Derivation of the Bose–Einstein condensation for trapped bosons

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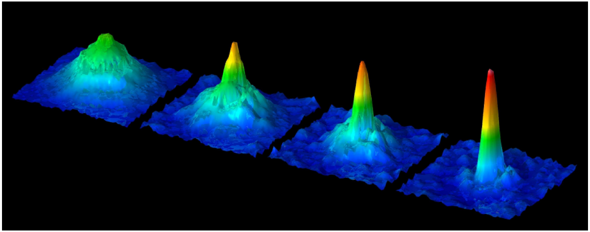
Bose-Einstein condensation

In 1995, the **Bose-Einstein condensation** (BEC) was observed in experiments:

- Many bosons occupy a common quantum state at low temperatures
- Macroscopic quantum effects: superfluidity, quantized vortices, ...



Cornell, Wieman, Ketterle (2001 Nobel Prize in Physics)

Mathematical analysis of **Bose** and **Einstein** (1924-25) for **non-interacting gas**

$$\frac{N_0}{N} = \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]_+$$

Our goal: Rigorous understanding of the BEC for **interacting systems**

Many-body quantum mechanics

Wave function Ψ_N: a normalized function in $L^2((\mathbb{R}^3)^N)$

- $|\Psi_N|^2$ position probability density, $|\widehat{\Psi}_N|^2$ momentum probability density
- Indistinguishable particles/bosonic symmetry

$$\Psi_N(..., x_k, ..., x_\ell, ...) = \Psi_N(..., x_\ell, ..., x_k, ...)$$

Hamiltonian H_N: a self-adjoint operator on $L^2_s((\mathbb{R}^3)^N)$

$$\langle \Psi_N, H_N \Psi_N \rangle = \text{energy expectation}$$

Schrödinger equations

$$H_N \Psi_N = E_N \Psi_N \quad \text{(stationary states)}$$

$$i\partial_t \Psi_N(t) = H_N \Psi_N(t) \quad \text{(dynamical states)}$$

Schrödinger equations are **linear** but have **too many variables**. For practical computation, people rely on effective equations which are **nonlinear** but have **fewer variables**. Our task is to justify these approximations when $N \to \infty$

A dilute Bose gas

Consider a quantum gas of N bosons in $\mathbb{R}^3$ described by the Hamiltonian

$$H_N = \sum_{i=1}^{N}(-\Delta_{x_i} + U(x_i)) + \sum_{i<j}^{N} V_N(x_i - x_j) \quad \text{on} \quad L^2_s((\mathbb{R}^3)^N)$$

- External potential $0 \le U \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$ is trapping

$$U(x) \to \infty, \quad |x| \to \infty$$

- Interaction potential $0 \le V_N \in C^\infty_c(\mathbb{R}^3)$ is short-range

$$V_N(x) = N^2 V(Nx) \sim \frac{b}{N}\delta_0(x)$$

The Hamiltonian H_N is bounded from below and can be extended to be a self-adjoint operator on $L^2_s(\mathbb{R}^3)$ by Friedrichs's method. We are interested in the **stationary solutions**

$$H_N \Psi_N = E_N(k)\Psi_N$$

where $E_N(k)$ is the k-th lowest eigenvalue of H_N. In particular, when $k = 1$ we have the **ground state energy**

$$E_N = \inf \mathrm{spec}(H_N) = \inf_{\|\Psi_N\|_{L^2}=1} \langle \Psi_N, H_N \Psi_N \rangle$$

Gross–Pitaevskii theory

Mean–field approximation: restriction to uncorrelated states

$$u^{\otimes N}(x_1, ..., x_N) = u(x_1)...u(x_N)$$

Formally replacing V_N by $(b/N)\delta_0$ leads to the **Gross–Pitaevskii functional**

$$\mathcal{E}_{\rm GP}(u) = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + U|u|^2 + \frac{b}{2}|u|^4 \right)$$

The variational problem

$$e_{\rm GP} = \inf_{\|u\|_{L^2(\mathbb{R}^3)}=1} \mathcal{E}_{\rm GP}(u)$$

has a **unique minimizer** $u_0 \geq 0$ which solves the nonlinear **Gross–Pitaevskii equation**

$$-\Delta u_0 + U u_0 + b|u_0|^2 u_0 = \mu_0 u_0, \quad \mu \in \mathbb{R}.$$

By standard regularity theory, u_0 is smooth if U is smooth

Scattering length

In the formal approximation

$$V_N(x) = N^2 V(Nx) \sim \frac{b}{N}\delta_0(x)$$

a natural guest is $b = \int V$. However, this choice does not describes correctly the dilute Bose gas because strong correlations at short distances lead to a **nonlinear correction**. The right choice is the scattering energy

$$b = \inf\left\{ \int_{\mathbb{R}^3} 2|\nabla f|^2 + V|f|^2, \quad \lim_{|x|\to\infty} f(x) = 1 \right\}$$

