Finitistic Higher Order Logic

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Introduction

Aim:

- A finititstic understanding of mathematics
- Forerunner: Jan Mycielski (Local finite theories, JSL '86)

Our approach: Use model theory

- FOL: Adopt Tarskian semantics
- HOL: Adopt models of STT

Main idea:

Infinite sets are not actual infinite, but indefinitely extensible —
This applies also to the syntax. No notion of computability.

First order logic I

Basic change:

Use a family $(\mathcal{M}_i)_{i \in \mathcal{I}}$ with finite sets $\mathcal{M}_i$ and a directed index set $\mathcal{I}$ instead of an infinite set $\mathcal{M}$.

Approximation declaration:

Introduce approximation declarations $C \vdash \mathrm{t} : i$ and $C \vdash \Phi$ with approximation context $C = (i_0, \ldots, i_{n-1})$ being a list of indices.

Sufficiently large:

Use a relation $i \gg C$ meaning "i is sufficiently large relative to C". Restrict contexts to those satisfying $i_k \gg (i_0, \ldots, i_{k-1})$.

Interpretation:

Interpret $\models_{\gg} \forall \mathrm{x} \Phi[\boldsymbol{a} : C]$ as: There is some "sufficiently large" index $i \gg C$ such that $\models_{\gg} \Phi[\boldsymbol{a}, b : C.i]$ holds for all $b \in \mathcal{M}_i$.

First order logic II

Main result:

- A model $(\mathcal{M}_i)_{i \in \mathcal{I}}$ validates the same formulas as the limit model $\bigcup_{i \in \mathcal{I}} \mathcal{M}_i$.
- We have soundness and completeness.
- New metatheory: e.g. no unavoidable non-standard models, categoricity of first order PA becomes possible.
- Applicable also to Kripke models and intuitionistic logic.

First order logic II

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No infinity at all:

Model theory is applicable to background theory: Model of model theory is also finitistic (in particular: Index $\mathcal{I}$ is not infinitely large).

Towards higher order logic I

New in HOL:

Infinite objects are approximated, too. No infinite functions as $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, but approximations $f : (\mathbb{N}_i \to \mathbb{N}_j) \to \mathbb{N}_k$ with $\mathbb{N}_i = \{0, \ldots, i-1\}$ and $i, j, k \in \mathbb{N} \setminus \{0\}$.

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Total or partial functions?

Domain theory provides a notion of approximation of infinite objects. But it uses partial functions. We us total functions:

- Partial functions not suitable for a HOL (Type *bool* has partial truth values).
- Since we only have finite sets $\mathcal{M}_i$, there is no need to approximate functions by finite partial functions.

Towards higher order logic II

Partial surjections:

Partial surjections $\overset{p}{\mapsto}$ arise when embeddings on base types are extended to higher type as logical relations. But at higher types the partial surjections are not transitive (composable):

$$f'' \overset{p}{\mapsto} f' \overset{p}{\mapsto} f \not\Rightarrow f'' \overset{p}{\mapsto} f\,.$$

Use instead property:

$$\exists f''\;\; f'' \overset{p}{\mapsto} f' \text{ and } f'' \overset{p}{\mapsto} f \iff f' \overset{p}{\mapsto} f\,. \tag{1}$$

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Embedding and projection:

Property (1) too weak to extend it to higher types. Auxiliary structure of embedding-projection pair around $\stackrel{p}{\mapsto}$ necessary. But: Several choices possible.

Towards higher order logic III

Limit construction:

New limit construction necessary: In FOL the direct limit suffices, in HOL we need more, e.g. $id_{i,j} : \mathbb{N}_i \to \mathbb{N}_j$ with $i \leq j$ has limit $id : \mathbb{N} \to \mathbb{N}$.
Idea: Use partial surjections for the limit construction with property (1) formulated as a universal property.

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Sufficiently many:

Additionally required: A notion of "sufficiently many" ("almost all"), a kind of Fréchet filter $\mathfrak{F}$.

Two views

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(FIN) Dynamic finitistic multiverse e.g. $id_{i,j} : \mathbb{N}_i \to \mathbb{N}_j$.

(INF) Static universe with actual infinity e.g. $id : \mathbb{N} \to \mathbb{N}$.

From (FIN) to (INF):

Limit construction with $\overset{p}{\mapsto}$.

Structure on (INF):

Family $\approx_i$ of partial equivalence relations (PERs) over $\mathcal{I}$, generated by $\overset{p}{\mapsto}$. Idea: $\approx_i$ approximates equality.

From (INF) to (FIN):

Take the equivalence classes of $\approx_i$.

Universes with families of PERs

Auxiliary structure:

Embedding $\leftrightarrow$ Points (representatives of equivalence classes).
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Universe view:

Set $\mathcal{M}$ with PERs $\approx_i$ for $i \in \mathcal{I}$, approximating equality on $\mathcal{M}$.
Properties:

- $\approx_i$ dense in $\mathcal{M}$, i.e., $\{i \in \mathcal{I} \mid a \approx_i a\} \in \mathfrak{F}$.
- $\approx_{i'}$ finer than $\approx_i$ for $i' \geq i$, but not necessarily $\approx_{i'} \subseteq \approx_i$.
- Further properties in combination with points and extension.
- Functions are extensional (due to universal property of the limit).
- Limit is complete (define convergent families $(a^i)_{i \in \mathcal{I}}$ w.r.t. $\approx_i$ and limit elements).

One-to-one correspondence

Main idea:

- To each multiverse with partial surjections there is exactly one (extensional and complete) limit universe with PERs.
- To each universe with PERs the equivalence classes form a multiverse with partial surjections.

Hence: The limit structure satisfies the principle of finite support.

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However:

From a finititstic perspective we need only multiverse with partial surjections. Nevertheless, the limit can be seen as an indefinitely large stage in $(\mathcal{M}_i)_{i \in \mathcal{I}}$.

Open questions

Limit structure:

What exactly is this limit structure? Are they the (hereditarily) total continuous functionals (Kleene-Kreisel)?

Continuity:

What is the notion of continuity with respect to $\approx_i$?

$$f \text{ is } i\text{-}j\text{-continuous iff } a \approx_i b \text{ implies } f(a) \approx_j f(b).$$

Define f is uniform continuous iff there are sufficiently many (i, j) such that f is i-j-continuous. Too restrictive?

Approximation declaration:

What is the corresponding notion in HOL?

Any questions?

Thank you for your attention.