Mathematisches Institut LMU https://www.math.lmu.de/~nam/Numerics24.php
Phan Thành Nam, Giao Ky Duong SoSe 2024

Numerics II

Homework Sheet 11

(Released 11.7.2024 – Discussed 16.7.2024)

E11.1 Let A be a positive definite operator A on a finite dimensional Hilbert space V.

(a) Prove that

$$\|A\|_{\text{op}} = \sup_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\|x\|^2}.$$

(b) Prove that $\|A^{-1}\|_{\text{op}} = \lambda_1^{-1}$ where λ_1 is the smallest eigenvalue of A.

E11.2 Let $T \subset \mathbb{R}^2$ be a triangle with the vertices p_1, p_2, p_3 and the edge-midpoints

$$m_1 = \frac{p_2 + p_3}{2}, \quad m_2 = \frac{p_1 + p_3}{2}, \quad m_3 = \frac{p_1 + p_2}{2}.$$

Let v be a linear function. Prove that

$$\frac{1}{|T|} \int_T v(x)^2 \mathrm{d}x = \frac{v(m_1)^2 + v(m_2)^2 + v(m_3)^2}{3}.$$

E11.3 Prove that for all $m \in \mathbb{N}$ we have

$$\sup_{t \in [0,1]} t(1-t)^m \leq \frac{1}{m}.$$

E11.4 Recall the notation from the multigrid method: A_k is a positive definite operator on V_k, $|||v|||_{t,k}^2 = \langle A_k^t v, v \rangle_k$, $R_k = 1 - A_k / \Lambda_k$ with $\Lambda_k \geq \|A_k\|_{\text{op}} > 0$. Prove that

$$|||R_k^m v|||_{1+t,k} \leq C_t m^{-1/2} |||v|||_{t,k}$$

for every $t \geq 0$. (The case $t = 1$ was discussed in the lecture.)

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Homework Sheet 10

(Released 27.6.2024 – Discussed 2.7.2024)

E10.1 Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Define

$$V = \{f \in L^2(\Omega) \mid \int_\Omega f = 0\}.$$

(a) Prove that V is a closed subspace of $L^2(\Omega)$.

(b) What is the dimension of $V^\perp$, with $L^2(\Omega) = V \oplus V^\perp$?

(c) Prove that $C_c^\infty(\Omega) \cap V$ is dense in V (with the L^2-topology).

E10.2 Let D be the unit disk in $\mathbb{R}^2$. Let

$$u(x) = u(x_1, x_2) = x_1 x_2 \ln(|x|), \quad x = (x_1, x_2) \in D.$$

(a) Prove that $\Delta u \in L^\infty(D)$.

(b) Does it hold $u \in W^2_\infty(D)$?

E10.3 In polar coordinate, let

$$\Omega = \{(r, \theta) \in (0,1) \times (0, \pi)\} \subset \mathbb{R}^2, \quad \Gamma = \partial\Omega \setminus \{(r, \pi), r \in (0,1)\}.$$

(a) Prove that $u(r, \theta) = (1 - r^2) r^{\frac{1}{2}} \sin\left(\frac{\theta}{2}\right)$ solves the the following

$$\begin{cases} -\Delta u &= 6 r^{\frac{1}{2}} \sin(\frac{\theta}{2}) \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma, \\ \frac{\partial u}{\partial \eta} &= 0 \text{ on } \partial\Omega \setminus \Gamma. \end{cases}$$

(b) Does u belong to $H^2(\Omega)$?

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Homework Sheet 9

(Released 20.6.2024 – Discussed 25.6.2024)

E9.1 Let $\{\mathcal{T}^h\}$ be a family of triangulations of a given domain $\Omega \subset \mathbb{R}^2$. Show that $\{\mathcal{T}^h\}$ is non-degenerate if and only if all the angles of the triangles in $\{\mathcal{T}^h\}$ are bounded below by a positive constant θ, namely

$$\inf_{T\in\mathcal{T}^h} \alpha(T) \geq \theta > 0$$

where $\alpha(T)$ is the smallest angle of the triangle T.

