(p,e)) := \{f \in \mathrm{Hom}(k,p) | e \circ id_a \times f = t\},$$
- $\forall a \in \mathrm{fam}C_0 \forall (k,e), (p,t), (u,v) \in \mathrm{fHom}(a) : \circ$ the regular composition,
- $\forall a \in \mathrm{fam}C_0 \forall (k,e) : id_{(k,e)} :\ id_k \in \mathrm{Hom}(k,k)$ the regular identity and
- $\forall a, b \in \mathrm{fam}C_0 \forall (k,e), (p,t) \in \mathrm{fHom}(a) : \circ : (h,f) \mapsto h$

is a 2-fam-category.

Proof.
Notice, that the above defined identity 2-fam-arrows actually do fulfill the condition on 2-fam-arrows above.
Now we have to show the conditions from Definition 3.1.1.

(fcat) Since we are just using the regular composition as composition between the 2-fam-arrows, using the regular identity, it is clear that $\forall \mathrm{fam}C_0 : \mathrm{CfHom}(a)$ a category.

($2fh_1$) Here, like in previous examples, the composition is constant in the second variable, so ($2fh_1$) holds immediately by definition.

($2fh_2$) Again, the constantness of the composition in the second variable makes ($2fh_2$) obviously hold. We have seen arguments to that end throughout this document.

($2fv$) Same story as in the previous cases, due to the composition being constant in the second variable, both sides of the equation immediately simplify to the same expression. □

Example 3.1.6 (A 2-$\mathtt{fam}$-structure for a commutative ring)**.**
Let R be a commutative ring.
Let fR be the $\mathtt{fam}$-category from Example 2.1.7.
Then, $2R$ consisting of

- $\mathtt{famC}$,
- $\forall (a,b),(c,d) \in \mathtt{fHom}(*) : \ \mathtt{Hom}((a,b),(c,d)) := \{\eta \in \mathtt{Hom}(R,R) | \eta(a-b) = c-d\}$,
- $\forall (a,b),(c,d),(e,f) \in \mathtt{fHom}(*) : \ (\eta,\theta) \mapsto \eta \circ \theta$ the regular function composition,
- $\forall (a,b) \in \mathtt{fHom}(*) : \ \mathtt{id}_{(a,b)} := \mathtt{id}_R$ and
- $\forall (a,b),(c,d) \in \mathtt{fHom}(*) : \ \circ_v : (\eta, f) \mapsto \eta$

is a 2-$\mathtt{fam}$-category.

Proof.
To show this, we have to observe the conditions in the definition.

($\mathtt{fcat}$) To see that we have a category $\mathtt{CfHom}(*)$ as in Definition 3.1.1, we have to check the conditions 1. and 2. in Definition 1.0.1.

1. This is clear, as function composition respects the identity $\mathtt{id}_R$.
2. This is also clear, as function composition is associative.

Hence $\mathtt{CfHom}(*)$ is a category.

($2fh_1$) Let $(a,b),(c,d) \in \mathtt{fHom}(*)$ and $\eta \in \mathtt{Hom}((a,b),(c,d))$.
We have to show
$$\eta \circ_v 0 = \eta.$$
This holds by definition of $\circ_v$.

($2fh_2$) Let $(a,b),(c,d) \in \mathtt{fHom}(*), \ \eta \in \mathtt{Hom}((a,b),(c,d))$ and $r,f \in \mathtt{Hom}(*,*)$.
We have to show
$$\eta \circ_v (r \circ f) = (\eta \circ_v r) \circ_v f.$$
But by definition, both sides of the equation reduce to η, making them equal.

($2fv$) Let $(a,b),(c,d),(e,f) \in \mathtt{fHom}(*), \ \eta \in \mathtt{Hom}((a,b),(c,d)), \ \theta \in \mathtt{Hom}((c,d),(e,f))$ and $r \in \mathtt{Hom}(*,*)$. We have to show
$$(\theta \circ \eta) \circ_v r = (\theta \circ_v r) \circ (\eta \circ_v r).$$
Again, applying the definition of $\circ_v$, both sides of the equation reduce to $(\theta \circ \eta)$.
Hence they are equal. □

3.2 $(2\text{-}\mathrm{fam}, \Sigma)$-Categories

$(2\text{-}\mathrm{fam}, \Sigma)$-categories combine the structures of a 2-fam- and a (fam, Σ)-category, much like (dep, Σ)-categories did for dep- and (fam, Σ)-categories. Also just like then, it allows us to add a little bit of additional structure, that can only be defined if you have both. For (dep, Σ)-categories, this was the second projection arrow, here it will be a second kind of Σ-arrow, which is induced by a 2-fam-arrow. These will interact with the first projection arrows in a similar way to the interaction between the first kind of Σ-arrow and second projection arrows. The two kinds of Σ-arrows will also be required to commute with each other in a certain way. The exact details for this, we can see in the definition:

Definition 3.2.1 ($(2\text{-}\mathrm{fam}, \Sigma)$-Category)**.**
$2\Sigma C$ consisting of

- a 2-fam-Category $2C$,
- a (fam, Σ)-Category ΣC, and
- an arrow $\Sigma_{\lambda,\mu}\eta$, $\forall a \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(\lambda, \mu)$

is called a $(2\text{-}\mathrm{fam}, \Sigma)$-category, iff

(fam) $2C$ and ΣC have the same underlying fam-structure, and

(diag) $\forall a, b \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b, a) \forall \eta \in \mathrm{Hom}(\lambda, \mu)$: the diagram

$$\begin{tikzcd}
\sum_b \lambda \circ f \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\Sigma_{\lambda\circ f, \mu\circ f}\eta\circ f"] \arrow[dd, bend right=60, "\mathrm{pr}_1^{b,\lambda\circ f}"'] & \sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"'] \arrow[dd, bend left=60, "\mathrm{pr}_1^{a,\lambda}"] \\
\sum_b \mu \circ f \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{b,\mu\circ f}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"'] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

is commutative,

(VS_1) $\forall a \in 2C_0 \forall \lambda \in \mathrm{fHom}(a)$:

$$\Sigma_{\lambda,\lambda}\mathrm{id}_\lambda = \mathrm{id}_{\Sigma_a \lambda},$$

and

(VS_2) $\forall a \in 2C_0 \forall \lambda, \mu, \nu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(\lambda, \mu) \forall \theta \in \mathrm{Hom}(\mu, \nu)$:

$$\Sigma_{\mu,\nu}\theta \circ \Sigma_{\lambda,\mu}\eta = \Sigma_{\lambda,\nu}(\theta \circ \eta).$$

We also define

- $2\Sigma C_0 := \Sigma C_0 = 2C_0$,
- $2\Sigma C_1 := \Sigma C_1 = 2C_1$,
- $2\Sigma C_2 := \Sigma C_2 = 2C_2$,

- $2\Sigma C_4 := 2C_4$.

Remark 3.2.2 (Pullbacks in $(\mathtt{diag})$)**.**
Notice that in $(\mathtt{diag})$, the upper square is a pullback. We can see this by the pullback lemma, as both the lower and outer square are pullbacks by the definition of $(\mathtt{fam}, \Sigma)$-categories.

Here we again consider a trivial example first. Similarly to Example 2.5.3, we would like to take a general 2-$\mathtt{fam}$- and a general $(\mathtt{fam}, \Sigma)$-structure and then add trivial Σ-arrows of the second kind to those. However, similarly to Example 2.5.3, this is not trivially possible. This is because, a priori, we do not have arrows between arbitrary objects. So instead, we will use a general 2-$\mathtt{fam}$-structure and the trivial $(\mathtt{fam}, \Sigma)$-category from Example 2.2.3.

Example 3.2.3 (A $(2\text{-}\mathtt{fam}, \Sigma)$-category with trivial Σ-objects)**.**
Let $2C$ be a 2-$\mathtt{fam}$-category. Let ΣC be the $(\mathtt{fam}, \Sigma)$-category from Example 2.2.3 using the $\mathtt{fam}$-structure of $2C$. Then, $2\Sigma C$ consisting of

- $2C$,
- ΣC and
- $\forall a \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(k, p) :\ \Sigma_{\lambda,\mu}\eta := \mathrm{id}_a$

is a $(2\text{-}\mathtt{fam}, \Sigma)$-category.

Proof.
So what we have to do is show the 4 conditions in Definition 3.2.1.

$(\mathtt{fam})$ That the 2-$\mathtt{fam}$- and $(\mathtt{fam}, \Sigma)$-categories have the same $\mathtt{fam}$-structure is immediately clear, by the constructions in Example 2.2.3.

$(\mathtt{diag})$ To show this let $a, b \in 2C_0,\ \lambda, \mu \in \mathrm{fHom}(a),\ f \in \mathrm{Hom}(b, a)$ and $\eta \in \mathrm{Hom}(\lambda, \mu)$. We have to show, that the diagram

$$\begin{tikzcd}
\sum_b \lambda \circ f \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\Sigma_{\lambda\circ f,\mu\circ f}\eta"] \arrow[dd, bend right, "pr_1^{b,\lambda\circ f}"'] & \sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"'] \arrow[dd, bend left, "pr_1^{a,\lambda}"] \\
\sum_b \mu \circ f \arrow[r, "\Sigma_\mu f"] \arrow[d, "pr_1^{b,\mu\circ f}"] & \sum_a \mu \arrow[d, "pr_1^{a,\mu}"'] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

is commutative.
Substituting with the above assignments, we have to show, that the diagram

$$\begin{tikzcd}
b \arrow[r, "f"] \arrow[d, "\mathrm{id}_b"] \arrow[dd, bend right, "\mathrm{id}_b"'] & a \arrow[d, "\mathrm{id}_a"'] \arrow[dd, bend left, "\mathrm{id}_a"] \\
b \arrow[r, "f"] \arrow[d, "\mathrm{id}_b"] & a \arrow[d, "\mathrm{id}_a"] \\
b \arrow[r, "f"] & a
\end{tikzcd}$$

is commutative.
But this is clearly the case.

(VS_1) follows immediately from the definition of the Σ-arrows of the second kind, as the equation for (VS_1) is exactly the defining equation for the relevant Σ-arrows.

(VS_2) The equation for (VS_2) is also immediately clear, as all of the Σ-arrows of the second kind are identities. $\square$

Example 3.2.4 (A $(2\text{-}\mathfrak{fam}, \Sigma)$-category with constant $\mathfrak{fam}$-arrows)**.**
Let C be a category with binary products.
Let 2C be the 2-$\mathfrak{fam}$-category form Example 3.1.3 using C.
Let ΣC be the $(\mathfrak{fam}, \Sigma)$-category given by Example 2.2.4 and C.
Then, 2ΣC consisting of

- 2C,
- ΣC and
- $\forall a \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(\lambda, \mu): \ \Sigma_{\lambda,\mu}\eta := \mathrm{id}_a \times \eta$

Proof.
We simply have to see that the conditions in Definition 3.2.1 hold.

$(\mathfrak{fam})$ That the categories form the Examples 3.1.3 and 2.2.4 have the same $\mathfrak{fam}$-structure is clear from their construction.

(diag) Let $a, b \in 2C_0, \ \lambda, \mu \in \mathrm{fHom}(a), \ f \in \mathrm{Hom}(b, a)$ and $\eta \in \mathrm{Hom}(\lambda, \mu)$.
Applying the definitions above, we have to show that the diagram:

$$\begin{tikzcd}
b \times \lambda \arrow[r, "f \times \mathrm{id}_\lambda"] \arrow[d, "\mathrm{id}_b \times \eta"] \arrow[dd, bend right=60, "\mathrm{pr}_1^{b,\lambda}"'] & a \times \lambda \arrow[d, "\mathrm{id}_a \times \eta"'] \arrow[dd, bend left=60, "\mathrm{pr}_1^{a,\lambda}"] \\
b \times \mu \arrow[r, "f \times \mathrm{id}_\mu"] \arrow[d, "\mathrm{pr}_1^{b,\mu}"] & a \times \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"'] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

is commutative.
From Example 2.2.4 we already know, that the outer and lower square are commutative, since they are pullbacks. Hence we only need to consider the upper square, the left triangle and the right triangle.

upper square
For this consider that

$$f \times \mathrm{id}_\mu \circ \mathrm{id}_b \times \eta = (f \circ \mathrm{id}_b) \times (\mathrm{id}_\mu \circ \eta) = f \times \eta$$

and

$$\mathrm{id}_a \times \eta \circ f \times \mathrm{id}_\lambda = (\mathrm{id}_a \circ f) \times (\eta \circ \mathrm{id}_\lambda) = f \times \eta.$$

With these, we can immediately see the commutativity of the upper square.

left triangle

As a property of product arrows, we know that

$$\mathrm{pr}_1^{b,\mu} \circ \mathrm{id}_b \times \eta = \mathrm{id}_b \circ \mathrm{p}_1^{b,\lambda}.$$

With this, the commutativity of the triangle immediately follows.

right triangle

For the right triangle we can argue analogously to the case of the left triangle.

Hence the diagram is commutative.

(VS_1) Let $a \in 2C_0$ and $\lambda \in \mathrm{fHom}(a)$. Here we have to show

$$\mathrm{id}_a \times \mathrm{id}_\lambda = \mathrm{id}_{a\times\lambda},$$

which of course holds.

