Tutorial 22 November

1) Find a sequence of functions $(f_n)_{n\in\mathbb{N}} \in L^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n(x)\,dx < \liminf_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx$$

2) Let A be a self-adjoint operator on $\mathcal{H}$ and $u \in \mathcal{D}(A)$. Denote by $d\mu_u$ the corresponding spectral measure on $\mathbb{R}$.

Show that $\lim\limits_{n\to\infty} \langle u, \sqrt{A^2 + \frac{1}{n}}\, u\rangle = \langle u, |A|\, u\rangle$.

Hint: Use that $\forall x,y \geq 0$: $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

3) Layer cake representation

$(X, \mathcal{A}, \mu)$ σ-finite measure space, $p \in [1,\infty)$, $f \in L^p(X)$

~~with $f \geq 0$~~

a) Show that if $f \geq 0$, $p = 1$, then

$$\int_X f\,d\mu = \int_{[0,\infty)} \mu(\{x \in X \mid f(x) > t\})\,dt$$

Hint: Tonell:

b) Show

$$\int_X |f|^p\,d\mu = p \int_{[0,\infty)} t^{p-1} \mu(\{x\in X \mid |f(x)|^p > t\})\,dt.$$

Helpful tools

$(X, \mathcal{A}, \mu)$, $f_n, f, g : X \longrightarrow \mathbb{C}$ measurable fcts

Monotone convergence

If $f_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for μ-a.e. $x \in X$,
~~then~~ and $f_{n+1}(x) \geq f_n(x)$ for all $n \in \mathbb{N}$, μ-a.e. $x \in X$

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

Dominated convergence

If $|f_n(x)| \leq g(x)$ for μ-a.e. $x \in X$ with $g \in L^1(X)$
and $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in X$, then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

Fatou

If $f_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\int_X \left(\liminf_{n\to\infty} f_n(x)\right) d\mu(x) \leq \liminf_{n\to\infty} \int_X f_n(x) \, d\mu(x) .$$

$(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ σ-finite measure spaces

$f: X \times Y \longrightarrow \mathbb{C}$ measurable

Tonelli :

If $f \geq 0$, then

$$(*) \quad \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x)$$

$$= \int_{X \times Y} f(x,y)\, d(\mu \otimes \nu)(x,y)$$

Fubini

If f is s.t. $\displaystyle\int_{X\times Y} |f(x,y)|\, d\mu\otimes\nu\,(x,y) < \infty$, then,

we also have $(*)$.