E9.2 Let $\{\mathcal{T}^h\}$ be a triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$ with

$$\max_{T\in\mathcal{T}^h} \operatorname{diam}(T) = h, \quad \inf_{T\in\mathcal{T}^h} \alpha(T) \geq \theta > 0.$$

For every $u \in H^2(\Omega)$, denote by $\mathcal{I}(u) \in V_h$ the interpolant of u. Prove that

$$\|u - \mathcal{I}(u)\|_{L^2(\Omega)} + h\|\nabla(u - \mathcal{I}(u))\|_{L^2(\Omega)} \leq C_\theta h^2 \|u\|_{H^2(\Omega)}.$$

E9.3 Let Ω is a ball in $\mathbb{R}^d$, and $1 \leq p < \infty$. Prove that the following are equivalent

$$\|u\|_{L^p(\Omega)} \leq C\left(\left|\int_\Omega u dx\right| + |u|_{W_p^1(\Omega)}\right), \forall u \in W_p^1(\Omega),$$

and

$$\|u - \frac{1}{|\Omega|}\int_\Omega u dx\|_{W_p^1(\Omega)} \leq C\,|u|_{W_p^1(\Omega)}, \forall u \in W_p^1(\Omega).$$

E9.4 Let $K = [0,1]$, and $V(K) = L^2(K)$. Let $(K, \mathcal{P}, \mathcal{N})$ is a finite element with $|\mathcal{N}| = \ell$. Let $\mathcal{A} = (a_{i,j})_{1\leq i,j\leq \ell}$ be an invertible square matrix, and we define

$$\tilde{\mathcal{N}} = \{\tilde{N}_i, 1 \leq i \leq \ell\} \text{ with } \tilde{N}_i = \sum_{j=1}^{\ell} a_{i,j} N_j, \text{ where } N_j \in \mathcal{N}.$$

Prove that $(K, P, \tilde{\mathcal{N}})$ is a finite element. Express the nodal basis $\{\tilde{\phi}_i\}$ in term of the nodal basis $\{\phi_i\}$ associating to $(K, \mathcal{P}, \mathcal{N})$. Verify that $\tilde{I}_K(v) = I_K(v)$ for all $v \in V(K)$.

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Homework Sheet 8

(Released 12.6.2024 – Discussed 18.6.2024)

E8.1 Let $\Omega \subset \mathbb{R}^2$ be the the unit square with vertices $z_1 = (0,0)$, $z_2 = (1,0)$, $z_3 = (1,1)$, $z_4 = (0,1)$. Consider the finite element $(\Omega, \mathcal{Q}_1, \mathcal{N}_1)$ where $\mathcal{Q}_\infty$ contains finite linear combinations of functions $(x + \text{const})(y + \text{const})$ and $\mathcal{N}_1 = (N_i)_{i=1}^4$ be the corresponding Lagrange element, namely

$$N_i(v) = v(z_i), \quad \forall i = 1,2,3,4.$$

Compute the nodal basis $(\phi_i)_{i=1}^4 \subset \mathcal{Q}_1$ such that

$$N_i(\phi_j) = \delta_{ij}, \forall i,j = 1,2,3,4.$$

E8.2 Let $f \in C^m(\mathbb{R})$. Prove the Taylor expansion

$$f(1) = \sum_{k=0}^{m-1} \frac{1}{k!} f^{(k)}(0) + \int_0^1 \frac{s^{m-1}}{(m-1)!} f^{(m)}(1-s)\mathrm{d}s.$$

E8.3 For $u \in C^{m-1}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, denote the Taylor polynomial

$$T_y^m u(x) = \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha.$$

Prove that for all $|\beta| < m$ we have

$$D_x^\alpha T_y^m u(x) = T_y^{m-|\beta|} D_x^\beta u(x).$$

E8.4 Let $B \subset \mathbb{R}^d$ be a ball of radius R. Prove that for $1 < p < \infty$ and $m - d/p = s > 0$, then

$$\left| \int_B |x-y|^{m-d} f(y)\mathrm{d}y \right| \le C_{d,p} R^s \|f\|_{L^p(B)}.$$

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Numerics II

Homework Sheet 7

(Released 7.6.2024 – Discussed 11.6.2024)

E7.1 Let $K = [0,1]$, $\mathcal{P}_2 = \{\text{polynomials of degree} \le 2\}$, and $\mathcal{N} = \{N_1, N_2, N_3\} \subset \mathcal{P}_2'$ with

$$N_1(v) = v(0), \quad N_2(v) = 2v(1/2) - v(0) - v(1), \quad N_3(v) = v(1),$$

for all $v \in \mathcal{P}_2$. Prove that $(K, \mathcal{P}, \mathcal{N})$ is a finite element. Determine the nodal basis $\{\phi_1, \phi_2, \phi_3\}$ for $\mathcal{P}_2$, namely $N_i(\phi_j) = \delta_{ij}$.