The unique minimizer $0 \le f \le 1$ solves the zero–scattering equation

$$(-2\Delta + V)f = 0, \quad f(x) = 1 - \mathfrak{a}|x|^{-1} + o(|x|^{-1})_{|x|\to\infty}$$

Equivalently $b = 8\pi\mathfrak{a}$ with $\mathfrak{a}$ called the **scattering length** of V

- If V is the **hard sphere potential** of $B(0,R)$, then $\mathfrak{a} = R$
- If V is smooth, we have **Born's series**

$$8\pi\mathfrak{a} = \int_{\mathbb{R}^3} Vf = \int_{\mathbb{R}^3} V - \int_{\mathbb{R}^3} V(2\Delta + V)^{-1}V = \ldots$$

- The scattering length of $V_N = N^2 V(N\cdot)$ is $\mathfrak{a}/N$

Main result

Consider the N-body Hamiltonian

$$H_N = \sum_{i=1}^{N}(-\Delta_{x_i} + U(x_i)) + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \quad \text{on} \quad L^2_s((\mathbb{R}^3)^N)$$

and the Gross–Pitaevskii functional

$$\mathcal{E}_{\mathrm{GP}}(u) = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + U|u|^2 + 4\pi\mathfrak{a}|u|^4\right)$$

Theorem (N.–Napiórkowski–Ricaud–Triay 2020, arXiv:2001.04364)

Assume $0 \le U \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$, $U(x) \to \infty$ as $|x| \to \infty$. Assume $0 \le V \in L^3(\mathbb{R}^3)$ radial, compactly supported with the scattering length $\mathfrak{a} > 0$ small. Then:

- The eigenvalues of H_N satisfies $|E_N(k) - Ne_{\mathrm{GP}}| \le C_k$ for any fixed $k \in \mathbb{N}$
- The eigenfunctions of H_N satisfies the Bose–Einstein condensation

$$\left\langle \Psi_N, \sum_{i=1}^{N} P_i \Psi_N \right\rangle = N + \mathcal{O}(1).$$

Here $P = |u_0\rangle\langle u_0|$ with u_0 the unique Gross–Pitaevskii minimizer.

History

- Leading order ground state energy: **Lieb–Seiringer–Yngvason** ('00)

$$E_N = Ne_{\mathrm{GP}} + o(N)$$

- Leading order BEC: **Lieb–Seiringer** ('02-06), **N.–Rougerie–Seiringer** ('16)

$$\left\langle \Psi_N, \sum_{i=1}^{N} P_i \Psi_N \right\rangle = N + o(N)$$

- Dynamical results: **Erdös–Schlein–Yau** ('09-10), **Benedikter–de Oliveira–Schlein** ('14), **Pickl** (2015), **Brennecke–Schlein** ('19)

If $\Psi_N \approx u(0)^{\otimes N}$, then $\Psi_N(t) = e^{-itH_N}\Psi_N \approx u(t)^{\otimes N}$ with GP equation

$$i\partial_t u(t,x) = (-\Delta_x + U(x) + 8\pi\mathfrak{a}|u(t,x)|^2)u(t,x)$$

E. H. Lieb, R. Seiringer, J. Yngvason. Bosons in a trap: A rigorous derivation of the Gross–Pitaevskii energy functional. Phys. Rev. A 61 (2000).

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Homogeneous case

In the simpler case when the particles live in the torus $\mathbb{T}^3 = [0,1]^3$ with $U = 0$, the system is **translation-invariant** and the Gross–Pitaevskii minimizer is

$$u_0(x) = 1, \quad \forall x \in [0,1]^3.$$

Next order is known by **Boccato-Brennecke-Cenatiempo-Schlein** ('19)

$$E_N = 4\pi\mathfrak{a}N - \sum_{0 \neq p \in 2\pi\mathbb{Z}^3} \left[p^2 + 8\pi\mathfrak{a} - \sqrt{p^4 + 16\pi a p^2} - \frac{(8\pi\mathfrak{a})^2}{2p^2} \right] + c_a + o(1)_{N\to\infty}$$

