Ground state for the hydrogen atom

Let $$-C_0 = \inf_{\substack{u\in H^1(\mathbb{R}^3)\\ u\neq 0}} \frac{\int |\nabla u|^2 - \int \frac{|u|^2}{|x|} = Q(u)}{\int |u|^2} \quad (*)$$

We assume that we already know that $C_0 > 0$ (can be shown using the Hardy inequality, see old lecture notes Sect. 2.1).

GOAL: Show that there exists $u \in H^1(\mathbb{R}^3)$ with $u \neq 0$ and

$$Q(u) = -C_0 \int |u|^2 .$$

Method of calculus of variations

1. Take a minimising sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $\int |u_n|^2 = 1$, i.e. s.t. $Q(u_n) \xrightarrow{n\to\infty} -C_0$.

2. Show that in some (weak) sense, $u_n \longrightarrow u$ (up to a subsequence) for some $u \in H^1(\mathbb{R}^3)$. (u is our candidate for the minimiser)

3. Show that $Q(u) \leq -C_0$ and argue that $u \neq 0$

4. Show that $Q(u) \leq -C_0 \int |u|^2$.

Hints:

- If $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ is bdd, then up to a subsequence,

$$u_n \xrightarrow{H^1} u \in H^1(\mathbb{R}^3)$$

$$u_n \xrightarrow{L^p_{loc}} u \quad \text{for all } p \in [2,6)$$

$$u_n(x) \longrightarrow u(x) \quad \text{a.e } x \in \mathbb{R}^3$$

- If $u_n \xrightarrow{H^1} u$, then $\int |\nabla u|^2 \leq \liminf\limits_{n\to\infty} \int |\nabla u_n|^2$

- For the $\int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|}$ part, split the integral $\int_{\mathbb{R}^3} \dots = \int_{B_R} \dots + \int_{B_R^c} \dots$ for $R>0$ large.

- Use Cauchy-Schwarz and $u_n \longrightarrow u$ for the $\int_{B_R}$ part.

- Show that $\int_{B_R} \frac{|u_n|^2}{|x|} \xrightarrow{n\to\infty} \int \frac{|u|^2}{|x|}$.

- For the $\int_{B_R^c}$ part, use that $\|u_n\|_{L^2} = 1$