E7.2 Let $K = [0,1]$, $\mathcal{P} = \{v \in C^1(K) : \text{piecewise quadratic polynomial on } [0, \frac{1}{2}] \text{ and } [\frac{1}{2}, 1]\}$ Define $\mathcal{N} = \{N_1, N_2, N_3, N_4\}$ where

$$N_1(v) = v(0); \quad N_2(0) = v'(0) \quad N_3(v) = v(1), \text{ and } N_4(v) = v'(1).$$

Prove $(K, \mathcal{P}, \mathcal{N})$ is a finite element. Determine the nodal basis for $\mathcal{P}$.

E7.3 Let K be the triangle with vertices $z_1 = (0,0)$, $z_2 = (1,0)$, $z_3 = (1,0)$. Let $\mathcal{P}_2 = \{\text{polynomials of 2 variables of degree} \le 2\}$ and $\mathcal{N}_2 = \{N_1, N_2, N_3, N_4, N_5, N_6\}$ be the Lagrange element.

(a) Compute the nodal basis of $\mathcal{P}_2$.

(b) Compute the interpolant $\mathcal{I}_K f$ where $f(x,y) = e^{xy}$.

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Numerics II

Homework Sheet 6

(Released 23.5.2024 – Discussed 28.5.2024)

E6.1 Let $V = H^1(0,1)$, $f \in L^2(0,1)$ and $k \in \mathbb{R}$. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be the bilinear form associated to the equation

$$-u'' + ku' + u = f$$

with Neumann boundary condition.

(a) Prove that a is continuous for all $k \in \mathbb{R}$.

(b) Prove that a is coercive *if and only if* $|k| < 2$.

(c) For $k = 2$, find $0 \neq v \in H^1(0,1)$ such that $a(v,v) = 0$.

(d) Prove for $k = 2$, $V_0 = H_0^1(0,1)$ that the restriction $a(\cdot,\cdot) : V_0 \times V_0 \to \mathbb{R}$ is coercive.

E6.2 Let V be a Hilbert space and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bilinear form which is symmetric, continuous, and coercive. Let $F \in V'$ and $u \in V$. Prove that the following statements are equivalent:

(a) $a(u,v) = F(v)$ for all $v \in V$.

(b) u is the minimizer for the functional $g : V \to R$ defined by

$$g(v) = \frac{1}{2}a(v,v) - F(v).$$

E6.3 Let $\mathcal{P}_k$ be the set of all polynomials in two variables of degree $\leq k$. Prove that

$$\dim \mathcal{P}_k = \frac{1}{2}(k+1)(k+2).$$

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Homework Sheet 5

(Released 16.5.2024 – Discussed 22.5.2024)

E5.1 Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Recall that

$$\overset{\circ}{W}{}_p^1(\Omega) = \{u \in W_p^1(\Omega) \mid u_{|\partial\Omega} = 0\}, \quad 1 < p < \infty.$$

(a) Prove that $\overset{\circ}{W}{}_p^1(\Omega)$ is a closed subspace of $W_p^1(\Omega)$.

(b) Prove that if $u \in W_p^1(\Omega)$, then $u \in \overset{\circ}{W}{}_p^1(\Omega)$ if and only if for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\max_{i=1,\dots,d} \left| \int_\Omega u \partial_{x_i} \varphi(x) \mathrm{d}x \right| \leq C \|\varphi\|_{L^{p'}(\mathbb{R}^d)}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and the constant C is independent of φ.

E5.2 Let $k > d \geq 1$ and $1 < p < \infty$. Define the Dirac delta mapping $\delta_0 : W_p^k(\mathbb{R}^d) \to \mathbb{R}$ by

$$\delta_0(f) = f(0), \quad \forall f \in W_p^k(\mathbb{R}^d).$$

(a) Prove that $\delta_0 \in W_p^{-k}(\Omega)$.