This is related to the **Lee–Huang–Yang formula** for the thermodynamic energy

$$\lim_{\substack{N\to\infty \\ N/\mathrm{Vol}=\rho}} \frac{E_N}{N} = 4\pi\mathfrak{a}\rho \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho\mathfrak{a}^3} + o(1)_{\rho\mathfrak{a}^3\to 0} \right)$$

Dyson (1957), **Lieb-Yngvason** ('98), **Yau-Yin** (2009), **Fournais-Solovej** ('19)

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H.-T. Yau, J. Yin. The Second Order Upper Bound for the Ground Energy of a Bose Gas. J. Stat. Phys. 136, (2009)

C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein. Bogoliubov Theory in the Gross–Pitaevskii Limit. Acta Math. 222 (2019)

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Upper bound

Applying the variational principle to **uncorrelated states** $\Psi_N = u^{\otimes N}$ gives

$$\frac{E_N}{N} \leq \inf_{\|u\|_{L^2}=1} \int_{\mathbb{R}^3} |\nabla u|^2 + U|u|^2 + \frac{1}{2}(NV_N * |u|^2)|u|^2,$$

not enough as $NV_N = N^3 V(Nx) \rightharpoonup \delta_0 \int V$ and $\int V > 8\pi\mathfrak{a}$. **A better choice** is

$$\Psi_N(x_1, ..., x_N) = \prod_{j=1}^{N} u(x_j) \prod_{k<\ell}^{N} f_N(x_k - x_\ell)$$

where

$$f_N(x) = f(Nx), \quad (-\Delta + \frac{1}{2}V_N)f_N(x) = 0, \quad \lim_{|x|\to\infty} f_N(x) = 1$$

Since the probability to have 3 particles very close ($\mathcal{O}(N^{-1})$) is negligible,

$$\frac{\langle \Psi_N, H_N \Psi_N \rangle}{N} \leq \inf_{\|u\|_{L^2}=1} \int_{\mathbb{R}^3} |\nabla u|^2 + U|u|^2 + \frac{1}{2}(g_N * |u|^2)|u|^2 + o(1)$$

where $g_N = N^3 g(Nx) \rightharpoonup \delta_0 \int g, \quad g = 2|\nabla f|^2 + V|f|^2, \quad \int g = 8\pi\mathfrak{a}$

This gives $E_N \leq N e_{\rm GP} + o(N)$. The bound $E_N \leq N e_{\rm GP} + \mathcal{O}(1)$ needs **a refined trial state** constructed using **Bogoliubov transformations** on Fock space.

Second quantization

To describe particles outside the condensate, we work on **Fock space**

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} L_s^2((\mathbb{R}^3)^n) = \mathbb{C} \oplus L^2(\mathbb{R}^3) \oplus L_s^2((\mathbb{R}^3)^2) \oplus ...$$

For $g \in L^2(\mathbb{R}^d)$, define the **creation and annihilation operators**

$$(a^*(g)\Psi)(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} g(x_j)\Psi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$$

$$(a(g)\Psi)(x_1, \ldots, x_{n-1}) = \sqrt{n} \int \overline{g(x_n)}\Psi(x_1, \ldots, x_n)\mathrm{d}x_n$$

They satisfy the **canonical commutation relations** (CCR)

$$[a(g_1), a(g_2)] = [a^*(g_1), a^*(g_2)] = 0, \quad [a(g_1), a^*(g_2)] = \langle g_1, g_2 \rangle$$

Take an orthonormal basis $\{u_n\}_{n \geq 0}$ for $L^2(\mathbb{R}^d)$ and denote $a_n = a(u_n)$. Then

$$[a_m, a_n] = [a^*m, a^*n] = 0, \quad [a_m, a_n^*] = \langle g_1, g_2 \rangle = \delta_{m=n}$$

Reformulation the problem

The many-body Hamiltonian can be rewritten as

$$H_N = \sum_{m,n\geq 0} h_{mn} a_m^* a_n + \frac{1}{2} \sum_{m,n,p,q\geq 0} W_{mnpq} a_m^* a_n^* a_p a_q$$

where

$$h_{mn} = \langle u_m, (-\Delta + U) u_n \rangle, \quad W_{mnpq} = \iint \overline{u_m(x) u_n(y)} V_N(x-y) u_p(x) u_q(y)$$

