Homework Sheet 14

(Released 29.1.2025 – Discussed 5.2.2025)

E14.1 Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Let $u_0 \in \mathcal{H}$ and assume that

$$M_T = \frac{1}{T}\int_0^T |e^{-itA}u_0\rangle\langle e^{-itA}u_0| \rightharpoonup M_\infty$$

weakly in the Hilbert–Schmidt topology when $T \to \infty$, namely $\mathrm{Tr}[M_T B] \to \mathrm{Tr}[M_\infty B]$ for every Hilbert–Schmidt operator B on $\mathcal{H}$.

(a) Prove that $e^{-itA}M_\infty e^{itA} = M_\infty$ for all $t \in \mathbb{R}$.

(b) Prove that M_∞ commutes with A.

(c) Deduce that there exist $\lambda_n \geq 0$ and an orthonormal family $\{u_n\}$ of eigenfunctions of A such that

$$M_\infty = \sum_{n=1}^\infty \lambda_n |u_n\rangle\langle u_n|.$$

(d) Conclude that if u_0 is orthogonal to all eigenfunctions of A, then $M_\infty = 0$.

Hint: E13.3 is helpful for (d). We used these ingredients in the proof of RAGE theorem.

E14.2 Let $d \geq 1$, $p \in [1, 2]$ and $1/p + 1/p' = 1$. Prove the dispersive estimate

$$\|e^{it\Delta}\varphi\|_{L^{p'}(\mathbb{R}^d)} \leq |t|^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}\|\varphi\|_{L^p(\mathbb{R}^d)}.$$

Hint: You may interpolate from two cases $p = 1$ and $p = 2$.

E14.3 Let $V \in L^2(\mathbb{R}^3, \mathbb{R}) + L^p(\mathbb{R}^3, \mathbb{R})$ for some $2 < p < 3$. Prove that the wave operators

$$\Omega_\pm = \lim_{t\to\pm\infty} e^{-itA}e^{-it\Delta}$$

are well defined on $L^2(\mathbb{R}^d)$.

Hint: You can use Cook method and E14.1

E14.4 $A : D(A) \to L^2(\mathbb{R}^d)$ is a self-adjoint operator and $\varphi \in L^2(\mathbb{R}^d)$ such that the limit

$$\Omega_\pm\varphi = \lim_{t\to\pm\infty} e^{-itA}e^{-it\Delta}\varphi$$

exists. Prove that the vectors $\Omega_\pm\varphi$ are orthogonal to all eigenfunctions of A.

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Homework Sheet 13

(Released 22.1.2025 – Discussed 29.1.2025)

E13.1 Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$.

(a) Prove that for every $x_0 \in \mathcal{H}$ and $t \in \mathbb{R}^d$, the vector $x(t) = e^{-itA}x_0$ satisfies that

$$\frac{d}{dt}\langle \varphi, \mathbf{i}x(t)\rangle = \langle A\varphi, x(t)\rangle, \quad \forall \varphi \in D(A).$$

(b) Prove that $\|x(t)\| = \|x_0\|$ for all $t \in \mathbb{R}$ if $x_0 \in \mathcal{H}$. Moreover, $\|Ax(t)\| = \|Ax_0\|$ for all $t \in \mathbb{R}$ if $x_0 \in D(A)$.

(c) Prove that for every $x_0 \in \mathcal{H}$, if $y(t)$ satisfies

$$\frac{d}{dt}\langle \varphi, \mathbf{i}y(t)\rangle = \langle A\varphi, y(t)\rangle, \quad \forall \varphi \in D(A), \quad \forall t \in \mathbb{R}$$

and $y(0) = x_0$, then $y(t) = x(t)$ for all $t \in \mathbb{R}$ (namely the weak solution is unique).

E13.2 Consider the free Schrödinger evolution $e^{\mathbf{i}t\Delta}$ on $L^2(\mathbb{R}^d)$. Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

(a) Prove that for all $\varepsilon > 0$, we have

$$e^{-(\mathbf{i}t+\varepsilon)|2\pi k|^2}\widehat{f}(k) = \widehat{G_\varepsilon * f}(k), \quad \text{with} \quad G_\varepsilon(x) = \frac{1}{(4\pi(\mathbf{i}t+\varepsilon))^{d/2}}\exp\left(-\frac{|x|^2}{4(\mathbf{i}t+\varepsilon)}\right)$$

