Tutorial 24.1.2025

1) Consider the operator $\tilde{A}$ on $L^2((0,\infty))$ given by

$$\tilde{A}: C_c^\infty((0,\infty)) \longrightarrow L^2((0,\infty))$$

$$\tilde{A}u = -\partial_x(x^2 \partial_x u)$$

a) Prove that

- $\tilde{A}$ is symmetric
- $\langle u, \tilde{A}u\rangle \geq 0$ for all $u \in C_c^\infty((0,\infty))$

Deduce that $\tilde{A}$ has a Friedrichs extension, which we will call A in the following.

Hint: Integration by parts.

b) Is A a bounded operator? Prove your claim.

c) For every $\lambda > 0$ and $u \in L^2((0,\infty))$ define $u_\lambda \in L^2((0,\infty))$ by

$$u_\lambda(x) = u(\lambda x) \quad \text{for a.e. } x \in (0,\infty).$$

Show that for every $\lambda > 0$ and every $u \in \mathcal{D}(A)$

$$(*) \qquad (Au_\lambda)(x) = \underbrace{(Au)(\lambda x)}_{=(Au)_\lambda(x)} \quad \text{for a.e. } x \in (0,\infty)$$

Hint:
- First show $(*)$ for all $u \in C_c^\infty((0,\infty))$.
- Then use this and $A = A^*$ to show for any $\varphi \in C_c^\infty((0,\infty))$ and any $u \in \mathcal{D}(A)$:

$$\langle \varphi, Au_\lambda\rangle = \langle \varphi, (Au)_\lambda\rangle$$

d) Do we have $\sigma_{disc}(A) \neq \emptyset$? Prove your claim.

Hint: Suppose that $\mu \geq 0$ is an eigenvalue of A with eigenfunction v. What can you say about v_λ for any $\lambda > 0$? What is the dimension of the eigenspace $\{u \in \mathcal{D}(A) \mid Au = \mu u\}$?

e) What can you deduce about $\sigma_{ess}(A)$?

2) For a smooth function $f : \mathbb{R} \to [0, \infty)$ with $\lim_{|x| \to \infty} f(x) = 0$ and $\lim_{|x| \to \infty} f'(x) = 0$ consider the operator

$$\hat{A} : C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$$

$$\hat{A}u = -\partial_x (f(x) \partial_x u)$$

and its corresponding Friedrichs extension, which we will call A.

Show that $0 \in \sigma_{ess}(A)$

Hint: Fix $\varphi \in C_c^\infty((0,1))$ with $\|\varphi\|_{L^2} = 1$ and define

$$\varphi_n(x) := \varphi(x-n)$$

Show that $(\varphi_n)_{n \in \mathbb{N}}$ is a suitable Weyl sequence