(VS_2) Let $a \in 2C_0,\ \lambda, \mu, \nu \in \mathrm{fHom}(a),\ \eta \in \mathrm{Hom}(\lambda, \mu)$ and $\theta \in \mathrm{Hom}(\mu, \nu)$.
We have to show

$$\mathrm{id}_a \times \theta \circ \mathrm{id}_a \times \eta = \mathrm{id}_a \times (\theta \circ \eta).$$

This again from the basic properties of product arrows. □

Proposition 3.2.5 (Small types form a $(2\text{-}\mathrm{fam}, \Sigma)$-category).
Let U be a type universe. Let $\mathrm{Type}^2(U)$ be the 2-fam-category given by Proposition 3.1.4 and U. Let $\mathrm{Type}^\Sigma(U)$ be the (fam, Σ)-category given by Proposition 2.2.5 and U. Then, $\mathrm{Type}^{2\Sigma}(U)$ consisting of

- $\mathrm{Type}^2(U)$,
- $\mathrm{Type}^\Sigma(U)$ and
- $\forall A \in \mathrm{Type}^2(U)_0 \forall \nu, \mu \in \mathrm{fHom}(A) \forall \eta \in \mathrm{Hom}(\nu, \mu)$:

$$\Sigma_{\lambda,\mu}\eta := \lambda x.\lambda y.(x, \eta(x)(y)) : \sum_a \nu \to \sum_a \mu$$

is a $(2\text{-}\mathrm{fam}, \Sigma)$-category.

Proof.
We again just have to show the conditions (fam), (diag), (VS_1) and (VS_2) in Definition 3.2.1.

(fam) This is clear from the construction of the 2-fam- and (fam, Σ)-category in the Propositions 3.1.4 and 3.2.5 respectively.

(diag) Let $A, B \in \mathrm{Type}^2(U)_0,\ \nu, \mu \in \mathrm{fHom}(A),\ f \in \mathrm{Hom}(b, a)$ and $\eta \in \mathrm{Hom}(\nu, \mu)$.
We have to show that the diagram

$$\begin{tikzcd}
\sum_b \lambda \circ f \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\Sigma_{\lambda\circ f, \mu\circ f}\eta"] \arrow[dd, bend right=60, "\mathrm{pr}_1^{b,\lambda\circ f}"'] & \sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"'] \arrow[dd, bend left=60, "\mathrm{pr}_1^{a,\lambda}"] \\
\sum_b \mu \circ f \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{b,\mu\circ f}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"'] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

is commutative.
Since $\mathrm{Type}^{\Sigma}(\mathrm{U})$ is a (fam, Σ)-category, by (pull) in Definition 2.2.1, we already know, that the outer square and the inner lower square in the diagram are commutative.
So it suffices to demonstrate that the inner upper square and the left and right triangles are commutative, then the entire diagram also is.
All of this can easily be shown using function extensionality.

(VS_1) This can easily be shown with function extensionality, similarly to the proof for (s_1) in Proposition 2.2.5.

(VS_2) This can also be shown easily with function extensionality, similarly to the proof for (s_2) in Proposition 2.2.5. □

Lemma 3.2.6 (Toposes as $(2\text{-}\mathrm{fam}, \Sigma)$-categories)**.**
Let C be a topos. Let 2C be the 2-fam-category from Lemma 3.1.5 using C.
Let ΣC be the $(\mathrm{fam}\Sigma)$-category given by Lemma 2.2.6 and C.
Then, $2\Sigma C$ consisting of

- 2C,
- ΣC and
- $\forall a \in C_0 \forall (k, e), (p, t) \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(k, p) : \Sigma_{(k,e),(p,t)}\eta$ the arrow given by the universal pullback property:

$$\begin{tikzcd}
\sum_a(k,e) \arrow[rr] \arrow[dr, dashed, "\Sigma_{(k,e),(p,t)}\eta"] \arrow[ddr, bend right] & & a \times k \arrow[d, "\mathrm{id}_a \times \eta"] \\
& \sum_a(p,t) \arrow[r] \arrow[d] & a \times p \arrow[d, "t"] \\
& 1 \arrow[r, "\top"] & \Omega
\end{tikzcd}$$

is a $(2\text{-}\mathrm{fam}, \Sigma)$-category.

Proof.
Again, we just have to show the 4 conditions in the definition.

(fam) This is clear from the definition of the 2-fam- and (fam, Σ)-category.

(diag) Let $a, b \in 2C_0$, $(k, e), (p, t) \in \mathrm{fHom}(a)$, $f \in \mathrm{Hom}(b, a)$ and $\eta \in \mathrm{Hom}((k, e), (p, t))$. we have to show that the diagram

$$\begin{tikzcd}
\sum_b(k,e)\circ f \arrow[r, "\Sigma_{(k,e)}f"] \arrow[d, "\Sigma_{(k,e)\circ f,(p,t)\circ f}\eta"] \arrow[dd, bend right=60, "\mathrm{pr}_1^{b,(k,e)\circ f}"'] & \sum_a(k,e) \arrow[d, "\Sigma_{(k,e),(p,t)}\eta"'] \arrow[dd, bend left=60, "\mathrm{pr}_1^{a,(k,e)}"] \\
\sum_b(p,t)\circ f \arrow[r, "\Sigma_{(p,t)}f"] \arrow[d, "\mathrm{pr}_1^{b,(p,t)\circ f}"] & \sum_a(p,t) \arrow[d, "\mathrm{pr}_1^{a,(p,t)}"'] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

is commutative.
Since ΣC is a (fam, Σ)-category, by (pull) in Definition 2.2.1, we already know, that the outer square and the inner lower square in the diagram are commutative. So it suffices to demonstrate that the inner upper square and the left and right triangle are commutative, then the entire diagram also is.
First, we show the commutativity of the inner upper square.
For this, let first be $\forall c \in C_0 \forall (d,h) \in \mathrm{fHom}(c) : p_{c,(d,h)} \in \mathrm{Hom}\big(\sum_c(d,h), c \times d\big)$ the morphism that comes with the pullback:

$$
\begin{tikzcd}
\sum_c(d,h) \arrow[r, "p_{c,(d,h)}"] \arrow[d] & c \times d \arrow[d, "h"] \\
1 \arrow[r, "\top"] & \Omega
\end{tikzcd}
$$

Now we know since 1 is a terminal object, that there is $1_\Omega \in \mathrm{Hom}(\Omega, 1)$ and that $1_\Omega \circ \top = \mathrm{id}_1$. Hence $\top$ is a monomorphism.
This since we are working with a pullback, we also find that $p_{c,(d,h)}$ is a monomorphism.
Now, consider the following diagram:

$$
\begin{tikzcd}
\sum_b(k,e) \circ f \arrow[rrr, "\Sigma_{(k,e)} f"] \arrow[dr, "p_{b,(k,e)\circ f}"] \arrow[ddd, "\Sigma_{(k,e)\circ f,(p,t)\circ f}\eta"'] & & & \sum_a(k,e) \arrow[dl, "p_{a,(k,e)}"'] \arrow[ddd, "\Sigma_{(k,e),(p,t)}\eta"] \\
& b \times k \arrow[r, "f \times \mathrm{id}_k"] \arrow[d, "\mathrm{id}_b \times \eta"] & a \times k \arrow[d, "\mathrm{id}_a \times \eta"] & \\
& b \times p \arrow[r, "f \times \mathrm{id}_p"] & a \times p & \\
\sum_b(p,t) \circ f \arrow[ur, "p_{b,(p,t)\circ f}"'] \arrow[rrr, "\Sigma_{(p,t)} f"] & & & \sum_a(p,t) \arrow[ul, "p_{a,(p,t)}"]
\end{tikzcd}
$$

Using the defining diagrams for the Σ-arrows, it is easy to verify, that the upper, left, lower and right square in this diagram are all commutative. It is also clear, that the centre square is commutative, using basic properties of the product arrows.
Using these commutativities, we get

$$p_{a,(p,t)} \circ \Sigma_{(k,e),(p,t)}\eta \circ \Sigma_{(k,e)} f = p_{a,(p,t)} \circ \Sigma_{(p,t)} f \circ \Sigma_{(k,e)\circ f,(p,t)\circ f}\eta.$$

Since $p_{a,(p,t)}$ is a monomorphism, this gives us

$$\Sigma_{(k,e),(p,t)}\eta \circ \Sigma_{(k,e)} f = \Sigma_{(p,t)} f \circ \Sigma_{(k,e)\circ f,(p,t)\circ f}\eta,$$

which is what we wanted to show.

Now we still have to show commutativity for the triagles.
For this, consider the diagram:

$$
\begin{tikzcd}
\sum_a(k,e) \arrow[r] \arrow[d, "\Sigma_{(k,e),(p,t)}\eta"] & a \times k \arrow[d, "\mathrm{id}_a \times \eta"] \arrow[dr] & \\
& & a \\
\sum_a(p,t) \arrow[r] & a \times p \arrow[ur] &
\end{tikzcd}
$$

By the defining diagram of $\Sigma_{(k,e),(p,t)}\eta$ we know that the square in this diagram is commutative. By the properties of the product, we know that the triangle in this diagram is commutative. Thus the entire diagram is commutative.
With this follows the commutativity of the right triangle in the above diagram.
For the left triangle the proof is analogous.

(VS_1) Let $a \in 2C_0$ and $(k, e) \in \mathrm{fHom}(a)$. Consider the pullback-competitor diagram defining $\Sigma_{(k,e),(k,e)}\mathrm{id}_{(k,e)}$.

$$\begin{tikzcd}
\sum_a(k,e) \arrow[rr] \arrow[dr, phantom, ""] \arrow[ddr, bend right] & & a \times k \arrow[dr, "\mathrm{id}_a \times \mathrm{id}_{(k,e)}"] & \\
& \sum_a(k,e) \arrow[rr] \arrow[d] & & a \times k \arrow[d, "t"] \\
& 1 \arrow[rr, "\top"] & & \Omega
\end{tikzcd}$$

Clearly the identity $\mathrm{id}_{\sum_a(k,e)}$ also fulfills the universal pullback property here. Thus (VS_1) holds.

(VS_2) Let $a \in 2C_0$, $\lambda, \mu, \nu \in \mathrm{fHom}(a)$, $\eta \in \mathrm{Hom}(\lambda, \mu)$ and $\theta \in \mathrm{Hom}(\mu, \nu)$.
We have to show that

$$\Sigma_{(p,t),(q,r)}\theta \circ \Sigma_{(k,e),(p,t)}\eta = \Sigma_{(k,e),(q,r)}(\theta \circ \eta). \tag{2}$$

To do so, we will use the universal pullback property. First consider $\Sigma_{(p,t),(q,r)}\theta \circ \Sigma_{(k,e),(p,t)}\eta$.
The arrow $\Sigma_{(p,t),(q,r)}\theta$ is defined by

$$\begin{tikzcd}
\sum_a(p,t) \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(p,t),(q,r)}\theta"] \arrow[ddr, bend right] & & a \times p \arrow[dr, "\mathrm{id}_a \times \theta"] & \\
& \sum_a(q,r) \arrow[rr] \arrow[d] & & a \times q \arrow[d, "r"] \\
& 1 \arrow[rr, "\top"'] & & \Omega
\end{tikzcd}$$

and $\Sigma_{(k,e),(p,t)}\eta$ is defined by

$$\begin{tikzcd}
\sum_a(k,e) \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(k,e),(p,t)}\eta"] \arrow[ddr, bend right] & & a \times k \arrow[dr, "\mathrm{id}_a \times \eta"] & \\
& \sum_a(p,t) \arrow[rr] \arrow[d] & & a \times p \arrow[d, "t"] \\
& 1 \arrow[rr, "\top"'] & & \Omega
\end{tikzcd}.$$

Putting these diagrams together, we get the following diagram, which we call $(*)$:

$$\begin{tikzcd}
\sum_a(k,e) \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(k,e),(p,t)}\eta"] \arrow[dddrr, bend right] & & a \times k \arrow[dr, "\mathrm{id}_a \times \eta"] & & \\
& \sum_a(p,t) \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(p,t),(q,r)}\theta"] \arrow[ddr, bend right] & & a \times p \arrow[dr, "\mathrm{id}_a \times \theta"] \arrow[ddr, bend right, "t"'] & \\
& & \sum_a(q,r) \arrow[rr] \arrow[d] & & a \times q \arrow[d, "r"] \\
& & 1 \arrow[rr, "\top"'] & & \Omega
\end{tikzcd} .$$

Now consider the other side of Equation 2, $\Sigma_{(k,e),(q,r)}(\theta \circ \eta)$ is defined by

$$\begin{tikzcd}
\sum_a(k,e) \arrow[rr] \arrow[dr, dashed, red, "\Sigma_{(k,e),(q,r)}(\theta\circ\eta)"] \arrow[ddr, bend right] & & a \times k \arrow[dr, "\mathrm{id}_a \times (\theta\circ\eta)"] & \\
& \sum_a(q,r) \arrow[rr] \arrow[d] & & a \times q \arrow[d, "r"] \\
& 1 \arrow[rr, "\top"'] & & \Omega
\end{tikzcd} .$$

We can see, that (except for the red arrow) this diagram is a subdiagram of $(*)$. Thus, since we are working with a pullback, whatever in $(*)$ has the place of the red arrow, is equal to the red arrow. Thus the equation follows. □

Example 3.2.7 (A $(2\text{-}\mathsf{fam}, \Sigma)$-structure for a commutative ring)**.**
Let R be a commutative ring. Let $2R$ be the 2-fam-category from Example 3.1.6 using R. Let ΣR be the (fam, Σ)-category from Example 2.2.7 using R. Then, $2\Sigma R$ consisting of

- $2R$,
- ΣR and
- $\forall (a,b),(c,d) \in \mathrm{fHom}(*) \forall \eta \in \mathrm{Hom}((a,b),(c,d)) : \ \Sigma_{(a,b),(c,d)}\eta := ab - cd$

is a $(2\text{-}\mathsf{fam}, \Sigma)$-category.