(b) Deduce that

$$e^{\mathbf{i}t\Delta}f(x) = \frac{1}{(4\pi\mathbf{i}t)^{d/2}}\int_{\mathbb{R}^d}\exp\left(-\frac{|x-y|^2}{4\mathbf{i}t}\right)f(y)\mathrm{d}y.$$

E13.3 Let A be a self-adjoint operator on $L^2(\mathbb{R}^d)$ and $f \in \overline{\text{span}\{\text{eigenfunctions of } A\}}$. Prove that for every $\varepsilon \in (0,1)$, there exists $R = R_\varepsilon > 0$ such that for all $t \in \mathbb{R}$,

$$\int_{|x|\le R}|(e^{-\mathbf{i}tA}f)(x)|^2\mathrm{d}x \ge (1-\varepsilon)\int_{\mathbb{R}^d}|f|^2.$$

E13.4 In $\mathcal{H} = L^2([0,1])$, we have seen in the lecture that the momentum operator $p_\alpha = i\frac{d}{dx}$ is self-adjoint when defined on

$$D(p_\alpha) = \{f \in \mathcal{J} \colon f \text{ absolutely continious with derivative in } \mathcal{H}, f(0) = e^{\mathbf{i}\alpha}f(1)\}$$

for a phase $0 \le \alpha < 2\pi$. Find an explicit expression for the one parameter group $U_\alpha(t) = e^{\mathbf{i}tp_\alpha}$ provided by Stone's theorem.

Hint: Note that U_α is defined on all of $\mathcal{H}$ but taking the t-derivative must yield the different p_α with their respective domains!

Homework Sheet 12

(Released 15.1.2025 – Discussed 22.1.2025)

In this homework we only consider the dimension $d \geq 3$.

E12.1 Let $\{u_n\}_{n=1}^N \subset H^1(\mathbb{R}^d)$ such that $\{\nabla u_n\}_{n=1}^N$ are orthonormal in $L^2(\mathbb{R}^d)$.

(a) Prove that $\{\sqrt{-\Delta} u_n\}_{n=1}^N$ are orthonormal.

(b) Prove that for all $E > 0$ we have

$$\sum_{n=1}^N \left| \int_{|k|^2 \leq E} e^{2\pi \mathbf{i} k\cdot x} \hat{u}_n(k) \mathrm{d}k \right|^2 \leq C_d E^{\frac{d-2}{2}}.$$

(c) Prove that

$$N = \sum_{n=1}^N \|\nabla u_n\|_{L^2}^2 \geq K_d \int_{\mathbb{R}^d} \rho^{\frac{d}{d-2}}.$$

Here $C_d > 0$ and $K_d > 0$ are constants depending only on d.

E12.2 Let $V \in L^{d/2}(\mathbb{R}^d, \mathbb{R})$ and let $A = -\Delta + V(x)$ be the self-adjoint operator on $L^2(\mathbb{R}^d)$ by Friedrichs' method (c.f. E8.3).

(a) Use the characterization in E11.5 to prove directly that $\inf_{\mathrm{ess}}(A) \geq 0$.

(b) Assume that A has N negative eigenvalues and let W be the space spanned by the corresponding eigenfunctions. Prove that $\langle \varphi, A\varphi \rangle \leq 0$ for all $\varphi \in W$ and that $\dim \sqrt{-\Delta} W = N$. Deduce that we can find N functions $\{u_n\}_{n=1}^N$ such that $\{\sqrt{-\Delta} u_n\}_{n=1}^N$ are orthonormal and that $\langle u_n, A u_n \rangle < 0$ for all $n = 1, 2, ..., N$.

(c) Use E12.1 to prove that the number of negative eigenvalues of A is bounded by $C_d \int_{\mathbb{R}^d} |V_-|^{d/2}$. This is the CLR inequality. Here $V_-(x) = \min(V(x), 0)$.

E12.3 Consider $A = -\Delta + |x|^2$. We know that it is a self-adjoint operator with eigenvalues $0 < \mu_1 \leq \mu_2 \leq ...$ and $\mu_n \to \infty$.

(a) Use the CLR inequality to prove that for all $\lambda > 0$, the number of eigenvalues $\{\mu_n\}$ in the interval $(0, \lambda)$ is bounded from above by $C_d \lambda^d$.

(b) Prove that $\mu_n \geq c_d n^{1/d}$ for all $n = 1, 2, ...$

Here $C_d > 0, c_d > 0$ are constants depending only on d.