The complete BEC can be reformulated as

$$\left\langle \Psi_N, \sum_{i=1}^N (|u_0\rangle\langle u_0|)_i \Psi_N \right\rangle = \left\langle \Psi_N, a_0^* a_0 \Psi_N \right\rangle = N + \mathcal{O}(1)$$

which is equivalent to

$$\left\langle \Psi_N, \mathcal{N}_+ \Psi_N \right\rangle = \mathcal{O}(1), \quad \mathcal{N}_+ := \sum_{n>0} a_n^* a_n.$$

The energy lower bound and BEC in the main theorem follows from the operator bound

$$H_N \geq N e_{\mathrm{GP}} + C^{-1} \mathcal{N}_+ - C$$

Bogoliubov theory

Bogoliubov approximation (1947)

- Ignore all terms with 3 or 4 operators $a^{\#}_{n\neq 0}$ in

$$H_N = \sum_{m,n\geq 0} h_{mn} a_m^* a_n + \frac{1}{2} \sum_{m,n,p,q\geq 0} W_{mnpq} a_m^* a_n^* a_p a_q$$

- Replace any $a_0^{\#}$ by $\sqrt{N_0}$ (**c-number substitution**). Heuristically,

$$N_0 = \langle \Psi_N, a_0^* a_0 \Psi_N \rangle \gg [a_0, a_0^*] = 1$$

- Diagonalize the resulting **quadratic Hamiltonian** by a symplectic/Bogoliubov transformation

$$a_p \mapsto \widetilde{a}_p = \cosh(K)_{p,q} a_q + \sinh(K)_{p,q} a_q^*, \quad [\widetilde{a}_p, \widetilde{a}_q^*] = \delta_{p,q}$$

- Anytime when you see $\int V$, replace it by $8\pi\mathfrak{a}$ (**Landau's correction**)

All this leads to

$$H_N \approx N e_{\rm GP} + e_{\rm Bog} + \sum_{p,q\geq 1} \xi_{p,q} \widetilde{a}_p^* \widetilde{a}_q,$$

The complete BEC follows from the **spectral gap** $\inf \mathrm{spec}_{\{u_0\}^\perp}(\xi) > 0$

Justifying Bogoliubov's approximation is nontrivial!

Lower bound: homogeneous case I

We **complete the square**: on the two-particle space since $V_N \geq 0$,

$$(\mathbb{1} - P \otimes P f_N) V_N (\mathbb{1} - f_N P \otimes P) \geq 0$$

where $V_N = N^2 V(N(x-y))$, $f_N = f(N(x-y))$, $P = |u_0\rangle\langle u_0|$. Consequently,

$$H_N \geq \sum_{p \neq 0} \left(|p|^2 a_p^* a_p + \frac{1}{2} \widehat{f_N V_N}(p) a_p^* a_{-p}^* a_0 a_0 + \frac{1}{2} \widehat{f_N V_N}(p) a_0^* a_0^* a_p a_{-p} \right)$$

$$+ \frac{1}{2} \left(\int (2 f_N - f_N^2) V_N \right) a_0^* a_0^* a_0 a_0$$

Here $a_p = a(e^{ip \cdot x})$, $p \in 2\pi \mathbb{Z}^3$. This implements parts of Bogoliubov's argument:

- First step of removing **cubic and quartic terms** in $a_{p \neq 0}^{\#}$
- Last step of **Landau's correction**: V_N replaced by $f_N V_N$ where

$$\int_{\mathbb{R}^3} f_N V_N = \frac{1}{N} \int_{\mathbb{R}^3} f V = \frac{8\pi \mathfrak{a}}{N}$$

Similar ideas used by **Brietzke–Fournais–Solovej** (2019) for LHY formula

Lower bound: homogeneous case II

The rest of Bogoliubov's argument (**c-number substitution** and **symplectic diagonalization**) can be implemented by **completing the square** again

$$A(b_p^* b_p + b_{-p}^* b_{-p}) + B(b_p^* b_{-p}^* + b_p b_{-p}) \geq (\sqrt{A^2 - B^2} - A)\frac{[b_p, b_p^*] + [b_{-p}, b_{-p}^*]}{2}$$

which is equivalent to $d^* d \geq 0$ for some d (**Lieb–Solovej** 2001). Taking

$$b_p = \frac{a_0^* a_p}{\sqrt{N}}, \quad b_p^* b_p \leq a_p^* a_p, \quad [b_p, b_p^*] \leq 1, \quad \forall 0 \neq p \in 2\pi\mathbb{Z}^3$$

