Tutorial 10.1.2025

1) Let $A = A^*$ be ~~s~~ a self-adj. op. on $\mathcal{H}$. Let $\lambda \in \sigma_{disc}(A)$ (discrete spectrum) and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ with $\|(A-\lambda)u_n\| \xrightarrow{n\to\infty} 0$ and $\|u_n\| = 1$ for all $n \in \mathbb{N}$.

Show that $(u_n)_{n\in\mathbb{N}}$ is pre-compact, i.e. there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $\tilde{u} \in \mathcal{H}$ with $\|\tilde{u} - u_{n_k}\| \xrightarrow{k\to\infty} 0$.

2) Let $A = A^*$ be bounded from below and let $\mu_n(A)$, $n \in \mathbb{N}$ denote the min-max values of A:

$$\mu_n(A) = \inf_{\substack{M \subset \mathcal{D}(A) \\ \dim(M) = n}} \sup_{\substack{u \in M \\ \|u\| = 1}} \langle u, Au \rangle$$

Suppose that $B = B^*$ is an operator on the same Hilbert space $\mathcal{H}$ with $A \leq B$.

Show that for all $n \in \mathbb{N}$, $\mu_n(A) \leq \mu_n(B)$.

3) Let A be as in 2) and define for $n \in \mathbb{N}$

$$\nu_n(A) := \sup_{\substack{M \subset \mathcal{D}(A) \\ \dim(M) = n-1}} \inf_{\substack{u \in M^\perp \\ \|u\| = 1}} \langle u, Au \rangle$$

Show that for all $n \in \mathbb{N}$,

$$\mu_n(A) = \nu_n(A).$$