(b) Prove that $\delta_0 \notin L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

E5.3 Let $f \in L^2(0,1)$ and let $u \in V = W_2^1(0,1)$ be the unique solution to the variational problem

$$\int (u'v' + uv) = \int_0^1 fv, \quad \forall v \in V$$

(the existence of u was proved in the lecture).

(a) Prove that $u \in W_2^2(0,1)$ and the weak derivative u'' solves the equation

$$-u''(x) + u(x) = f(x), \quad a.e. \quad x \in (0,1).$$

(b) Prove that $u'(0) = u'(1) = 0$.

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Homework Sheet 4

(Released 10.5.2024 – Discussed 14.5.2024)

E4.1 Consider the function $f : (-1,1) \to \mathbb{R}$

$$f(x) = \begin{cases} x, & x < 0 \\ 1, & x > 0. \end{cases}$$

(a) Prove that

$$\inf_{g \in C(-1,1)} \|g - f\|_{L^\infty(-1,1)} = 1/2.$$

(b) Does f has a weak derivative f' in $L^1(-1,1)$?

E4.2 Let $u \in W_1^2(0,1)$ with $u(0) = u'(0) = 0$. Prove that $v(x) = u(x)/x$ belongs to $W_1^1(0,1)$. What is $v(0)$?

E4.3 As in the lecture, we consider

$$Q = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \,|\, |x'| < 1, |x_d| < 1\}, \quad Q_+ = \{(x', x_d) \in Q \,|\, x_d > 0\}.$$

For any function $f : Q_+ \to \mathbb{R}$, we define $f^*, \widetilde{f} : Q \to \mathbb{R}$ by

$$f^*(x', x_d) = \begin{cases} f(x', x_d), & x_d > 0 \\ f(x', -x_d), & x_d \le 0 \end{cases} \quad , \quad \widetilde{f}(x', x_d) = \begin{cases} f(x', x_d), & x_d > 0 \\ -f(x', -x_d), & x_d \le 0 \end{cases}$$

Prove that if $u \in W_p^1(Q_+)$ with $1 \le p \le \infty$, then we have the weak derivative

$$\frac{\partial u^*}{\partial x_d} = \widetilde{\left(\frac{\partial u}{\partial x_d}\right)}$$

E4.4 Let $u \in W_p^1(0,1)$ with $1 < p < \infty$ such that $u(0) = u(1) = 0$. We define the extension $\tilde{u} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{u}(x) = \begin{cases} u(x), & x \in (0,1), \\ 0, & x \notin (0,1). \end{cases}$$

Prove that $\tilde{u} \in W_p^1(\mathbb{R})$ and $\|\tilde{u}\|_{W_p^1(\mathbb{R})} = \|u\|_{W_p^1(0,1)}$.

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Homework Sheet 3

(Released 2.5.2024 – Discussed 7.5.2024)

E3.1 Show that $C^1(0,1)$ is not dense in L^∞, with L^∞ norm.

E3.2 Let $f \in W^{1,1}(0,1) \cap C([0,1])$. Assume that we can find $\{f_n\} \subset C_c^\infty(0,1)$ such that $f_n \to f$ in $W^{1,1}(0,1)$. Prove that $f(0) = f(1) = 0$. Deduce that $C_c^\infty(0,1)$ is not dense in $W^{1,1}(0,1)$.

E3.3 Construct a function in $W^{1,2}(0,1)$, but not in $W^{2,2}(0,1)$ and explain the answer.

E3.4 Let $\Omega = \{(x,y) \in \mathbb{R}^2, 0 < x < 1, |y| < x^{10}\}$ and define

$$u(x,y) = x^{-\frac{1}{2}}.$$

(i) Prove that $u \in L^2(\Omega)$.
(ii) Prove that the weak derivatives $\partial_x u$, $\partial_y u$ exist and determinate them.
(iii) Prove that $u \in W^{1,2}(\Omega)$ and $u \notin L^\infty(\Omega)$.