Proof.
We have to show the conditions (fam), (diag), (VS_1) and (VS_2).

(fam) To see that the fam-structures of $2R$ and ΣR are the same, simply consider the respective definitions, both use the fam-category from Example 2.1.7.

(diag) Let $(a,b),(c,d) \in \mathrm{fHom}(*)$, $\eta \in \mathrm{Hom}((a,b),(c,d))$ and $r \in \mathrm{Hom}(*,*)$.

We have to show that the diagram

$$\begin{tikzcd}
* \arrow[r, "r(1+r+a+b)"] \arrow[d, "(a+r)(b+r)-(c+r)(d+r)"] \arrow[dd, bend right=60, "(a+r)(b+r)"'] & * \arrow[d, "ab-cd"] \arrow[dd, bend left=60, "ab"] \\
* \arrow[r, "r(1+r+c+d)"'] \arrow[d, "(c+r)(d+r)"] & * \arrow[d, "cd"] \\
* \arrow[r, "r"'] & *
\end{tikzcd}$$

is commutative.
Since we already know that the lower square is a pullback, due to the Σ-structure, we only need to show that the right triangle, left triangle and upper square are commutative. For the triangles this is easy.
But we still have to consider the upper square.
We have to show, that

$$r(1+r+c+d)+(a+r)(b+r)-(c+r)(d+r) = ab-cd+r(1+r+a+b).$$

For this we can just simplify both sides of the equation, this gives us:

$$\begin{aligned}
&r(1+r+c+d)+(a+r)(b+r)-(c+r)(d+r)\\
&= r+r^2+rc+rd+ab+ar+br+r^2-cd-cr-dr-r^2\\
&= r+ab+ar+br+r^2-cd
\end{aligned}$$

and

$$ab-cd+r(1+r+a+b) = ab-cd+r+r^2+ar+br.$$

Then, using commutativity of addition, we can see that the rightmost terms in these two equations are equal. Hence the square and thus the entire diagram is commutative.

(VS_1) Let $(a,b) \in \mathrm{fHom}(*)$. We have to show

$$\Sigma_{(a,b),(a,b)}\mathrm{id}_{(a,b)} = 0.$$

Applying the definition for $\Sigma_{(a,b),(a,b)}\mathrm{id}_{(a,b)}$, we have to show

$$ab - ab = 0,$$

which clearly holds.

(VS_2) Let $(a,b),(c,d),(e,f) \in \mathrm{fHom}(*)$, $\eta \in \mathrm{Hom}((a,b),(c,d))$ and $\theta \in \mathrm{Hom}((c,d),(e,f))$. We have to show that

$$cd - ef + ab - cd = ab - ef.$$

Using the commutativity of addition in R, this is clear. $\square$

3.3 2-fam- and $(2\text{-}\mathsf{fam}, \Sigma)$-Functors

Now we will take a brief look at the 2-versions of fam- and (fam, Σ)-functors, 2-fam- and $(2\text{-}\mathsf{fam}, \Sigma)$-functors. We will start by giving a definition for 2-fam-functors:

Definition 3.3.1 (2-fam-Functor)**.**
Given 2-fam-categories $2C$ and $2D$,
$2F$ consisting of

- a fam-functor $\mathsf{famF} : C \to D$, and
- an assignment $2F_3^{a,\lambda,\mu} : \mathrm{Hom}(\lambda, \mu) \to \mathrm{Hom}(\mathsf{famF}(\lambda), \mathsf{famF}(\mu))$, $\forall a \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a)$,

is called a 2-fam-functor, iff

1. $\forall a \in 2C_0 : (\mathsf{famF}_2^a, (2F_3^{a,\lambda,\mu})_{\lambda,\mu \in \mathrm{fHom}(a)})$ is a functor and
2. $\forall a, b \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall f \in \mathrm{Hom}(b, a) \forall \eta \in \mathrm{Hom}(\lambda, \mu) :$
$$2F_3^{b,\lambda\circ f,\mu\circ f}(\eta \circ f) = 2F_3^{a,\lambda,\mu}(\eta) \circ \mathsf{famF}(f)$$

Remark 3.3.2 (Typing in Definition 3.3.1)**.**
The equation 2. in the definition is well-typed.
To see this, consider the types of the 2 sides:

- $2F(\eta \circ f) \in \mathrm{Hom}(2F(\lambda \circ f), 2F(\mu \circ f))$, and
- $2F(\eta) \circ 2F(f) \in \mathrm{Hom}(2F(\lambda) \circ 2F(f), 2F(\mu) \circ 2F(f))$.

And of course, due to the axioms of fam-functors, we know

$$2F(\mu \circ f) = 2F(\mu) \circ 2F(f),$$

and

$$2F(\lambda \circ f) = 2F(\lambda) \circ 2F(f).$$

Hence we can see that 2. is well-typed.

Example 3.3.3 (A fam-functor from $\mathrm{Type}(U)$ to $\mathrm{Type}(U)$)**.**
Let U be a type universe, $A : U$ and $a : A$.
Let $\mathrm{Type}^2(U)$ be the 2-fam-category given by Proposition 3.1.4 and U.
Let $\mathrm{Hom}^{f,a}(A, -)$ be the fam-functor given by Example 2.3.2 and U, A and a.
Then, $\mathrm{Hom}^{2,a}(A, -)$ consisting of

- $\mathrm{Hom}^{f,a}(A, -)$ and
- $\forall B \in 2C_0 \forall \mu, \nu \in \mathrm{fHom}(B) : \ 2F_3^{B,\mu,\nu} := \lambda\eta.\lambda f.\lambda y.\big(\eta(f(a))(y)\big)$

is a 2-fam-functor from $\mathrm{Type}^2(U)$ to $\mathrm{Type}^2(U)$.

Proof.
We have to show the conditions form Definition 3.3.1.

1. Let $B : U$. We have to show that $(\mathsf{famF}_2^B, \lambda\mu.\lambda\nu.2F_3^{B,\mu,\nu})$ is a functor.
For this we can simply show the conditions in Definition 1.0.4.

(a) Let $\mu \in \mathsf{fHom}(B)$.
We have to show
$$2F_3^{B,\mu,\mu}(\mathrm{id}_\mu) = \mathrm{id}_{\mathsf{fam}F_2^B(\mu)}.$$
To do so, we first substitute with the definition and simplify the right sides of the equation:
$$\lambda f.\lambda y.\big(\mathrm{id}_\mu(f(a))(y)\big) = \lambda f.\lambda y.y.$$
By definition we know that
$$\mathrm{id}_{\mathsf{fam}F_2^B(\mu)} = \lambda f.\lambda y.y.$$

(b) Let $\mu, \nu, \iota \in \mathsf{fHom}(B)$, $\eta \in \mathsf{Hom}(\mu, \nu)$ and $\theta \in \mathsf{Hom}(\nu, \iota)$.
We have to show
$$2F_3^{B,\mu,\iota}(\theta \circ \eta) = 2F_3^{B,\nu,\iota}(\theta) \circ 2F_3^{B,\mu,\nu}(\eta).$$
For this, we just simplify both sides of the equation, which yields:
$$2F_3^{B,\mu,\iota}(\theta \circ \eta) = \lambda f.\lambda y.\big((\theta \circ \eta)(f(a))(y)\big) = \lambda f.\lambda y.\Big(\big(\theta(f(a)) \circ \eta(f(a))\big)(y)\Big)$$
$$= \lambda f.\lambda y.\Big(\theta(f(a))\big(\eta(f(a))(y)\big)\Big)$$
and
$$2F_3^{B,\nu,\iota}(\theta) \circ 2F_3^{B,\mu,\nu}(\eta) = \big(\lambda f.\lambda y.\big(\theta(f(a))(y)\big)\big) \circ \big(\lambda f.\lambda y.\big(\eta(f(a))(y)\big)\big)$$
$$= \lambda x.\Big(\big(\lambda f.\lambda y.\big(\theta(f(a))(y)\big)\big)(x) \circ \big(\lambda f.\lambda y.\big(\eta(f(a))(y)\big)\big)(x)\Big)$$
$$= \lambda x.\Big(\big(\lambda y.\big(\theta(x(a))(y)\big)\big) \circ \big(\lambda y.\big(\eta(x(a))(y)\big)\big)\Big)$$
$$= \lambda x.\lambda z.\Big(\big(\lambda y.\big(\theta(x(a))(y)\big)\big)\big(\big(\lambda y.\big(\eta(x(a))(y)\big)\big)(z)\big)\Big)$$
$$= \lambda x.\lambda z.\Big(\big(\lambda y.\big(\theta(x(a))(y)\big)\big)\big(\eta(x(a))(z)\big)\Big)$$
$$= \lambda x.\lambda z.\Big(\theta(x(a))\big(\eta(x(a))(z)\big)\Big).$$

2. Let $B, C : \mathcal{U}$, $\mu, \nu \in \mathsf{fHom}(B)$, $f \in \mathsf{Hom}(C, B)$ and $\eta \in \mathsf{Hom}(\mu, \nu)$.
We have to show that
$$2F_3^{C,\mu\circ f,\nu\circ f}(\eta \circ f) = 2F_3^{B,\mu,\nu}(\eta) \circ \mathsf{Hom}^{f,a}(A, f).$$
For this, we again just simplify both sides and get:
$$2F_3^{C,\mu\circ f,\nu\circ f}(\eta \circ f) = \lambda h.\lambda y.\big(\eta(f(h(a)))(y)\big)$$
and
$$2F_3^{B,\mu,\nu}(\eta) \circ \mathsf{Hom}^{f,a}(A, f) = \big(\lambda h.\lambda y.\big(\eta(h(a))(y)\big)\big) \circ \lambda h.(f \circ h)$$
$$= \lambda x.\Big(\big(\lambda h.\lambda y.\big(\eta(h(a))(y)\big)\big)\big(\lambda h.(f \circ h)(x)\big)\Big)$$
$$= \lambda x.\Big(\big(\lambda h.\lambda y.\big(\eta(h(a))(y)\big)\big)(f \circ x)\Big)$$
$$= \lambda x.\lambda y.\big(\eta((f \circ x)(a))(y)\big)$$
$$= \lambda x.\lambda y.\big(\eta(f(x(a)))(y)\big). \quad \square$$

Example 3.3.4 (A 2-$\mathfrak{fam}$-functor induced by a ring homomorphism)**.**
Let R, Q be a commutative ring and $f : R \to Q$ a ring isomorphism.
Let $2R$ and $2Q$ be the 2-$\mathfrak{fam}$-categories given by R and Q via Example 3.1.6 respectively.
Let $\mathfrak{fam}F$ be the $\mathfrak{fam}$-functor given by Example 2.3.3 and R, Q and f.
Then, $2F$ consisting of

- $\mathfrak{fam}F$ and
- $\forall (a,b),(d,c) \in \mathrm{fHom}(*) : 2F_3^{(a,b),(c,d)} : \eta \mapsto f \circ \eta \circ f^{-1}$

is a 2-$\mathfrak{fam}$-functor from $2R$ to $2Q$.

Proof.
First, we have to convince ourselves, that the assignment $2F_3^{(a,b),(c,d)}$ is actually of the kind demanded by Definition 3.3.1. For this, let $(a,b),(d,c) \in \mathrm{fHom}(*)$ and $\eta \in \mathrm{Hom}((a,b),(c,d))$. We have to show that

$$f \circ \eta \circ f^{-1} \in \mathrm{Hom}((f(a),f(b)),(f(c),f(d))).$$

To show this, we need to show that

$$f \circ \eta \circ f^{-1} : Q \to Q$$

and

$$(f \circ \eta \circ f^{-1})(f(a) - f(b)) = f(c) - f(d).$$

The first of these can be immediately seen by considering the typing of η and f.
For the second, we can just calculate:

$$(f \circ \eta \circ f^{-1})(f(a) - f(b)) = f(\eta(f^{-1}(f(a) - f(b)))) = f(\eta(a-b)) = f(c-d) = f(c) - f(d).$$

What remains now is to show the conditions from Definition 3.3.1.

1. We have to show that $(f, (2F_3^{(a,b),(c,d)})_{(a,b),(d,c) \in \mathrm{fHom}(*)})$ is a functor, hence that it fulfills the conditions from Definition 1.0.4.

