1) Definition of $f(A)$ for $f \in L^2$ – a density argument

Known from the lecture (first part of Th. 4.18 (Spectral measure) in the lecture notes from 2018/2019):

Let $A = A^*$ be a bounded operator on $\mathcal{H}$. Then for all $u \in \mathcal{H}$, there exists a unique Borel measure μ_u on $\sigma(A)$ s.t. for all $f \in C(\sigma(A))$:

$$\langle u, f(A) u \rangle = \int_{\sigma(A)} f(x) \, d\mu_u(x) \qquad (*)$$

and

$$\| f(A) u \|_{\mathcal{H}} = \| f \|_{L^2(\sigma(A), d\mu_u)} \qquad (**)$$

Exercise:

Show that there exists a unique linear map

$$T: L^2(\sigma(A), d\mu_u) \longrightarrow B(\mathcal{H})$$
$$f \longmapsto Tf =: f(A)$$

such that $(*)$ and $(**)$ are satisfied.

I.e. extend the result from the lecture for $f \in C_0(\sigma(A))$ to all $f \in L^2(\sigma(A), d\mu_u(x))$

Hint: You may use that $C(\sigma(A))$ is dense in $L^2(\sigma(A), d\mu_u(x))$, i.e. for any $f \in L^2(\sigma(A), d\mu_u)$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C(\sigma(A))$ with

$$\| f_n - f \|_{L^2(\sigma(A), d\mu_u)} \xrightarrow{n\to\infty} 0.$$

2) Definition of $f(-i\partial_x)$

a) Show that for any $u \in \mathcal{S}(\mathbb{R})$, we have

Schwartz space, you can also take $u \in C_c^\infty(\mathbb{R})$

$$[\mathcal{F}^{-1}\xi\, \mathcal{F}u](x) = (-i\partial_x u)(x)$$

Fourier transform

~~What~~ Hint: First try to show $\xi \cdot (\mathcal{F}u)(\xi) = \mathcal{F}[-i\partial_x u](\xi)$.

b) How can you express for $n \in \mathbb{N}$

$[(-i\partial_x)^n u](x)$ using the Fourier transformation?

Hint: Use a) several times

What about $P(-i\partial_x)u$ for any polynomial P?

c) How would you define $f(-i\partial_x)u$ for any ~~vector~~ $f \in C(\mathbb{R})$? How about measurable f?

Hint: Functional calculus ~~e.g.~~ and b).

d) What would be a good choice of $\mathcal{D}(f(-i\partial_x))$ that is natural and makes $f(-i\partial_x)$ self-adjoint if f is real-valued?

Hint: Homework E 3.5

3) Properties of operators ~~of~~ that are unitarily equivalent

Let $A: \mathcal{D}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be a densely defined operator.

Let $U: \mathcal{H} \to \mathcal{H}$ be unitary. Define

$$B: U^* \mathcal{D}(A) \longrightarrow \mathcal{H}$$

$$f \longmapsto U^* A U f$$

Show that

a) B is a ~~~~ densely defined linear operator

b) $\sigma(A) = \sigma(B)$

c) A is a bounded operator $\iff$ B is a bounded operator

d) A is compact $\iff$ B is compact.

4) Properties of $f(-i\partial_x)$ - continuation of exercise 2).

For measurable $f: \mathbb{R} \to \mathbb{C}$, we let

$$f(-i\partial_x) : \mathcal{D} \longrightarrow L^2(\mathbb{R})$$
$$u \longmapsto \mathcal{F}^{-1} f(\xi)\, \mathcal{F}u$$

with domain $\mathcal{D} = \{ u \in L^2(\mathbb{R}) \mid f(\xi)(\mathcal{F}u)(\xi) \in L^2(\mathbb{R}) \}$

a) Let $t > 0$ and consider $e^{-t\partial_x^2}$. What is the corresponding f s.t.

$$e^{-t\partial_x^2} = f(-i\partial_x) \quad ?$$

b) What is the spectrum $\sigma(e^{-t\partial_x^2})$?

Hint: Use exercise 3b) and homework E3.5

c) Is $e^{-t\partial_x^2}$ a compact operator?

Hint: Exercise 3d)