Quantum dynamics

Let T be a self-adj. operator on $\mathcal{H}$

$U(t) := e^{-itH}$, $t \in \mathbb{R}$

1. Prove that

$$\left.\begin{array}{l} \bullet\ \forall t \in \mathbb{R}:\ U(t) \text{ is unitary} \\ \bullet\ \forall t,s \in \mathbb{R}\quad U(t+s) = U(t)\,U(s) \end{array}\right\} (*)$$

1.5) Show that for any $\varphi \in \mathcal{H}$, the map $\begin{array}{l} \mathbb{R} \to \mathcal{H} \\ t \mapsto U(t)\varphi \end{array}$ is continuous

2 Show:

$$\begin{array}{l} \mathbb{R} \to B(\mathcal{H}) \\ t \longmapsto U(t) \end{array} \text{ is continuous} \iff T \in B(\mathcal{H})$$

3. Let $\varphi \in \mathcal{D}(T)$. Show that

- $\forall t \in \mathbb{R}:\quad U(t)\varphi \in \mathcal{D}(T)$
- $\dfrac{d}{dt}\big(U(t)\varphi\big) := \lim\limits_{h \to 0} \dfrac{U(t+h)\varphi - U(t)\varphi}{h}$

  exists as a limit in $\mathcal{H}$ for all $t \in \mathbb{R}$

4. Suppose $(U(t))_{t \in \mathbb{R}}$ satisfies $(*)$ (i.e. $(U(t))_{t\in\mathbb{R}}$ is a <u>unitary group</u>). Show

- $U(0) = \mathbb{1}$
- $U(t)^* = U(-t) \quad \forall t \in \mathbb{R}$.

<u>Hints</u>

1. Use the spectral theorem

~~2. First show that~~

1.5) Dominated convergence and spectral theorem.

2.) $\Leftarrow$ Try to show that $\begin{array}{l} \mathbb{R} \to B(\mathcal{H}) \\ t \mapsto U(t) \end{array}$ is Lipschitz cont.

$\Rightarrow$

3) Use dominated convergence, the spectral thm and $\varphi \in \mathcal{D}(T)$

4) • Take $s=0$

• Take $s=-t$.