   1. Let $(a,b) \in \mathrm{fHom}(*)$. We have to show that

      $$2F_3^{(a,b),(a,b)}(\mathrm{id}_{(a,b)}) = \mathrm{id}_{(f(a),f(b))}.$$

      We know:

      $$2F_3^{(a,b),(a,b)}(\mathrm{id}_{(a,b)}) = f \circ \mathrm{id}_{(a,b)} \circ f^{-1}.$$

      We also know that

      $$\mathrm{id}_{(a,b)} = \mathrm{id}_R$$

      and

      $$\mathrm{id}_{(f(a),f(b))} = \mathrm{id}_Q.$$

      Hence

      $$f \circ \mathrm{id}_{(a,b)} \circ f^{-1} = f \circ f^{-1} = \mathrm{id}_Q = \mathrm{id}_{(f(a),f(b))}.$$

   2. Let $(a,b),(c,d),(m,n) \in \mathrm{fHom}(*)$, $\eta \in \mathrm{Hom}((a,b),(c,d))$ and $\theta \in \mathrm{Hom}((c,d),(m,n))$.
      We have to show that

      $$2F_3^{(a,b),(m,n)}(\theta \circ \eta) = 2F_3^{(c,d),(m,n)}(\theta) \circ 2F_3^{(a,b),(c,d)}(\eta).$$

      By simply applying the definition, we can see that

      $$2F_3^{(a,b),(m,n)}(\theta \circ \eta) = f \circ \theta \circ \eta \circ f^{-1} = f \circ \theta \circ f \circ f^{-1} \eta \circ f^{-1} = 2F_3^{(c,d),(m,n)}(\theta) \circ 2F_3^{(a,b),(c,d)}(\eta).$$

2. Let $(a, b), (c, d) \in \mathrm{fHom}(*)$, $\eta \in \mathrm{Hom}((a, b), (c, d))$ and $r \in \mathrm{Hom}(*, *)$.
   We have to show
   $$2F_3^{(a+r,b+r),(c+r,d+r)}(\eta \circ_v r) = 2F_3^{(a,b),(c,d)}(\eta) \circ_v f(r).$$
   Using that $\circ_v$ is constant in the second variable, this equation simplifies to:
   $$2F_3^{(a+r,b+r),(c+r,d+r)}(\eta) = 2F_3^{(a,b),(c,d)}(\eta).$$
   We furthermore, have that
   $$2F_3^{(a+r,b+r),(c+r,d+r)}(\eta) = f \circ \eta \circ f^{-1}$$
   and
   $$2F_3^{(a,b),(c,d)}(\eta) = f \circ \eta \circ f^{-1}. \quad \square$$

Next we will have a look at $(2\text{-}\mathrm{fam}, \Sigma)$-functors. Similar to how it was for (fam, Σ)-functors, these will just be 2-fam-functors with two additional conditions.

Definition 3.3.5 ($(2\text{-}\mathrm{fam}, \Sigma)$-Functor)**.**
Let $2\Sigma C$ and $2\Sigma D$ be $(2\text{-}\mathrm{fam}, \Sigma)$-categories, with underlying 2-fam- and (fam, Σ)-categories $2C, \Sigma C$ and $2D, \Sigma D$ respectively. Then, a 2-fam-functor
$$2F = (\mathrm{fam}F, 2F_3) : 2C \longrightarrow 2D$$
is called a $(2\text{-}\mathrm{fam}, \Sigma)$-functor $2\Sigma C \longrightarrow 2\Sigma D$, iff

1. $\mathrm{fam}F$ is a (fam, Σ)-functor $\Sigma C \longrightarrow \Sigma D$ and
2. $\forall a \in 2\Sigma C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(\lambda, \mu) : \mathrm{fam}F(\Sigma_{\lambda,\mu}\eta) = \Sigma_{\mathrm{fam}F(\lambda),\mathrm{fam}F(\mu)} 2F_3^{a,\lambda,\mu}(\eta)$.

Example 3.3.6 (A (fam, Σ)-functor induced by a ring homomorphism)**.**
Let R, Q be a commutative ring and $f : R \longrightarrow Q$ a ring isomorphism.
Let $2F$ be the 2-fam-functor given by Example 3.3.4 and R, Q and f.
Let $2\Sigma R$ and $2\Sigma Q$ be the $(2\text{-}\mathrm{fam}, \Sigma)$-categories given by Example 3.2.7.
$2F = (\mathrm{fam}F, 2F_3)$ is a $(2\text{-}\mathrm{fam}, \Sigma)$-functor from $2\Sigma R$ to $2\Sigma Q$.

Proof.
We just have to show that the conditions in Definition 3.3.5 are fulfilled.

1. That $\mathrm{fam}F$ is a (fam, Σ)-functor, we know from Example 2.3.5.
2. Let $(a, b), (c, d) \in \mathrm{dHom}(*)$ and $\eta \in \mathrm{Hom}((a, b), (c, d))$.
   We have to show that
   $$\mathrm{fam}F(\Sigma_{(a,b),(c,d)}\eta) = \Sigma_{(f(a),f(b)),(f(c),f(d))} 2F_3^{(a,b),(c,d)}(\eta).$$
   For this, first consider that we know
   $$\mathrm{fam}F(\Sigma_{(a,b),(c,d)}\eta) = f(ab - cd)$$
   and
   $$\Sigma_{(f(a),f(b)),(f(c),f(d))} 2F_3^{(a,b),(c,d)}(\eta) = f(a)f(b) - f(c)f(d).$$
   Using then that f is a ring isomorphism, we can see that the rightmost sides of these two equations are equal. $\square$

3.4 2-dep-Categories

In this subsection, we combine the notions of 2-fam- and dep-categories into 2-dep-categories. As with (dep, Σ)- and $(2\text{-}\mathsf{fam}, \Sigma)$-categories, this will allow us to add a bit of structure that we couldn't before. This structure is going to be a composition of 2-fam-arrows with dep-arrows. This composition will be required to satisfy the obvious conditions on interactions with identities and other compositions:

Definition 3.4.1 (2-dep-Category)**.**
$2dC$ consisting of

- a 2-fam-category $2C$,
- a dep-category dC, and
- an operation, $\forall a \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a)$, as shown in the diagram:

$$\begin{tikzcd} a \arrow[r, bend left=25, "\lambda"] \arrow[r, bend right=25, "\mu"'] \arrow[r, Rightarrow, bend left=70, "\phi"] \arrow[r, Rightarrow, bend right=70, "\eta\circ\phi"'] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \end{tikzcd}$$

is called a 2-dep-category iff,

(fam) $2C$ and dC have the same underlying fam-category.

$(2dv_1)$ $\forall a \in 2C_0 \forall \lambda \in \mathrm{fHom}(a) \forall \phi \in \mathrm{dHom}(a, \lambda):$

$$\mathrm{id}_\lambda \circ \phi = \phi$$

$(2dv_2)$ $\forall a \in 2C_0 \forall \lambda, \mu, \nu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(\lambda, \mu) \forall \theta \in \mathrm{Hom}(\mu, \nu) \forall \phi \in \mathrm{dHom}(a, \lambda):$

$$(\theta \circ \eta) \circ \phi = \theta \circ (\eta \circ \phi),$$

$$\begin{tikzcd} a \arrow[r, bend left=50, "\lambda"] \arrow[r, "\mu"] \arrow[r, bend right=50, "\nu"'] \arrow[r, Rightarrow, bend left=60, "\phi"] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\theta"] & \cdot \end{tikzcd},$$

$$\begin{tikzcd} a \arrow[r, "\lambda"] \arrow[d, "\mu"'] \arrow[dr, "\nu"] \arrow[r, Rightarrow, bend left=50, "\phi"] \arrow[d, Rightarrow, bend right=50, "\eta\circ\phi"'] & \cdot \\ \cdot & \cdot \end{tikzcd}$$

and

(2dh) $\forall a, b \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(\lambda, \mu) \forall \phi \in \mathrm{dHom}(a, \lambda) \forall f \in \mathrm{Hom}(b, a)$:

$$(\eta \circ \phi) \circ f = (\eta \circ f) \circ (\phi \circ f),$$

$$\begin{tikzcd} b \arrow[r, "f"] & a \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] \arrow[r, Rightarrow, bend left=70, "\phi"] & \cdot \end{tikzcd}.$$

$$\begin{tikzcd} b \arrow[rr, bend left, "\lambda\circ f"] \arrow[rr, bend right, "\mu\circ f"'] \arrow[rr, phantom, "\Downarrow\eta\circ f"] \arrow[rr, Rightarrow, bend left=70, "\phi(f)"] \arrow[rr, Rightarrow, bend right=70, "(\eta\circ\phi)\circ f"'] & & \cdot \end{tikzcd}.$$

We also define

- $C_0 := dC_0 = 2C_0$,
- $C_1 := dC_1 = 2C_1$,
- $C_2 := dC_2 = 2C_2$,
- $C_3 := dC_3$ and
- $C_4 := 2C_4$.

Now, just like previously, we look at a simple example. Also just like for the Example 2.5.3 and 3.2.3, one of the component structures is going to be trivial, the other general.

Example 3.4.2 (A 2-dep-category with trivial dep-arrows)**.**
Let $2C$ be a 2-fam-category. Let A be any collection. Let dC be the dep-category given by Example 2.4.2 and the fam-structure of $2C$. Then, $2dC$ consisting of

- $2C$,
- dC and
- $\forall a \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) : \circ := ((n, \phi) \mapsto \phi)$

is a 2-dep-category.

Proof.
We have to show that the conditions (fam), ($2dv_1$), ($2dv_1$) and (2dh) from Definition 3.4.1 hold:

(fam) That $2C$ and dC have the same fam-structure is immediately clear, since we explicitly use the fam-structure of $2C$ in the construction of dC.

($2dv_1$) Since the composition is constant in the first variable, identities are respected by definition.

($2dv_2$) Because of the constantness of the composition in the first variable, we can immediately see that both sides of the relevant equation for ($2dv_2$) simplify to the same dep-arrow.

($2dh$) Just like for ($2dv_2$), using the respective constantnesses of the operations we immediately see that both sides of the equation for ($2dh$) simplify to the same dep-arrow.
A relevant difference to the previous case, is that we need not just the constantness for the composition of 2-fam-arrows with dep-arrows, but also the constantness of the application of dep-arrows. □

Example 3.4.3 (A 2-dep-category with constant fam-arrows).
Let C be a category. Let $2C$ be the 2-fam-category from Example 3.1.3 using C. Let dC be the dep-category given by Example 2.4.3 and C. Then, $2dC$ consisting of

- $2C$,
- dC and
- $\forall a \in 2C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) :\ \circ$ the regular composition of arrows

is a 2-dep-category.

Proof.
We have to show that the conditions (fam), ($2dv_1$), ($2dv_1$) and ($2dh$) hold.

(fam) That $2C$ and dC have the same fam-structure is clear by how they where defined in Example 3.1.3 and 2.4.3 respectively.

($2dv_1$) Since regular composition of arrows respects identities, ($2dv_1$) holds.

($2dv_2$) Since regular composition of arrows is associative, ($2dv_2$) also holds.

($2dh$) Here, let $a, b \in 2C_0$, $\lambda, \mu \in \mathrm{fHom}(a)$, $\eta \in \mathrm{Hom}(\lambda, \mu)$, $\phi \in \mathrm{dHom}(a, \lambda)$ and $f \in \mathrm{Hom}(b, a)$.
We have to show
$$(\eta \circ \phi) \circ f = (\eta \circ f) \circ (\phi \circ f).$$
We know, from Example 3.1.3, that
$$\eta \circ f = \eta.$$
Thus we have to show
$$(\eta \circ \phi) \circ f = \eta \circ (\phi \circ f),$$
which immediately follows from the associativity of regular composition of arrows. □

Proposition 3.4.4 (Small types form a 2-dep-category).
Let U be a type universe. Let $\mathrm{Type}^2(U)$ be the 2-fam-category from Proposition 3.1.4 using U. Let $\mathrm{Type}^d(U)$ be the dep-category from Proposition 2.4.4 using U.
Then, $\mathrm{Type}^{2d}(U)$ consisting of

- $\mathrm{Type}^2(U)$,
- $\mathrm{Type}^d(U)$ and
- $\forall A \in \mathrm{Type}^2(U)_0 \forall \nu, \mu \in \mathrm{fHom}(A) :$
$$\circ := \lambda\eta.\lambda\phi.\lambda x.(\eta(x)(\phi(x))) : \Big(\prod_{x:A} \nu(x) \to \mu(x)\Big) \to \Big(\prod_{x:A} \nu(x)\Big) \to \prod_{x:A} \mu(x).$$

is a 2-dep-category.

Proof.
We again just have to show the 4 conditions in Definition 3.4.1.

(fam) By considering the definitions of $\mathrm{Type}^2(U)$ and $\mathrm{Type}^d(U)$ in Example 3.1.4 and 2.4.4 respectively, we can immediately see, that both have the fam-structure given by Proposition 2.1.4 and U.

