Tutorial 17.1.2025



1) Consider the quadratic form

$$Q(u) = \int_0^{\cancel{1}\,\infty} \frac{1}{x^2} |u'(x)|^2 \, dx \qquad \text{for} \quad u \in C_c^\infty((0,\cancel{1}\,\infty))$$

and let $A$ be the corresponding self-adjoint operator on $L^2((0,\cancel{1}\,\infty))$ defined via the Friedrichs extension

Question: Do we have $\sigma_{ess}(A) = \emptyset$ or $\sigma_{ess}(A) \neq \emptyset$?

Prove your claim.

Hints: • Weyl sequences

• For $\lambda > 0$ and $\varphi \in C_c^\infty((0,\cancel{1}\,\infty))$, define $\varphi_\lambda(x) = \varphi(\lambda x)$.

Compute $\dfrac{\langle \varphi_\lambda, A\varphi_\lambda \rangle}{\|\varphi_\lambda\|^2} \left( = \dfrac{Q(\varphi_\lambda)}{\|\varphi_\lambda\|^2} \right)$. How does it depend on $\lambda$?

2) Show that

$$A_a = (1-\Delta)^{-1} \frac{1}{1+|x|^a}$$

is a compact operator for all $a > 0$.

Hint: Theorem from the lecture on $f(P)g(X)$

3) What about

$$B_a = (-\Delta)^{-1} \frac{1}{1+|x|^a} \quad ?$$

For which $a > 0$ is $B_a$ compact?

Hint: In the lecture it was shown that for $0 < a < 2$, $B_a$ has infinitely many eigenvalues