we find that for $0 < \mu < 4\pi^2 - 8\pi\mathfrak{a}$

$$H_N - \frac{1}{2}\left(\int (2f_N - f_N^2) V_N\right) a_0^* a_0^* a_0 a_0 - \mu\mathcal{N}_+$$

$$\geq \frac{1}{2}\sum_{p\neq 0}\Big((|p|^2 - \mu)(b_p^* b_p + b_{-p}^* b_{-p}) + N\widehat{f_N V_N}(p) b_p^* b_{-p}^* + \widehat{f_N V_N}(p) b_p b_{-p}\Big)$$

$$\geq \frac{1}{2}\sum_{p\neq 0}\left(\sqrt{(|p|^2 - \mu)^2 - |N\widehat{f_N V_N}(p)|^2} - |p|^2 + \mu\right) = -\frac{N}{2}\int_{\mathbb{R}^3} Vf(1-f) + \mathcal{O}(1)$$

The last equality follows from the scattering equation $(-2\Delta + V_N)f_N = 0$

Lower bound: homogeneous case III

We have proved that for any $0 < \mu < 4\pi^2 - 8\pi\mathfrak{a}$

$$H_N \geq \mu \mathcal{N}_+ + \frac{1}{2}\left(\int (2f_N - f_N^2) V_N\right) a_0^* a_0^* a_0 a_0 - \frac{N}{2}\int_{\mathbb{R}^3} Vf(1-f) + \mathcal{O}(1)$$

Finally, using

$$a_0^* a_0^* a_0 a_0 = a_0^* a_0 (a_0^* a_0 - 1) = (N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1)$$

we find that

$$H_N \geq (\mu - 16\pi\mathfrak{a})\mathcal{N}_+ + \frac{N}{2}\int (2f - f^2) V - \frac{N}{2}\int_{\mathbb{R}^3} Vf(1-f) + \mathcal{O}(1)$$

In the homogeneous case

$$\frac{1}{2}\int (2f - f^2) V - \frac{1}{2}\int_{\mathbb{R}^3} Vf(1-f) = \frac{1}{2}\int fV = 4\pi\mathfrak{a} = e_{\mathrm{GP}}$$

If $\mathfrak{a} < \pi/6$, we can choose

$$16\pi\mathfrak{a} < \mu < 4\pi^2 - 8\pi\mathfrak{a}$$

and obtain the desired operator lower bound

$$H_N \geq (\mu - 16\pi\mathfrak{a})\mathcal{N}_+ + N e_{\mathrm{GP}} + \mathcal{O}(1)$$

Lower bound: general case I

Using the two-body inequality

$$(\mathbb{1} - P \otimes P f_N) V_N (\mathbb{1} - f_N P \otimes P) \geq 0$$

we obtain

$$H_N \geq \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2$$

where

$$\mathcal{H}_0 = \int_{\mathbb{R}^3} (|\nabla u_0|^2 + U|u_0|^2) a_0^* a_0 + \frac{1}{2} \int_{\mathbb{R}^3} (((2f_N - f_N^2) V_N) * |u_0|^2) |u_0|^2 a_0^* a_0^* a_0 a_0$$

$$\mathcal{H}_1 = \sum_{m \geq 1} \Big(\langle u_m, (-\Delta + U) u_0 \rangle a_m^* a_0 + \langle u_m, ((f_N V_N) * |u_0|^2) u_0 \rangle a_m^* a_0^* a_0 a_0 \Big) + \mathrm{h.c.}$$

$$\mathcal{H}_2 = \frac{1}{2} \sum_{m,n \geq 1} \Big(\langle u_m, (-\Delta + U) u_n \rangle a_m^* a_n + \langle u_m u_0, (f_N V_N) * (u_n u_0) \rangle a_m^* a_n^* a_0 a_0 \Big) + \mathrm{h.c.}$$

Thus we have removed cubic and quartic terms in $a_{p \neq 0}^{\#}$, and replace V_N by $f_N V_N$

Lower bound: general case II

Now consider the **c-number substitution**, i.e. replacing $a_0^\#$ by $\sqrt{N}$