($2\mathrm{dv}_1$) For this let $A \in \mathrm{Type}^2(U)_0,\ \lambda \in \mathrm{fHom}(A)$ and $\phi \in \mathrm{dHom}(A, \lambda)$. We have to show

$$\mathrm{id}_\lambda \circ \phi = \phi.$$

We can reduce this to pointwise equality, using function extensionality.
Thus let $x : A$. We have

$$(\mathrm{id}_\lambda \circ \phi)(x) = \mathrm{id}_\lambda(x)(\phi(x)) = \phi(x).$$

Hence ($2\mathrm{dv}_1$) holds.

($2\mathrm{dv}_2$) Let $A \in \mathrm{Type}^2(U)_0,\ \lambda, \mu, \nu \in \mathrm{fHom}(A),\ \phi \in \mathrm{dHom}(A, \lambda),\ \eta \in \mathrm{Hom}(\lambda, \mu)$ and $\theta \in \mathrm{Hom}(\mu, \nu)$. We have to show, that

$$(\theta \circ \eta) \circ \phi = \theta \circ (\eta \circ \phi).$$

Again we can use function extensionality.
Thus let $x : A$. We get

$$\big((\theta\circ\eta)\circ\phi\big)(x) = (\theta\circ\eta)(x)(\phi(x)) = \theta(x)\big(\eta(x)(\phi(x))\big) = \theta(x)((\eta\circ\phi)(x)) = (\theta\circ(\eta\circ\phi))(x),$$

with which we can see ($2\mathrm{dv}_2$).

($2\mathrm{dh}$) Let $A, B \in \mathrm{Type}^2(U)_0,\ \mu, \nu \in \mathrm{fHom}(A),\ \phi \in \mathrm{dHom}(A, \nu),\ \eta \in \mathrm{Hom}(\nu, \mu)$ and $f \in \mathrm{Hom}(B, A)$. We have to show, that

$$(\eta \circ \phi) \circ f = (\eta \circ f) \circ (\phi \circ f).$$

And yet again, we use function extensionality.
Thus let $x : A$. We get

$$((\eta\circ\phi)\circ f)(x) = (\eta\circ\phi)(f(x)) = \eta(f(x))\big(\phi(f(x))\big) = (\eta\circ f)(x)((\phi\circ f)(x)) = \big((\eta\circ f)\circ(\phi\circ f)\big)(x). \quad \square$$

Example 3.4.5 (A 2-dep-structure for a commutative Ring)**.**
Let R be a commutative ring. Let $\forall r \subseteq R :\ I(r)$ be the ideal generated by r.
Let $2R$ be the 2-fam-category from Example 3.1.6 using R.
Let dR be the dep-category from Example 2.4.8 using R.
Then, $2dR$ consisting of

- $2R$,
- dR and
- $\forall (a, b), (c, d) \in \mathrm{fHom}(*) :\ (\eta, J) \mapsto I(\eta(J))$

is a 2-dep-category.

Proof.
First, notice that the composition is actually well-defined, as

$$\forall a \in R \forall f \in \mathrm{Hom}(R,R) \forall J \in \mathrm{Ideal}(R) : \big(a \in J \to f(a) \in I(f(J))\big).$$

What we now have to do is the same as in the previous examples. We simply have to show the conditions (fam), $(2dv_1)$, $(2dv_2)$ and $(2dh)$.

(fam) It it clear from the respective definitions, that $2R$ and dR have the same fam-structure.

$(2dv_1)$ Let $(a,b) \in \mathrm{fHom}(*)$ and $J \in \mathrm{dHom}(*, (a,b))$.
We have to show $\mathrm{id}_R \circ J = J$.
By considering the definition of the composition above, we can see:

$$\mathrm{id}_R \circ J = I(\mathrm{id}_R(J)) = I(J) = J.$$

Hence we see that $(2dv_1)$ holds.

$(2dv_2)$ Let $(a,b), (c,d), (e,f) \in \mathrm{fHom}(*)$, $\eta \in \mathrm{Hom}((a,b),(c,d))$, $\theta \in \mathrm{Hom}((c,d),(e,f))$ and $J \in \mathrm{dHom}(*, (a,b))$.
We have to show

$$(\theta \circ \eta) \circ J = \theta \circ (\eta \circ J). \tag{3}$$

Using the definition of the compositions, both we get

$$(\theta \circ \eta) \circ J = I\big((\theta \circ \eta)(J)\big) = I\big(\theta(\eta(J))\big)$$

and

$$\theta \circ (\eta \circ J) = \theta \circ I(\eta(J)) = I\big(\theta(I(\eta(J)))\big).$$

It is immediately clear, that $I\big(\theta(\eta(J))\big) \subseteq I\big(\theta(I(\eta(J)))\big)$, so we just have to show

$$I\big(\theta(I(\eta(J)))\big) \subseteq I\big(\theta(\eta(J))\big).$$

So let $r \in I\big(\theta(I(\eta(J)))\big)$.
Hence let $n \in \mathbb{N}$, $a_1, ..., a_n \in R$ and $r_1, ..., r_n \in \theta\big(I(\eta(J))\big)$, such that

$$r = \sum_{i=1}^{n} a_i * r_i.$$

Next then, let for every $i \in \{1, ..., n\}$ be $n_i \in \mathbb{N}$, $a_{i,1}, ..., a_{i,n_i}$ and $r_{i,1}, ..., r_{i,n_i}$, such that

$$r_i = \theta(\sum_{p=1}^{n_i} a_{i,p} r_{i,p}).$$

Since θ is a ring homomorphism, thus

$$r_i = \sum_{p=1}^{n_i} \theta(a_{i,p})\theta(r_{i,p}).$$

Thus

$$r = \sum_{i=1}^{n} \sum_{p=1}^{n_i} a_i \theta(a_{i,p})\theta(r_{i,p}).$$

Thus, $r \in I\big(\theta(\eta(J))\big)$.
Hence $I\big(\theta(I(\eta(J)))\big) \subseteq I\big(\theta(\eta(J))\big)$, and Equation 3 follows.

(2dh) Let $(a, b), (c, d) \in \mathtt{fHom}(*)$, $\eta \in \mathtt{Hom}((a, b), (c, d))$, $J \in \mathtt{dHom}(*, (a, b))$ and $r \in \mathtt{Hom}(*, *)$.
We have to show $(\eta \circ J) \circ r = (\eta \circ r) \circ (J \circ r)$.
But both sides of the equation simplify to $\eta \circ J$, hence they are equal. □

Now we get to the first major theorem of this section. We will now show, that the structure of a $(2\text{-}\mathtt{fam}, \Sigma)$-category is sufficient to construct a 2-dep-category with the same 2-fam-structure.

Theorem 3.4.6 (Every $(2\text{-}\mathtt{fam}, \Sigma)$-category induces a 2-dep-category)**.**
Let $2\Sigma C$ be a $(2\text{-}\mathtt{fam}, \Sigma)$-category, let $2C$ be its underlying 2-fam-category and let ΣC be its underlying $(\mathtt{fam}, \Sigma)$-category. Let dC be the dep-category given by Theorem 2.4.5 and ΣC.
Then, $2dC$ consisting of

- $2C$,
- dC and
- $\forall a \in 2C_0 \forall \lambda, \mu \in \mathtt{fHom}(a)$:
$$\circ_2 : (\eta, \phi) \mapsto \Sigma_{\lambda,\mu}\eta \circ \phi$$

is a 2-dep-category.

Proof.
Before we get to the conditions, we have to convince ourselves, that the the composition has correct type. For this, let $a \in 2C_0$, $\lambda, \mu \in \mathtt{fHom}(a)$, $\eta \in \mathtt{Hom}(\lambda, \mu)$ and $\phi \in \mathtt{dHom}(a, \lambda)$.
We need to show, that
$$\eta \circ_2 \phi \in \mathtt{dHom}(a, \mu).$$
For this, we first need to make sense of the term $\Sigma_{\lambda,\mu}\eta \circ \phi$, which is the definition of $\eta \circ_2 \phi$.
Recall, that
$$\mathtt{dHom}(a, \lambda) := \{x \in \mathtt{Hom}\Big(a, \sum_a \lambda\Big) | \mathtt{pr}_1^{a,\lambda} \circ x = \mathtt{id}_a\}.$$
Thus $\phi \in \mathtt{Hom}(a, \sum_a \lambda)$.
Next, recall form Definition 3.2.1, that
$$\Sigma_{\lambda,\mu}\eta \in \mathtt{Hom}\Big(\sum_a \lambda, \sum_a \mu\Big).$$
Thus $\Sigma_{\lambda,\mu}\eta \circ \phi \in \mathtt{Hom}\big(a, \sum_a \mu\big)$.

To see that $\Sigma_{\lambda,\mu}\eta \circ \phi \in \mathtt{dHom}(a, \mu)$, we must now show, that $\mathtt{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta \circ \phi = \mathtt{id}_a$.
For this, recall $(\mathtt{diag})$ from Definition 3.2.1, and consider only the right triangle.
This yields us a commutative diagram as follows:

$$\begin{tikzcd}
\sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"'] \arrow[dd, bend left=60, "\mathtt{pr}_1^{a,\lambda}"] \\
\sum_a \mu \arrow[d, "\mathtt{pr}_1^{a,\mu}"'] \\
a
\end{tikzcd}$$

Thus, we get

$$\mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta \circ \phi = \mathrm{pr}_1^{a,\lambda} \circ \phi = \mathrm{id}_a.$$

Hence we know

$$\eta \circ_2 \phi = \Sigma_{\lambda,\mu}\eta \circ \phi \in \mathrm{dHom}(a, \mu).$$

Now, what remains to be seen are the conditions (fam), $(2\mathrm{dv}_1)$, $(2\mathrm{dv}_2)$ and $(2\mathrm{dh})$ from Definition 3.4.1.

(fam) The fam-structure of $2\Sigma C$ is the same as the fam-structure of ΣC, by Definition 3.2.1. By the Theorem 2.4.5, the fam-structure of ΣC is the same as the fam-structure of $\mathrm{d}C$. Thus $2C$ and $\mathrm{d}C$ have the same underlying fam-category.

$(2\mathrm{dv}_1)$ Let $a \in 2C_0$, $\lambda \in \mathrm{fHom}(a)$ and $\phi \in \mathrm{dHom}(a, \lambda)$. We have to show, that

$$\mathrm{id}_\lambda \circ_2 \phi = \phi.$$

We know by definition of $\circ_2$,

$$\mathrm{id}_\lambda \circ_2 \phi = \Sigma_{\lambda,\lambda}\mathrm{id}_\lambda \circ \phi.$$

By Definition 3.2.1 (VS_1), we know

$$\Sigma_{\lambda,\lambda}\mathrm{id}_\lambda = \mathrm{id}_{\sum_a \lambda}.$$

Thus

$$\Sigma_{\lambda,\lambda}\mathrm{id}_\lambda \circ \phi = \phi.$$

$(2\mathrm{dv}_2)$ Let $a \in 2C_0$, $\lambda, \mu, \nu \in \mathrm{fHom}(a)$, $\eta \in \mathrm{Hom}(\lambda, \mu)$, $\theta \in \mathrm{Hom}(\mu, \nu)$ and $\phi \in \mathrm{dHom}(a, \lambda)$. We have to show, that

$$(\theta \circ \eta) \circ_2 \phi = \theta \circ_2 (\eta \circ_2 \phi).$$

We know by definition of $\circ_2$,

$$(\theta \circ \eta) \circ_2 \phi = \Sigma_{\lambda,\nu}(\theta \circ \eta) \circ \phi.$$

Using (VS_2) from Definition 3.2.1, we get

$$\Sigma_{\lambda,\nu}(\theta \circ \eta) \circ \phi = \Sigma_{\mu,\nu}\theta \circ \Sigma_{\lambda,\mu}\eta \circ \phi$$

By the definition of $\circ_2$ again, we then get

$$\Sigma_{\mu,\nu}\theta \circ \Sigma_{\lambda,\mu}\eta \circ \phi = \Sigma_{\mu,\nu}\theta \circ (\eta \circ_2 \phi) = \theta \circ_2 (\eta \circ_2 \phi).$$

$(2\mathrm{dh})$ Let $a, b \in 2C_0$, $\lambda, \mu \in \mathrm{fHom}(a)$, $\eta \in \mathrm{Hom}(\lambda, \mu)$, $\phi \in \mathrm{dHom}(a, \lambda)$ and $f \in \mathrm{Hom}(b, a)$. We have to show

$$(\eta \circ_2 \phi) \circ_d f = (\eta \circ f) \circ_2 (\phi \circ_d f).$$

By definition of $\circ_2$,

$$(\eta \circ_2 \phi) \circ_d f = (\Sigma_{\lambda,\mu}\eta \circ \phi) \circ_d f.$$

Now recall, the definition of the application of dependent arrows from Theorem 2.4.5. We know, that $(\Sigma_{\lambda,\mu}\eta \circ \phi) \circ_d f$ is the arrow fulfilling the following pullback-competitor diagram:

$$\begin{tikzcd}
b \arrow[dr, dashed, red, "(\Sigma_{\lambda,\mu}\eta\circ\phi)\circ_d f"] \arrow[drr, bend left, "\Sigma_{\lambda,\mu}\eta\circ\phi\circ f"] \arrow[ddr, bend right, "\mathrm{id}_b"'] & & \\
& \sum_b(\mu\circ f) \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{b,\mu\circ f}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"] \\
& b \arrow[r, "f"] & a
\end{tikzcd}$$

To show the equality, we will now simply show that $(\eta \circ f) \circ_2 (\phi \circ_d f)$ also fulfills the diagram.
For this, first note that

$$(\eta \circ f) \circ_2 (\phi \circ_d f) = \Sigma_{\lambda\circ f,\mu\circ f}(\eta \circ f) \circ (\phi \circ_d f).$$

$\phi \circ_d f$ is defined by the universal pullback property:

$$\begin{tikzcd}
b \arrow[dr, dashed, red, "\phi\circ_d f"] \arrow[drr, bend left, "\phi\circ f"] \arrow[ddr, bend right, "\mathrm{id}_b"'] & & \\
& \sum_b(\lambda\circ f) \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{b,\lambda\circ f}"] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& b \arrow[r, "f"] & a
\end{tikzcd}$$

Putting this together with $(\mathtt{diag})$ from Definition 3.2.1, we get that the diagram

$$\begin{tikzcd}
b \arrow[dr, dashed, red, "\phi\circ_d f"] \arrow[drr, bend left, "\phi\circ f"] \arrow[dddr, bend right=60, "\mathrm{id}_b"'] & & \\
& \sum_b(\lambda\circ f) \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\Sigma_{\lambda\circ f,\mu\circ f}(\eta\circ f)"] \arrow[dd, bend right=50, "\mathrm{pr}_1^{b,\lambda\circ f}"'] & \sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"'] \arrow[dd, bend left=50, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_b(\mu\circ f) \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{b,\mu\circ f}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"'] \\
& b \arrow[r, "f"] & a
\end{tikzcd}$$

is commutative.