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Homework Sheet 2

(Released 24.4.2024 – Discussed 30.4.2024)

E2.1 In the lecture, we proved that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} f\varphi = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ then $f = 0$ a.e. Prove the same result with $\mathbb{R}^d$ replaced by a general open subset $\Omega \subset \mathbb{R}^d$.

E2.2 In this exercise we consider functions mapping from $\Omega \to \mathbb{R}$ with $\Omega = (-1, 1)$.

(i) Prove that $f(x) = |x|^{\frac{3}{2}} - 1$ has a weak derivative in $L^1_{\text{loc}}(\Omega)$.

(ii) Is there a weak derivative in $L^1_{\text{loc}}(\Omega)$ of the following function?

$$g(x) = \begin{cases} x & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

E2.3 Consider the Cantor set $C_\infty = \bigcap_{k=1}^\infty C_k$, where C_k satisfy the following recurrence

$$C_0 = [0, 1], \quad C_{k+1} = \frac{1}{3}C_k \cup \left\{ \frac{2}{3} + \frac{1}{3}C_k \right\}, k \geq 1.$$

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0, 1] \backslash C_\infty, \\ 2x + 1 & \text{if} \quad x \in C_\infty. \end{cases}$$

(i) Find $\sup f, \inf f$, and $\|f\|_{L^\infty}$.

(ii) Find its weak derivative f' and compute

$$f(x) - \int_0^x f'(s)ds.$$

E2.4 Let $u \in L^1_{\text{loc}}(\mathbb{R})$ which has a second order weak derivative $f'' \in L^1_{\text{loc}}(\mathbb{R})$. Show that it has also the first order weak derivative $f' \in L^1_{\text{loc}}(\mathbb{R})$.

E2.5 Let $f \in W^{1,p}(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi(x) = 1$ if $|x| \leq 1$. Denote $f_n(x) = \chi(nx)f(x)$. Prove that

$$f_n \to f \text{ in } W^{1,p}(\mathbb{R}^d) \text{ as } n \to +\infty.$$

Recall that $\|f\|^p_{W^{1,p}} = \|f\|^p_{L^p} + \sum_{|\alpha|=1} \|D^\alpha f\|^p_{L^p}$.

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Numerics II

Homework Sheet 1

(Released 18.4.2024 – Discussed 23.4.2024)

E1.1 Define

$$V = \{u \in L^2(0,1) \,|\, a(u,u) < \infty \text{ and } u(0) = 0\}, \quad a(u,v) = \int_0^1 u'(x)v'(x)\mathrm{d}x.$$

(i) Prove the Cauchy-Schwarz inequality

$$|a(u,v)| \le \sqrt{a(u,u)}\sqrt{a(v,v)}, \quad \forall u,v \in V.$$

When does the equality hold?

(ii) Define $\|u\|_E = \sqrt{a(u,u)}$. Verify that $\|\cdot\|_E$ is a norm. If we don't have the condition $u(0) = 0$, is $\|\cdot\|_E$ a norm?

(iii) Prove the "Parallelogram law"

$$\|u\|_E + \|v\|_E = \frac{1}{2}\left[\|u+v\|_E^2 + \|u-v\|_E^2\right] \quad \forall u,v \in V.$$

E1.2 Based on the lecture, formulate the weak formulation of the following problem

$$\begin{cases} -u'' + u &= f, \\ u(0) = u(1) &= 0, \end{cases}$$

where $f \in L^2[0,1]$.

E1.3 Let V and $a(u,v)$ be given in E1.1. Prove that for $u \in V \cap C^1[0,1]$ we have

(i) $\|u\|_{L^2(0,1)}^2 \le \frac{1}{2}\|u'\|_{L^2(0,1)}^2$

(ii) $\|u\|_{L^2(0,1)}^2 \le \frac{1}{\sqrt{8}}\|u'\|_{L^2(0,1)}^2$ if we assume further $u(1) = 0$

(iii) $\|u\|_{L^2(0,1)}^2 \le \frac{1}{6}\|u'\|_{L^2(0,1)}^2$ if we assume further $\int_0^1 u = 0$

(iv) $\max_{x\in[0,1]} |u(x)|^2 \le 2u^2(1) + 2\|u'\|_{L^2}^2$

(v) $\max_{x\in[0,1]} |u(x)|^2 \le 2(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2)$