On the Computational Content of the Vitali Covering Theorem

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joint work with

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Arbeitstagung Bern-München

LMU München December 2017

Outline

Heine-Borel Covering Theorem

The Heine-Borel Covering Theorem

Theorem (Heine-Borel)

Any open covering $(U_n)_{n\in\mathbb{N}}$ of $[0,1]$ has a finite subcover, i.e.,

$$[0,1] \subseteq \bigcup_{n=0}^{\infty} U_n \Longrightarrow (\exists m \in \mathbb{N})\ [0,1] \subseteq \bigcup_{n=0}^{m} U_n.$$

- The Theorem counts as computable in computable analysis and as non-constructive in constructive analysis.
- How can this difference be explained?

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Theorem (Friedman's and Simpson's reverse mathematics 1983)

Using recursive comprehension RCA_0 *and using second-order arithmetic and classical logic the Heine-Borel Theorem is equivalent to Weak Kőnig's Lemma* WKL_0.

Theorem (Ishihara's constructive reverse mathematics 1990)

Using intuitionistic logic (and countable and dependent choice) the Heine-Borel Theorem is equivalent to Weak Kőnig's Lemma WKL *and to the Lesser Limited Principle of Omniscience* LLPO.

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Formalize this as

- $\mathrm{HBT}_0 :\subseteq \mathcal{O}([0,1])^{\mathbb{N}} \rightrightarrows \mathbb{N}, (U_n)_n \mapsto \{m : [0,1] \subseteq \bigcup_{n=0}^{m} U_n\}$,
- $\mathrm{dom}(\mathrm{HBT}_0) := \{(U_n)_n : [0,1] \subseteq \bigcup_{n=0}^{\infty} U_n\}$.

Proposition

HBT_0 *is computable.*

Proof. Just use the classical Heine-Borel Theorem and search for a suitable $m \in \mathbb{N}$. $\square$

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Theorem (Heine-Borel - contrapositive form)

$$(\forall m \in \mathbb{N})\ [0,1] \not\subseteq \bigcup_{n=0}^{m} U_n \Longrightarrow [0,1] \not\subseteq \bigcup_{n=0}^{\infty} U_n.$$

Formalize this as

- $\mathsf{HBT}_1 :\subseteq \mathcal{O}([0,1])^{\mathbb{N}} \rightrightarrows [0,1], (U_n)_n \mapsto [0,1] \setminus \bigcup_{n=0}^{\infty} U_n$,
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Proposition

$\mathsf{HBT}_1 \equiv_{\mathrm{sW}} \mathsf{WKL}$.

Proof. We obtain $\mathsf{HBT}_1 \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WKL}$. □

- Note that this statement makes a stronger claim than the mere classical reverse mathematics result in that it is uniform and resource sensitive.

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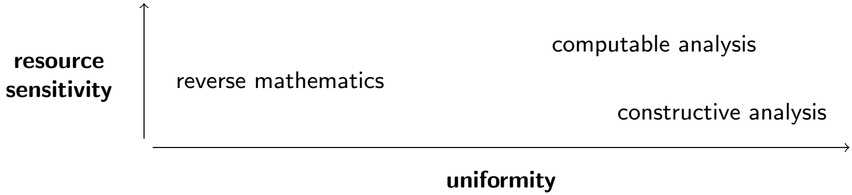
Varieties of Constructivism and Computability

- Reverse mathematics in the Friedman-Simpson style is neither uniform nor resource-sensitive. For instance, products and compositions are allowed. Since classical logic is used, theorems and their contrapositive forms are equivalent.
- Constructive mathematics in Bishop's style is uniform since intuitionistic logic is used, but even less resources sensitive than reverse mathematics since countable and dependent choice is allowed. Certain computable operations are not allowed (Markov's principle, BD-$\mathbb{N}$, etc.).
- Computable analysis in the Weihrauch lattice is fully uniform and resource sensitive. All computable operations are allowed.



Vitali Covering Theorem

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- A point $x \in \mathbb{R}$ is captured by a sequence $\mathcal{I} = (I_n)_n$ of open intervals, if for every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ with $\operatorname{diam}(I_n) < \varepsilon$ and $x \in I_n$.
- $\mathcal{I}$ is a Vitali cover of $A \subseteq \mathbb{R}$, if every $x \in A$ is captured by $\mathcal{I}$.
- $\mathcal{I}$ eliminates A, if the I_n are pairwise disjoint and $\lambda(A \setminus \bigcup \mathcal{I}) = 0$ (where λ denotes the Lebesgue measure).

Theorem (Vitali Covering Theorem)

If $\mathcal{I}$ is a Vitali cover of $[0,1]$, then there exists a subsequence $\mathcal{J}$ of $\mathcal{I}$ that eliminates $[0,1]$.

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Theorem (Brown, Giusto and Simpson 2002)

Over RCA_0 *the Vitali Covering Theorem is equivalent to Weak Weak Kőnig's Lemma* WWKL_0.

- Weak Weak Kőnig's Lemma is Weak Kőnig's Lemma restricted to trees whose set of infinite paths has positive measure.

Theorem (Diener and Hedin 2012)

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- $\mathcal{I}$ is called saturated, if $\mathcal{I}$ is a Vitali cover of $\bigcup \mathcal{I} = \bigcup_{n=0}^{\infty} I_n$.