Then, taking the following subdiagram

$$\begin{tikzcd}
b \arrow[dr, dashed, red, "\phi \circ_d f"] \arrow[drr, bend left, "\phi \circ f"] \arrow[ddd, bend right=60, "\mathrm{id}_b"'] & & \\
& \sum_b(\lambda \circ f) \arrow[d, "\Sigma_{\lambda \circ f, \mu \circ f}(\eta \circ f)"] & \sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"'] \\
& \sum_b(\mu \circ f) \arrow[r, "\Sigma_\mu f"] \arrow[d, "\mathrm{pr}_1^{b,\mu \circ f}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"'] \\
& b \arrow[r, "f"] & a
\end{tikzcd}$$

We see immediately, that $\Sigma_{\lambda \circ f, \mu \circ f}(\eta \circ f) \circ (\phi \circ_d f)$ fulfilles the same pullback-competitor diagram as $(\Sigma_{\lambda,\mu}\eta \circ \phi) \circ_d f$.
Hence they are equal. ∎

With this theorem, we have now achieved one of our central aims. What is left now, is to define $(2\text{-dep}, \Sigma)$-categories and show the analogous theorem for them. We also still want to talk a bit about toposes, which is what we do in this next lemma. Here we will have a look at the structure Theorem 3.4.6 generates on a topos and develop the notions from Lemma 2.4.6.

Lemma 3.4.7 (Toposes as 2-dep-categories)**.**
Let $2\Sigma C$ be the $(2\text{-fam}, \Sigma)$-category from Example 3.2.6.
$2\Sigma C$ induces a 2-dep-category $2dC$ by Theorem 3.4.6.
Let $a \in 2\Sigma C_0$ and $(k, e) \in \mathrm{fHom}(a)$.
Take the bijective assignments χ and δ from Lemma 2.4.6.
Then $\forall (p,t) \in \mathrm{fHom}(a) \forall \phi \in \mathrm{dHom}(a, (k,e)) \forall \eta \in \mathrm{Hom}((k,e),(p,t))$:

$$\chi(\eta \circ_2 \phi) = \eta \circ \chi(\phi).$$

Proof.
Let $(p,t) \in \mathrm{fHom}(a)$, $\phi \in \mathrm{dHom}(a, (k,e))$, $\eta \in \mathrm{Hom}((k,e),(p,t))$.
By definition,

$$\eta \circ_2 \phi = \Sigma_{(k,e),(p,t)}\eta \circ \phi.$$

Using the defining diagram for $\Sigma_{(k,e),(p,t)}\eta$, we thus get the following diagram, which is commutative:

$$\begin{tikzcd}
a \arrow[dr, "\phi"] & & & & \\
& \sum_a(k,e) \arrow[rr, "p_{(k,e)}"] \arrow[dr, "\Sigma_{(k,e),(p,t)}\eta"'] & & a \times k \arrow[dr, "\mathrm{id}_a \times \eta"] & \\
& & \sum_a(p,t) \arrow[rr, "p_{(p,t)}"] \arrow[d] & & a \times p \arrow[d, "e"] \\
& & 1 \arrow[rr, "\top"] & & \Omega
\end{tikzcd},$$

By the properties of the product, we also have the commutative diagram:

$$\begin{tikzcd}
a \times k \arrow[r, "\mathrm{pr}_2^{a,k}"] \arrow[d, "\mathrm{id}_a \times \eta"] & k \arrow[d, "\eta"] \\
a \times p \arrow[r, "\mathrm{pr}_2^{a,p}"] & p
\end{tikzcd}.$$

Putting these diagrams together, yields the following commutative diagram:

$$\begin{tikzcd}
a \arrow[dr, "\phi"] & & & & & & \\
& \sum_a(k,e) \arrow[rr, "p_{(k,e)}"] \arrow[dr, "\Sigma_{(k,e),(p,t)}\eta"'] & & a \times k \arrow[rr, "\mathrm{pr}_2^{a,k}"] \arrow[dr, "\mathrm{id}_a \times \eta"] & & k \arrow[dr, "\eta"] & \\
& & \sum_a(p,t) \arrow[rr, "p_{(p,t)}"] \arrow[d] & & a \times p \arrow[rr, "\mathrm{pr}_2^{a,p}"] \arrow[d, "e"] & & p \\
& & 1 \arrow[rr, "\top"] & & \Omega & &
\end{tikzcd}.$$

Now, remember, we have to show, that

$$\chi(\eta \circ_2 \phi) = \eta \circ \chi(\phi).$$

Using the definition of χ, this equation turns into

$$\mathrm{pr}_2^{a,p} \circ p_{(p,t)} \circ \Sigma_{(k,e),(p,t)}\eta \circ \phi = \eta \circ \mathrm{pr}_2^{a,p} \circ p_{(p,t)} \circ \phi,$$

Which then clearly follows from the diagram. □

Remark 3.4.8 (δ also commutes with composition)**.**
Note that, since δ and χ are inverse to each other, as an easy consequence of Lemma 3.4.7, we also get the analogous statement for δ.

Remark 3.4.9 (A 2-dep-structure on a commutative ring not given by Theorem 3.4.6)**.**
Using Proposition 2.4.9 it is easy to see that the 2-dep-category from Example 3.4.5 cannot be given, by the $(2\text{-}\mathtt{fam}, \Sigma)$-category from Example 3.2.7 through Theorem 3.4.6. This is because Theorem 3.4.6 uses the dep-structure from Theorem 2.4.5 and Example 3.4.5 uses the dep-structure from Example 2.4.8. We did see that these dep-structures were not the same in Proposition 2.4.9, making the 2-dep-categories different as well.

3.5 $(2\text{-}\mathtt{dep}, \Sigma)$-Categories

Now, we are in the last part of section 3. We will now wrap up everything that we have done and define $(2\text{-}\mathtt{dep}, \Sigma)$-categories, which are a structure that has 2-$\mathtt{fam}$-arrows, $\mathtt{dep}$-arrows and Σ-obejcts. Previously, we had already combined some of those notion, for example, we combined $\mathtt{dep}$-categories and $(\mathtt{fam}, \Sigma)$-categories into $(\mathtt{dep}, \Sigma)$-categories. However, $(2\text{-}\mathtt{dep}, \Sigma)$-categories are different from our previous efforts to combine these structures. The difference is, that for each of these combinations we still added a little bit of structure which we couldn't capture before. Like the second projection arrows or the Σ-arrows of the second kind. This time we will not add anything new. The only things we will need for a $(2\text{-}\mathtt{dep}, \Sigma)$-category are a 2-$\mathtt{dep}$-, a $(\mathtt{dep}, \Sigma)$- and a $(2\text{-}\mathtt{fam}\Sigma)$-category, as well as some conditions, to make their structures compatible.

Definition 3.5.1 ((2-dep, Σ)-Category)**.**
$2\mathrm{d}\Sigma\mathrm{C}$ consisting of

- a 2-dep-category $2\mathrm{dC}$,
- a (dep, Σ)-category $\mathrm{d}\Sigma\mathrm{C}$, and
- a $(2\text{-}\mathrm{fam}, \Sigma)$-category $2\Sigma\mathrm{C}$

is called a $(2\text{-dep}, \Sigma)$-category, iff

(dep) the underlying dep-categories of $2\mathrm{dC}$ and $\mathrm{d}\Sigma\mathrm{C}$ are the same,

$(2\text{-}\mathrm{fam})$ the underlying 2-fam-categories of $2\mathrm{dC}$ and $2\Sigma\mathrm{C}$ are the same,

(Σ) the underlying (fam, Σ)-categories of $2\Sigma\mathrm{C}$ and $\mathrm{d}\Sigma\mathrm{C}$ are the same, and

$(2\mathrm{ds})$ $\forall a \in C_0 \forall \lambda, \mu \in \mathrm{fHom}(a) \forall \eta \in \mathrm{Hom}(\lambda, \mu)$:

$$(\eta \circ \mathrm{pr}_1^{a,\lambda}) \circ \mathrm{pr}_2^{a,\lambda} = \mathrm{pr}_2^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta.$$

To illustrate this, consider the following diagrams:

$$\begin{tikzcd}
\sum_a \lambda \arrow[dr, "\mathrm{pr}_1^{a,\lambda}"'] \arrow[drr, "\lambda\circ\mathrm{pr}_1^{a,\lambda}", color=blue] \arrow[drr, Rightarrow, bend left=50, "\mathrm{pr}_2^{a,\lambda}", color=red] \arrow[dd, "\Sigma_{\lambda,\mu}\eta"'] & & \\
& a \arrow[r, bend left, "\lambda", color=blue] \arrow[r, bend right, "\mu"', color=blue] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \\
\sum_a \mu \arrow[ur, "\mathrm{pr}_1^{a,\mu}"] \arrow[urr, "\mu\circ\mathrm{pr}_1^{a,\mu}"', color=blue] \arrow[urr, Rightarrow, bend right=50, "\mathrm{pr}_2^{a,\mu}"', color=red] & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\sum_a \lambda \arrow[r, Rightarrow, bend left=70, "\mathrm{pr}_2^{a,\lambda}", color=red] \arrow[r, bend left=25, "\lambda\circ\mathrm{pr}_1^{a,\lambda}", color=blue] \arrow[r, bend right=25, "\mu\circ\mathrm{pr}_1^{a,\mu}\circ\Sigma_{\lambda,\mu}\eta"', color=blue] \arrow[r, Rightarrow, bend right=70, "\mathrm{pr}_2^{a,\mu}\circ\Sigma_{\lambda,\mu}\eta"', color=red] \arrow[r, phantom, "\Downarrow\eta\circ\mathrm{pr}_1^{a,\lambda}"] & \cdot
\end{tikzcd}$$

We also define

- $C_0 := 2\mathrm{dC}_0 = \mathrm{d}\Sigma\mathrm{C}_0 = 2\Sigma\mathrm{C}_0$,
- $C_1 := 2\mathrm{dC}_1 = \mathrm{d}\Sigma\mathrm{C}_1 = 2\Sigma\mathrm{C}_1$,
- $C_2 := 2\mathrm{dC}_2 = \mathrm{d}\Sigma\mathrm{C}_2 = 2\Sigma\mathrm{C}_2$,
- $C_3 := 2\mathrm{dC}_3 = \mathrm{d}\Sigma\mathrm{C}_3$ and

- $C_4 := 2dC_4 = 2\Sigma C_4$.

Remark 3.5.2 (Typing in Definition 3.5.1)**.**
Note, that the arrows on the 2 sides of the equation in $(2ds)$ indeed have the same type. For this, by definition, the types of the 2 arrows are (given variables as specified in $(2ds)$) are:

- $(\eta \circ pr_1^{a,\lambda}) \circ pr_2^{a,\lambda} \in dHom\left(\sum_a \lambda, \mu \circ pr_1^{a,\lambda}\right)$, and
- $pr_2^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta \in dHom\left(\sum_a \lambda, \mu \circ pr_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta\right)$.

Remember, from Definition 3.2.1, $pr_1^{a,\lambda} = pr_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta$.
With this we can clearly see, that the typings are the same.

Remark 3.5.3 (Compontents of $(2\text{-}dep, \Sigma)$-categories have the same fam-structure)**.**
Notice, that the fam-structures of the 3 components of a $(2\text{-}dep, \Sigma)$-category are the same. This can be seen, by using the conditions (dep), $(2\text{-}fam)$ and (Σ).

As always, we will now look at a trivial example. Here we will be combining the structures given by the Examples 3.4.2, 3.2.3 and 2.5.3 into a $(2\text{-}dep, \Sigma)$-category.

