Tutorial 31. 1. 2025

1) Let $A = A^*$ be a self-adjoint operator on some Hilbert space $\mathcal{H}$. Assume furthermore that A is bounded from below, that is, there is a constant $C > 0$ such that $A \geq -C$.

a) Is e^{-A} a bounded operator? Prove your claim.

b) Is e^{-A} a compact operator? Prova your claim.

2) Consider the operator $-\Delta - \frac{1}{|x|^{3/2}}$ on $L^2(\mathbb{R}^3)$.

Show that there exists $C > 0$ such that

$$-\Delta - \frac{1}{|x|^{3/2}} \geq -C .$$

Hint: Use Hardy's inequality and estimate $\frac{1}{|x|^{3/2}} \leq \frac{c_1}{|x|^2} + c_2$ for suitable $c_1, c_2 > 0$.

3) Let $(u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $\|u_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$ and suppose that there exists $C > 0$ such that for all $n \in \mathbb{N}$

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} |x| \, |u_n|^2 \leq C$$

Show that there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and a function $u \in L^2(\mathbb{R}^3)$ such that

$$\lim_{n \to \infty} \|u_n - u\|_{L^2} = 0 .$$

Hint: • Show that for every $\varepsilon > 0$ there is $R = R_\varepsilon > 0$ such that for all $n \in \mathbb{N}$

$$\int_{|x| \geq R_\varepsilon} |u_n|^2 \leq \varepsilon$$

- Prove that there is a subsequence $(u_{n_k})_{k\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ and $u \in H^1(\mathbb{R}^3)$ such that

$$u_{n_k} \xrightharpoonup[k\to\infty]{H^1} u \quad \text{(weak convergence in } H^1 \text{)}$$

- Show that for every $R>0$

$$\int_{|x|\le R} |u_{n_k} - u|^2 \xrightarrow{k\to\infty} 0$$

- Conclude by combining the first and third hint.

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4) Give an example of a sequence $(u_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R})$ wit

$$u_n \xrightharpoonup{L^2} 0 \quad \text{(weak convergence in } L^2\text{)}$$

but we do <u>not</u> have $u_n \xrightarrow[n\to\infty]{L^2} 0$ (strong convergence in L^2)

5) Let $A = A^*$ be a selfadjoint operator. Show that

$$\inf \sigma(A) = \inf_{\substack{u \in \mathcal{D}(A) \\ \|u\|=1}} \langle u, Au \rangle$$

<u>Hint:</u>
- Argue using the spectral theorem that you may WLOG assume that A is a multiplication operator
- How is $\sigma(A)$ related to the function a if $A = M_a$ (multiplication by a)?
- Show $\geq$ and $\leq$ separately.