Definition (Contrapositive versions of the Vitali Covering Theorem)

- VCT_0: Given a Vitali cover $\mathcal{I}$ of $[0,1]$, find a subsequence $\mathcal{J}$ of $\mathcal{I}$ that eliminates $[0,1]$.
- VCT_1: Given a saturated $\mathcal{I}$ that does not admit a subsequence that eliminates $[0,1]$, find a point that is not covered by $\mathcal{I}$.
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- $\mathsf{VCT}_0 : (A \wedge B) \to C$,
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Theorem

- VCT_0 *is computable,*
- $\mathrm{VCT}_1 \equiv_{\mathrm{sW}} \mathrm{WWKL}$,
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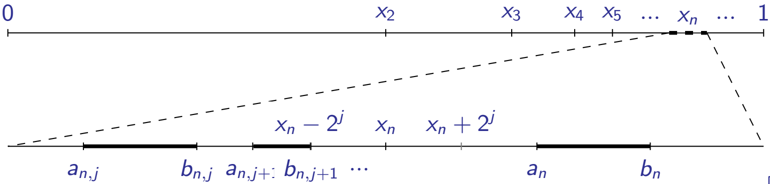
Proof.

- The proof of computability of VCT_0 is based on a construction that repeats steps of the classical proof of the Vitali Covering Theorem (and is not just based on a waiting strategy).
- The proof of $\mathsf{VCT}_1 \equiv_{\mathrm{sW}} \mathsf{WWKL}$ is based on the equivalence chain $\mathsf{VCT}_1 \equiv_{\mathrm{sW}} \mathsf{PC}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WWKL}$.
- We use a Lemma by Brown, Giusto and Simpson on "almost Vitali covers" in order to prove $\mathsf{VCT}_2 \leq_{\mathrm{sW}} \mathsf{WWKL} \times \mathsf{C}_{\mathbb{N}}$. The harder direction is the opposite one for which it suffices to show $\mathsf{C}_{\mathbb{N}} \times \mathsf{VCT}_2 \leq_{\mathrm{sW}} \mathsf{VCT}_2$ by an explicit construction:



$\square$

$\mathsf{C}_{\mathbb{N}}$ as the Weihrauch Lattice Counterpart of $\mathsf{I}\Sigma^0_1$

- $\mathsf{I}\Sigma^0_n$ (Σ^0_n–induction) corresponds to the least number principle $\mathsf{L}\Pi^0_n$ over a very weak system (Hájek, Pudlák 1993).
- $\mathsf{L}\Pi^0_1$ directly translates into the problem:

$$\mathsf{min}^{\mathrm{c}} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, p \mapsto \mathsf{min}\{n \in \mathbb{N} : (\forall k)\ p(k) \neq n\}$$

- It is easy to see that $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{min}^{\mathrm{c}}$.
- Hence $\mathsf{C}_{\mathbb{N}}^{(n)}$ can be seen as the Weihrauch lattice counterpart of $\mathsf{I}\Sigma^0_{n+1}$.

Vitali Covering Theorem in the Weihrauch Lattice

$$\begin{tikzcd}
\mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{WKL} \times \mathsf{C}_{\mathbb{N}} \arrow[dr] \arrow[dd] & & \\
 & \mathsf{VCT}_2 \equiv_{\mathrm{sW}} \mathsf{PC}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{WWKL} \times \mathsf{C}_{\mathbb{N}} \arrow[dd] \arrow[ddr] & \\
\mathsf{HBT}_1 \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WKL} \arrow[dr] \arrow[dddr] & & \\
 & \mathsf{VCT}_1 \equiv_{\mathrm{sW}} \mathsf{PC}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WWKL} \arrow[d] & \mathsf{C}_{\mathbb{N}} \arrow[ddl, dashed] \\
 & \mathsf{ACT} \equiv_{\mathrm{sW}} *\text{-}\mathsf{WWKL} \arrow[d, dashed] & \\
 & \mathsf{VCT}_0 &
\end{tikzcd}$$

Epilogue

Should we Consider Countable or Arbitrary Covers?

- Constructive and computable versions of covering theorems usually deal with countable covers, not arbitrary covers.
- However, this is *not* an artefact caused by codings.
- Most relevant spaces in analysis are actually (heriditarily) Lindelöf spaces, i.e., any open cover has a countable subcover.
- Hence classically the "countable cover" and "arbitrary cover" versions are equivalent for such spaces.
- There is an ontological problem in giving a meaning to an expression such as "given an arbitrary open cover".
- While type-2 objects, even though infinite, can still be presented in a concrete way, type-3 objects are rather elusive.
- This difference is the *common reason* for using countable covers and codings and one is not a consequence of the other.
- This discussion can help to identify such subtle points, but one should not jump to conclusions too quickly.

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- Vasco Brattka, Guido Gherardi, Arno Pauly, Weihrauch Complexity in Computable Analysis, arXiv 1707.03202 (2017)
- Vasco Brattka, Guido Gherardi, Rupert Hölzl, The Vitali Covering Theorem in the Weihrauch Lattice in: Day et al. (eds.), Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday, vol. 10010 of LNCS, Springer, Cham, 2017, 188–200
- Vasco Brattka, Guido Gherardi and Rupert Hölzl, Probabilistic Computability and Choice, *Information and Computation* 242 (2015) 249–286
- Vasco Brattka, Tahina Rakotoniaina, On the Uniform Computational Content of Ramsey's Theorem, *Journal of Symbolic Logic* (accepted for publication) (2017)
- Vasco Brattka, Rupert Hölzl, Rutger Kuyper, Monte Carlo Computability, in: Vollmer et al. (eds.), STACS 2017, vol. 66 of LIPIcs (2017) 17:1-17:14