Example 3.5.4 (A $(2\text{-}dep, \Sigma)$-category with trivial dep-arrows and Σ-objects)**.**
Let $2C$ be a $2\text{-}fam$-category. Let $2dC$ be the $2\text{-}dep$-category from Example 3.4.2, using $2C$. Let ΣC be the (fam, Σ)-category given by Example 2.2.3 and the fam-structure of $2C$. Let $d\Sigma C$ be the (dep, Σ)-category from Example 2.5.3, using ΣC. Let $2\Sigma C$ be the $(2\text{-}fam, \Sigma)$-category from Example 3.2.3, using $2C$. Then, $2d\Sigma C$ consisting of

- $2dC$,
- $d\Sigma C$ and
- $2\Sigma C$

is a $(2\text{-}dep, \Sigma)$-category.

Proof.
To show this, we have to show the conditions (dep), $(2\text{-}fam)$, (Σ) and $(2ds)$.
To see (dep), $(2\text{-}fam)$ and (Σ), simply go to the respective examples and compare the relevant structures. So what remains is to show $(2ds)$.
So let $a \in C_0$, $\lambda, \mu \in fHom(a)$ and $\eta \in Hom(\lambda, \mu)$. We have to show

$$(\eta \circ pr_1^{a,\lambda}) \circ pr_2^{a,\lambda} = pr_2^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta.$$

Using the definitions of the compositions in Example 3.4.2, we get

$$(\eta \circ pr_1^{a,\lambda}) \circ pr_2^{a,\lambda} = pr_2^{a,\lambda}$$

and

$$pr_2^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta = pr_2^{a,\mu}.$$

Furthermore, by Example 2.5.3 we know $pr_2^{a,\mu} = pr_2^{a,\lambda}$.
Thus $(2ds)$ follows, and we indeed have a $(2\text{-}dep, \Sigma)$-category. $\square$

Next we will show that we get $(2\text{-}dep, \Sigma)$-categories with constant fam-arrows, from type universes and from commutative rings. In all of these, we will take the respective examples from the previous section and simply show, that the given structures are compatible as per Definition 3.5.1.

Example 3.5.5 (A $(2\text{-}\mathsf{dep}, \Sigma)$-category with constant fam-arrows).
Let C be a category with binary products.
Let there be an assignment

$$\mathsf{fam}C_0 \times \mathsf{fam}C_0 \to \mathsf{fam}C_0, (a, b) \mapsto a \times b,$$

where $a \times b$ is a product of a and b.
Let $2dC$ be the $2\text{-}\mathsf{dep}$-category from Example 3.4.3, using C. Let $\mathsf{dsig}C$ be the (dep, Σ)-category from Example 2.5.4, using C. Let $2\mathsf{sig}C$ be the $(2\text{-}\mathsf{fam}, \Sigma)$-category from Example 3.2.4, using C. Then, $2d\Sigma C$ consisting of

- $2dC$,
- $\mathsf{dsig}C$ and
- $2\mathsf{sig}C$

is a $(2\text{-}\mathsf{dep}, \Sigma)$-category.

Proof.
We have to again show the conditions from Definition 3.5.1. Here it is again easy to see that (dep), $(2\text{-}\mathsf{fam})$ and (Σ) are fulfilled, using the definitions in the examples.
So we just have to show $(2\mathsf{ds})$. Hence let $a \in C_0$, $\lambda, \mu \in \mathsf{fHom}(a)$ and $\eta \in \mathsf{Hom}(\lambda, \mu)$.
We have to show that

$$(\eta \circ \mathsf{pr}_1^{a,\lambda}) \circ \mathsf{pr}_2^{a,\lambda} = \mathsf{pr}_2^{a,\mu} \circ (\mathsf{id}_a \times \eta).$$

We know by the definition in Example 3.2.4, that

$$(\eta \circ \mathsf{pr}_1^{a,\lambda}) \circ \mathsf{pr}_2^{a,\lambda} = \eta \circ \mathsf{pr}_2^{a,\lambda}.$$

As a basic property of products, we know that

$$\eta \circ \mathsf{pr}_2^{a,\lambda} = \mathsf{pr}_2^{a,\mu} \circ (\mathsf{id}_a \times \eta).$$

Thus we have shown $(2\mathsf{ds})$. □

Example 3.5.6 (Small types form a $(2\text{-}\mathsf{dep}, \Sigma)$-category).
Let U be a type universe. Let $\mathsf{Type}^{2d}(U)$ be the 2-dep-category from Example 3.4.4 using U. Let $\mathsf{Type}^{d\Sigma}(U)$ be the (dep, Σ)-category given by Example 2.5.5 and U. Let $\mathsf{Type}^{2\Sigma}(U)$ the $(2\text{-}\mathsf{fam}, \Sigma)$-category given by Example 3.2.5 and U. Then, $\mathsf{Type}^{2d\Sigma}(U)$ consisting of

- $\mathsf{Type}^{2d}(U)$,
- $\mathsf{Type}^{d\Sigma}(U)$ and
- $\mathsf{Type}^{2\Sigma}(U)$

is a $(2\text{-}\mathsf{dep}, \Sigma)$-category.

Proof.
We have to show the conditions in Definition 3.5.1 hold. By observing the constructions for $\mathsf{Type}^{2d}(U)$, $\mathsf{Type}^{d\Sigma}(U)$ and $\mathsf{Type}^{2\Sigma}(U)$, in the examples, we immediately see, that the conditions (dep), $(2\text{-}\mathsf{fam})$ and (Σ) hold. So what remains to be seen is $(2\mathsf{ds})$.

For this we will use function extensionality.

Let $A \in \mathsf{Type}^{2d\Sigma}(U)_0,\ \mu, \nu \in \mathsf{fHom}(A)$, and $\eta \in \mathsf{Hom}(\mu, \nu)$.
Hence $A : U,\ \mu, \nu : A \to U$, and $\eta : \prod_{x:A}(\mu(x) \to \nu(x))$. We need to show

$$(\eta \circ pr_1^{A,\mu}) \circ pr_2^{A,\mu} = pr_2^{A,\nu} \circ \Sigma_{\mu,\nu}\eta.$$

So let $(x, y) : \sum_{a:A} \mu(a)$.
By simplifying both sides of the equation we get:

$$\begin{aligned}((\eta \circ pr_1^{A,\mu}) \circ pr_2^{A,\mu})(x, y) &= (\eta \circ pr_1^{A,\mu})(x, y)(pr_2^{A,\mu}(x, y)) \\ &= \eta(pr_1^{A,\mu}(x, y))(pr_2^{A,\mu}(x, y)) = \eta(x)(y)\end{aligned}$$

and

$$(pr_2^{A,\nu} \circ \Sigma_{\mu,\nu}\eta)(x, y) = pr_2^{A,\nu}(\Sigma_{\mu,\nu}\eta(x, y)) = pr_2^{A,\nu}(x, \eta(x)(y)) = \eta(x)(y). \quad \square$$

Example 3.5.7 (A $(2\text{-dep}, \Sigma)$-structure for a commutative ring)**.**
Let R be a commutative ring. Let $\forall r \in R :\ I(r)$ be the ideal generated by r. Let $2dR$ be the 2-dep-category from Example 3.4.5 using R. Let $d\Sigma R$ be the (dep, Σ)-category given by Example 2.5.8 and R. Let $2\Sigma R$ the $(2\text{-fam}, \Sigma)$-category given by Example 3.2.7 and R.
Then, $2d\Sigma R$ consisting of

- $2dR$,
- $d\Sigma R$ and
- $2\Sigma R$

is a $(2\text{-dep}, \Sigma)$-category.

Proof.
As with the previous examples, the conditions (dep), (2-fam) and (Σ) are clear.
So what remains is to show $(2ds)$. So let $(a, b), (c, d) \in \mathsf{fHom}(*)$ and $\eta \in \mathsf{Hom}((a, b), (c, d))$.
We have to show

$$(\eta \circ pr_1^{a,\lambda}) \circ pr_2^{*,(a,b)} = pr_2^{*,(c,d)} \circ \Sigma_{(a,b),(c,d)}\eta.$$

Using the respective definitions, this equation simplifies to

$$I\big(\eta(I(a - b))\big) = I(c - d).$$

Since we have $\eta(a - b) = c - d$, we can see that $I\big(\eta(I(a - b))\big) \supseteq I(c - d)$ is clear, but we still have to show

$$I\big(\eta(I(a - b))\big) \subseteq I(c - d).$$

For this, let $r \in I\big(\eta(I(a - b))\big)$. Then, let $n \in \mathbb{N},\ s_1, ..., s_n \in R$ and $r_1, ..., r_n \in \eta(I(a - b))$, such that

$$r = \sum_{i=1}^{n} s_i r_i.$$

Now, for each $i \in \{1, ..., n\}$ let $p_i \in R$, such that $r_i = \eta(p_i(a - b))$.
Hence

$$r = \sum_{i=1}^{n} s_i \eta(p_i)\eta(a - b) = \sum_{i=1}^{n} s_i \eta(p_i)(c - b) \in I(c - d).$$

Thus $I\big(\eta(I(a - b))\big) = I(c - d)$. Thus $(2ds)$ holds. $\square$

Now we will wrap up our big theorems. We will see now that the structures from Theorem 2.5.6 and Theorem 3.4.6 are compatible, such that we do get a $(2\text{-}\mathsf{dep}, \Sigma)$-category from every $(2\text{-}\mathsf{fam}, \Sigma)$-category in the following theorem. Afterwards we will make two quick remarks about easy consequences of the Theorem.

Theorem 3.5.8 (Every $(2\text{-}\mathsf{fam}, \Sigma)$-category induces a $(2\text{-}\mathsf{dep}, \Sigma)$-category)**.**
Let $2\Sigma C$ be a $(2\text{-}\mathsf{fam}, \Sigma)$-category. Let $2dC$ the 2-dep-category induced by $2\Sigma C$ through Theorem 3.4.6. Let $d\Sigma C$ be the (dep, Σ)-category induced by $\Sigma 2\Sigma C$ through Theorem 2.5.6. Then, $2d\Sigma C$ consisting of

- $2dC$,
- $d\Sigma C$ and
- $2\Sigma C$

is a $(2\text{-}\mathsf{dep}, \Sigma)$-category.

Proof.
To see that the data gives a $(2\text{-}\mathsf{dep}, \Sigma)$-category, we must show that the conditions in Definition 3.5.1 hold.

(dep) Since both Theorem 3.4.6 and Theorem 2.5.6 use the dep-structure induced by $\Sigma 2\Sigma C$ through Theorem 2.4.5, $2dC$ and $d\Sigma C$ indeed have the same dep-structure.

$(2\text{-}\mathsf{fam})$ Since Theorem 3.4.6 uses $2 2\Sigma C$ for the $2\text{-}\mathsf{fam}$-structure, $2\Sigma C$ and $2dC$ have the same $2\text{-}\mathsf{fam}$-structure.

(Σ) Since Theorem 2.5.6 uses $\Sigma 2\Sigma C$ for the (fam, Σ)-structure, $2\Sigma C$ and $d\Sigma C$ have the same (fam, Σ)-structure.

$(2\mathsf{ds})$ Let $a \in 2d\Sigma C_0$, $\lambda, \mu \in \mathrm{fHom}(a)$ and $\eta \in \mathrm{Hom}(\lambda, \mu)$.
We have to show

$$(\eta \circ \mathrm{pr}_1^{a,\lambda}) \circ_2 \mathrm{pr}_2^{a,\lambda} = \mathrm{pr}_2^{a,\mu} \circ_d \Sigma_{\lambda,\mu}\eta. \tag{4}$$

To do this, we want to the universal pullback property. For this, first consider the definition of $\mathrm{pr}_2^{a,\mu} \circ_d \Sigma_{\lambda,\mu}\eta$ given by Theorem 2.4.5. By this, $\mathrm{pr}_2^{a,\mu} \circ_d \Sigma_{\lambda,\mu}\eta$ is given as the unique arrow from the universal pullback property:

$$\begin{tikzcd}
\sum_a \lambda \arrow[dr, dashed, red, "\mathrm{pr}_2^{a,\mu} \circ_d \Sigma_{\lambda,\mu}\eta"] \arrow[drr, bend left, "\mathrm{pr}_2^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta"] \arrow[ddr, bend right, "\mathrm{id}_{\Sigma_a \lambda}"'] & & \\
& \sum_{\Sigma_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta) \arrow[r, "\Sigma_{\mu \circ \mathrm{pr}_1^{a,\mu}} \Sigma_{\lambda,\mu}\eta"] \arrow[d, "\mathrm{pr}_1^{\Sigma_a \lambda, \mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta}"] & \sum_{\Sigma_a \mu}(\mu \circ \mathrm{pr}_1^{a,\mu}) \arrow[d, "\mathrm{pr}_1^{\Sigma_a \mu, \mu \circ \mathrm{pr}_1^{a,\mu}}"] \\
& \sum_a \lambda \arrow[r, "\Sigma_{\lambda,\mu}\eta"'] & \sum_a \mu
\end{tikzcd}$$