Using $a_0^* a_0 = N - \mathcal{N}_+$, then up to an error $\mathcal{O}(a)\mathcal{N}_+$ we have

$$\mathcal{H}_0 \approx N \int_{\mathbb{R}^3} (|\nabla u_0|^2 + U|u_0|^2) + \frac{N}{2} \Big(\int_{\mathbb{R}^3} (2f - f^2)V \Big) \int_{\mathbb{R}^3} |u_0|^4$$

and $\mathcal{H}_1 \approx 0$ thanks to the **Gross–Pitaevskii equation**

$$(-\Delta + U)u_0 + N((f_N V_N) * |u_0|^2)u_0 \approx \mu_0 u_0$$

The terms $a_0 a_0$ and $a_0^* a_0^*$ can be treated by a variational principle for **quasi-free states** on Fock space. This gives

$$\mathcal{H}_2 \geq \inf \mathrm{spec}(\mathbb{H}_{\mathrm{Bog}})$$

where

$$\mathbb{H}_{\mathrm{Bog}} = \frac{1}{2} \sum_{m,n \geq 1} \Big(\langle u_m, (-\Delta + U)u_n \rangle a_m^* a_n + N \langle u_m u_0, (f_N V_N) * (u_n u_0) \rangle a_m^* a_n^* \Big) + \mathrm{h.c.}$$

Lower bound: general case III

We can diagonalize the **quadratic Hamiltonian** $\mathbb{H}_{\rm Bog}$ and find that

$$\inf \mathrm{spec}(\mathbb{H}_{\rm Bog}) = -\frac{1}{4}\operatorname{Tr}((-\Delta+U)^{-1}K^2)+\mathcal{O}(1)$$

where the operator K has kernel $K(x,y)=u_0(x)u_0(y)(Nf_NV_N)(x-y)$, i.e.

$$K = u_0(x)N\widehat{f_NV_N}(p)u_0(x)$$

with $u_0(x)$ and $v(p)$ are multiplication operators in the position and momentum spaces. If the operators **commuted**, then by the **scattering equation**

$$\operatorname{Tr}((-\Delta)^{-1}K^2) = N^2\operatorname{Tr}\left(p^{-2}u_0(x)\widehat{f_NV_N}(p)u_0^2(x)\widehat{f_NV_N}(p)u_0(x)\right)$$

$$= N^2\operatorname{Tr}\left(u_0(x)p^{-2}\widehat{f_NV_N}(p)u_0^2(x)\widehat{f_NV_N}(p)u_0(x)\right)$$

$$= 2N^2\operatorname{Tr}\left(u_0^2(x)\widehat{1-f_N}(p)u_0^2(x)\widehat{f_NV_N}(p)\right) = 2N\int((1-f_N)f_NV_N*u_0^2)u_0^2$$

which together with $\mathcal{H}_0$ gives $Ne_{\rm GP}$. Rigorously, $u_0(x)$ and $|p|^{-2}$ do not commute, but the commutator can be controlled by the **Kato–Seiler–Simon inequality**

$$\|u(x)v(p)\|_{\mathfrak{S}^r}\le C_{d,r}\|u\|_{L^r(\mathbb{R}^d)}\|v\|_{L^r(\mathbb{R}^d)},\quad 2\le r<\infty.$$

An open problem

Our derivation of the BEC with optimal estimate is a step towards

Conjecture on the excitation spectrum (Bogoliubov 1947, Grech–Seiringer 2013)

Let $0 = e_0 < e_1 \leq e_2 \leq \ldots$ be eigenvalues of $(D^{1/2}(D + 16\pi\mathfrak{a}|u_0|^2)D^{1/2})^{1/2}$ on $L^2(\mathbb{R}^3)$, where

$$D := -\Delta + U + 8\pi\mathfrak{a}|u_0|^2 - \mu_0 \geq 0$$

Then all eigenvalues of $H_N - E_N$ in the interval $[0, o(N)]$ are the finite sums

$$\sum_{i\geq 1} n_i e_i (1 + o(1)_{N\to\infty}), \quad n_i \in \{0, 1, 2, \ldots\}$$

- Proved for the **mean-field regime** with V_N replaced by $N^{-1}V$ by **Seiringer** ('11), **Grech–Seiringer** ('13), **Lewin–N.–Serfaty–Solovej** ('14), **Dereziński–Napiórkowski** ('14)
- Proved for the **dilute homogeneous gas** by **Boccato–Brennecke –Cenatiempo–Schlein** ('19) where $e_p = \sqrt{|p|^4 + 16\pi\mathfrak{a}|p|^2}$, $\quad p \in 2\pi\mathbb{Z}^3$. This verifies Landau's criterion for **superfluidity**: a drop with velocity less than $\inf_{p\neq 0} e_p/|p|$ can move frictionlessly in the ground state of H_N

Thank you!