Furthermore, consider the definition of $\mathrm{pr}_2^{a,\mu}$ given by Theorem 2.5.6.
this yields us the following commutative diagram:

$$\begin{tikzcd}
\sum_a \mu \arrow[dr, dashed, "\mathrm{pr}_2^{a,\mu}"] \arrow[drr, bend left, "\mathrm{id}_{\sum_a \mu}"] \arrow[ddr, bend right, "\mathrm{id}_{\sum_a \mu}"'] & & \\
& \sum_{\sum_a \mu}(\mu \circ \mathrm{pr}_1^{a,\mu}) \arrow[r, "\Sigma_\mu \mathrm{pr}_1^{a,\mu}"] \arrow[d, "\mathrm{pr}_1^{\sum_a \mu, \mu \circ \mathrm{pr}_1^{a,\mu}}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"] \\
& \sum_a \mu \arrow[r, "\mathrm{pr}_1^{a,\mu}"'] & a
\end{tikzcd}$$

Putting these diagrams together results in the following commutative diagram $(*)$:

$$\begin{tikzcd}
\sum_a \lambda \arrow[rr, bend left, "\Sigma_{\lambda,\mu}\eta"] \arrow[dr, dashed, "\mathrm{pr}_2^{a,\mu} \circ_d \Sigma_{\lambda,\mu}\eta"] \arrow[ddr, bend right, "\mathrm{id}_{\sum_a \lambda}"'] & & \sum_a \mu \arrow[drr, bend left, "\mathrm{id}_{\sum_a \mu}"] \arrow[dr, dashed, "\mathrm{pr}_2^{a,\mu}"] \arrow[ddr, "\mathrm{id}_{\sum_a \mu}"'] & & \\
& \sum_{\sum_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta) \arrow[rr, "\Sigma_{\mu \circ \mathrm{pr}_1^{a,\mu}} \Sigma_{\lambda,\mu}\eta"'] \arrow[d, "\mathrm{pr}_1^{\sum_a \lambda, \mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta}"] & & \sum_{\sum_a \mu}(\mu \circ \mathrm{pr}_1^{a,\mu}) \arrow[r, "\Sigma_\mu \mathrm{pr}_1^{a,\mu}"] \arrow[d, "\mathrm{pr}_1^{\sum_a \mu, \mu \circ \mathrm{pr}_1^{a,\mu}}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"] \\
& \sum_a \lambda \arrow[rr, "\Sigma_{\lambda,\mu}\eta"] & & \sum_a \mu \arrow[r, "\mathrm{pr}_1^{a,\mu}"] & a
\end{tikzcd}$$

Next consider the other side of Equation 4.
The definition of $\mathrm{pr}_2^{a,\lambda}$ as given by Theorem 2.5.6 is

$$\begin{tikzcd}
\sum_a \lambda \arrow[dr, dashed, "\mathrm{pr}_2^{a,\lambda}"] \arrow[drr, bend left, "\mathrm{id}_{\sum_a \lambda}"] \arrow[ddr, bend right, "\mathrm{id}_{\sum_a \lambda}"'] & & \\
& \sum_{\sum_a \lambda}(\lambda \circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\lambda \mathrm{pr}_1^{a,\lambda}"] \arrow[d, "\mathrm{pr}_1^{\sum_a \lambda, \lambda \circ \mathrm{pr}_1^{a,\lambda}}"] & \sum_a \lambda \arrow[d, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_a \lambda \arrow[r, "\mathrm{pr}_1^{a,\lambda}"'] & a
\end{tikzcd}$$

Then, by the condition $(\mathtt{diag})$ from Definition 3.2.1, we have the commutative diagram:

$$\begin{tikzcd}
\sum_{\sum_a \lambda}(\lambda \circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\lambda \mathrm{pr}_1^{a,\lambda}"] \arrow[d, "\Sigma_{\lambda\circ \mathrm{pr}_1^{a,\lambda}, \mu\circ \mathrm{pr}_1^{a,\mu}}(\eta\circ \mathrm{pr}_1^{a,\lambda})"] \arrow[dd, bend right=60, "\mathrm{pr}_1^{\sum_a \lambda, \lambda\circ \mathrm{pr}_1^{a,\lambda}}"'] & \sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"] \arrow[dd, bend left=60, "\mathrm{pr}_1^{a,\lambda}"] \\
\sum_{\sum_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\mu \mathrm{pr}_1^{a,\lambda}"'] \arrow[d, "\mathrm{pr}_1^{\sum_a \lambda, \mu\circ \mathrm{pr}_1^{a,\lambda}}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"] \\
\sum_a \lambda \arrow[r, "\mathrm{pr}_1^{a,\lambda}"] & a
\end{tikzcd}$$

We can see that the square in the diagram for $\mathrm{pr}_2^{a,\lambda}$ is just the outer square in $(\mathtt{diag})$. Hence we can also put these diagrams together, yielding a commutative diagram $(**)$:

$$\begin{tikzcd}
\sum_a \lambda \arrow[dr, dashed, red, "\mathrm{pr}_2^{a,\lambda}"] \arrow[drr, bend left=40, "\mathrm{id}_{\sum_a \lambda}"] \arrow[dddr, bend right=40, "\mathrm{id}_{\sum_a \lambda}"'] & & \\
& \sum_{\sum_a \lambda}(\lambda \circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\lambda \mathrm{pr}_1^{a,\lambda}"] \arrow[d, "\Sigma_{\lambda\circ \mathrm{pr}_1^{a,\lambda}, \mu\circ \mathrm{pr}_1^{a,\mu}}(\eta\circ \mathrm{pr}_1^{a,\lambda})"] \arrow[dd, bend right=60, "\mathrm{pr}_1^{\sum_a \lambda, \lambda\circ \mathrm{pr}_1^{a,\lambda}}"'] & \sum_a \lambda \arrow[d, "\Sigma_{\lambda,\mu}\eta"] \arrow[dd, bend left=60, "\mathrm{pr}_1^{a,\lambda}"] \\
& \sum_{\sum_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\mu \mathrm{pr}_1^{a,\lambda}"'] \arrow[d, "\mathrm{pr}_1^{\sum_a \lambda, \mu\circ \mathrm{pr}_1^{a,\lambda}}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"] \\
& \sum_a \lambda \arrow[r, "\mathrm{pr}_1^{a,\lambda}"] & a
\end{tikzcd}$$

Now, we know that both of the inner squares of $(*)$ are pullback, thus by the pullback lemma, the rectangle these two form together is also a pullback. Since $(*)$ is commutative, we can get the following pullback-competitor diagram, as a subdiagram $(*')$ of $(*)$:

$$
\begin{tikzcd}
\sum_a \lambda \arrow[rrd, bend left, "\Sigma_{\lambda,\mu}\eta"] \arrow[rd, dashed, red, "\mathrm{pr}_2^{a,\mu} \circ_d \Sigma_{\lambda,\mu}\eta"] \arrow[rdd, bend right, "\mathrm{id}_{\sum_a \lambda}"'] & & \\
& \sum_{\sum_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta) \arrow[r, "\Sigma_\mu \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\mu \circ \mathrm{pr}_1^{a,\mu}} \Sigma_{\lambda,\mu}\eta"] \arrow[d, "\mathrm{pr}_1^{\sum_a \lambda, \mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"] \\
& \sum_a \lambda \arrow[r, "\mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta"'] & a
\end{tikzcd},
$$

For $(**)$, we know that the lower square is a pullback. Hence we get the following pullback-competitor diagram as a subdiagram $(**')$ of $(**)$:

$$
\begin{tikzcd}
\sum_a \lambda \arrow[rrd, bend left, "\Sigma_{\lambda,\mu}\eta"] \arrow[rd, dashed, red, "(\eta \circ \mathrm{pr}_1^{a,\lambda}) \circ_2 \mathrm{pr}_2^{a,\lambda}"] \arrow[rdd, bend right, "\mathrm{id}_{\sum_a \lambda}"'] & & \\
& \sum_{\sum_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\lambda}) \arrow[r, "\Sigma_\mu \mathrm{pr}_1^{a,\lambda}"] \arrow[d, "\mathrm{pr}_1^{\sum_a \lambda, \mu \circ \mathrm{pr}_1^{a,\lambda}}"] & \sum_a \mu \arrow[d, "\mathrm{pr}_1^{a,\mu}"] \\
& \sum_a \lambda \arrow[r, "\mathrm{pr}_1^{a,\lambda}"'] & a
\end{tikzcd}
$$

Notice that we used, that, by the definition in Theorem 3.4.6,

$$(\eta \circ \mathrm{pr}_1^{a,\lambda}) \circ_2 \mathrm{pr}_2^{a,\lambda} = \Sigma_{\lambda \circ \mathrm{pr}_1^{a,\lambda}, \mu \circ \mathrm{pr}_1^{a,\mu}}(\eta \circ \mathrm{pr}_1^{a,\lambda}) \circ \mathrm{pr}_2^{a,\lambda}.$$

Now we need to show, that $(*')$ and $(**')$ are the same diagram, then we are done. For this we need to show

(i) $\mathrm{pr}_1^{a,\lambda} = \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta$,

(ii) $\sum_{\sum_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\lambda}) = \sum_{\sum_a \lambda}(\mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta)$,

(iii) $\Sigma_\mu \mathrm{pr}_1^{a,\lambda} = \Sigma_\mu \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\mu \circ \mathrm{pr}_1^{a,\mu}} \Sigma_{\lambda,\mu}\eta$, and

(iv) $\mathrm{pr}_1^{\sum_a \lambda, \mu \circ \mathrm{pr}_1^{a,\lambda}} = \mathrm{pr}_1^{\sum_a \lambda, \mu \circ \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta}$.

(i) follows immediately from the right triangle in $(**)$, (ii) and (iv) then follow from (i). For (iii) use (i) and (s_2) from the definition of $(\mathtt{fam}, \Sigma)$-categories to see, that

$$\Sigma_\mu \mathrm{pr}_1^{a,\mu} \circ \Sigma_{\mu \circ \mathrm{pr}_1^{a,\mu}} \Sigma_{\lambda,\mu}\eta = \Sigma_\mu(\mathrm{pr}_1^{a,\mu} \circ \Sigma_{\lambda,\mu}\eta) = \Sigma_\mu \mathrm{pr}_1^{a,\lambda}.$$

Thus, $(*')$ and $(**')$ are the same, and with the uniqueness property for pullback, we get

$$(\eta \circ \mathrm{pr}_1^{a,\lambda}) \circ \mathrm{pr}_2^{a,\lambda} = \mathrm{pr}_2^{a,\mu} \circ_d \Sigma_{\lambda,\mu}\eta. \quad \square$$

Remark 3.5.9 (Toposes as $(2\text{-}\mathtt{dep}, \Sigma)$-categories)**.**
Taking the structures on toposes defined in Lemma 3.2.6, Lemma 3.4.7 and Remark 2.5.7, we can use Theorem 3.5.8 to see that every topos is a $(2\text{-}\mathtt{dep}, \Sigma)$-category.

Remark 3.5.10 (Not all $(2\text{-}\mathtt{dep}, \Sigma)$-categories are given by their $(2\text{-}\mathtt{fam}, \Sigma)$-structure)**.**
We have seen that we can combine the structures from the Examples 3.4.5, 2.5.8 and 3.2.7 into a $(2\text{-}\mathtt{dep}, \Sigma)$-category, in Example 3.5.7. Hence, by a similar argument as in the Remarks 2.5.9 and 3.4.9, we can see, that there exists a $(2\text{-}\mathtt{dep}, \Sigma)$-category, such that it is not induced through Theorem 3.5.8 by its own $(2\text{-}\mathtt{fam}, \Sigma)$-structure.

4 Conclusion

We have now seen the definitions associated with 2-$\mathtt{fam}$-arrows, several examples and that $(2\text{-}\mathtt{fam}, \Sigma)$-categories induce 2-$\mathtt{dep}$- and $(2\text{-}\mathtt{dep}, \Sigma)$-categories. But there are many further concepts that one might talk about, regarding this topic.
The most obvious of which might be 2-$\mathtt{dep}$- and $(2\text{-}\mathtt{dep}, \Sigma)$-functors. After all, when we define a new kind of object, we always want to know how a map between two objects of that kind might look like. Then with extended notions of functors of course comes the question, whether one can define corresponding extended notions of limits and, if it is possible, how these behave. Furthermore, with definitions of 2-$\mathtt{dep}$- and $(2\text{-}\mathtt{dep}, \Sigma)$-functors, it could be interesting to investigate whether the correspondences that we discussed for the topos example in Lemma 2.4.6 can be used to construct 2-$\mathtt{dep}$- and $(2\text{-}\mathtt{dep}, \Sigma)$-functors.
Going now beyond functors, another object one might look at are duals for 2-$\mathtt{fam}$-arrows. In [3] it is mentioned, that for $\mathtt{fam}$-arrows, $\mathtt{dep}$-arrows and Σ-objects, one can find duals of these, called $\mathtt{cofam}$-arrows, $\mathtt{codep}$-arrows and $\mathtt{co}\Sigma$-objects respectively. There is no reason to assume, that we wouldn't be able to define 2-$\mathtt{cofam}$-arrows, that point from a $\mathtt{cofam}$-arrow to another $\mathtt{cofam}$-arrow. These also might have interesting properties and example.
So, there clearly is more work to be done.

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