E12.4 (a) Let A and B be two self-adjoint operators on a separable Hilbert space $\mathcal{H}$ such that $D(A) \cap D(B)$ is dense in $\mathcal{H}$. Prove that if $A \geq B$, then the number of negative eigenvalues of A is smaller than or equal to the number of negative eigenvalues of B.

(b) Prove that $V \in L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R})$ and $V(x) \geq -\lambda |x|^{-2}$ for $|x|$ large, with a constant $0 < \lambda < (d-2)^2/4$, then the Schrödinger operator $-\Delta + V(x)$ has only finitely many negative eigenvalues.

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Homework Sheet 11

(Released 8.1.2025 – Discussed 15.1.2025)

E11.1 Let $A = M_a$ be the multiplication operator on $L^2(\mathbb{R}^d, \mu)$, with μ a Borel locally finite measure on $\mathbb{R}^d$ and $a \in L^\infty_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$. Let $\lambda \in \mathbb{R}$.

(a) Prove that λ is an eigenvalue of A if and only if $\mu(a^{-1}(\{\lambda\})) > 0$.

(b) Prove that if $\lambda \in \sigma(A)$ but λ is not an eigenvalue, then $\lim_{\varepsilon\to 0} \mu(a^{-1}((\lambda-\varepsilon, \lambda+\varepsilon)) = 0$. Deduce that there exists a sequence $\varepsilon_n \to 0$ such that

$$\mu(a^{-1}((\lambda - \varepsilon_n, \lambda + \varepsilon_n)\backslash(\lambda - \varepsilon_{n+1}, \lambda + \varepsilon_{n+1})) = 0, \quad \forall n \in \mathbb{N}.$$

(c) Prove that if λ is an isolated point of $\sigma(A)$, namely here exists $\varepsilon > 0$ such that $\sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}$, then λ is an eigenvalue and

$$|A - \lambda|^2 \geq \varepsilon^2 \mathbb{1}_{\text{Ker}_\lambda^\perp}, \quad K_\lambda = \text{Ker}(A - \lambda).$$

Hint: We used these properties to prove Weyl's criterion for spectrum. Here $a^{-1}(U) = \{x : a(x) \in U\}$. You can use $\sigma(A) = \text{ess-range}(a)$ (see E3.5).

E11.2 Use Weyl's criterion to prove that $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$. Here $-\Delta$ is the Laplacian in $L^2(\mathbb{R}^d)$.

E11.3 Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space and $B : D(A) \to \mathcal{H}$ be a symmetric operator. Assume that B is A-relatively compact, namely $B(A + \mathrm{i})^{-1}$ is a compact operator.

(a) Prove that

$$\lim_{n\to\infty} \|B(A + \mathrm{i}n)^{-1}\| = 0.$$

(b) Prove that B is A-relatively bounded with any relative bound $\varepsilon \in (0, 1)$, namely

$$\|Bu\| \leq \varepsilon\|Au\| + C_\varepsilon\|u\|, \quad \forall u \in D(A).$$

Hint: You can show that $A_n = (A + \mathrm{i})(A + \mathrm{i}n)^{-1}$ satisfies $\|A_n u\| \to 0$ for all $u \in \mathcal{H}$.

E11.4 Let $d \in \mathbb{N}$. Let $V \in L^p(\mathbb{R}^d)$ with $\max(2, d/2) \leq p < \infty$ and additionally $2 < p$ if $d = 4$. Prove that V is Δ-relatively compact.

E11.5 Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on an (infinite dimensional) separable Hilbert space $\mathcal{H}$ which is bounded from below. Prove that

$$\begin{aligned}\inf \sigma_{\text{ess}}(A) &= \inf\{\liminf_{n\to\infty} \langle u_n, Au_n\rangle : \{u_n\} \subset D(A) \text{ orthonormal}\}\\ &= \inf\{\liminf_{n\to\infty} \langle u_n, Au_n\rangle : \{u_n\} \subset D(A) \text{ normalized}, u_n \rightharpoonup 0\}.\end{aligned}$$

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Homework Sheet 10

(Released 20.12.2024 – Discussed 8.1.2025)

E10.1 Show that the space of states of $Mat(2 \times 2, \mathbb{C})$ as a convex space is the unit ball in $\mathbb{R}^3$. Hint: The identity $\mathbb{I}$ and the Pauli matrices σ_x, σ_y, σ_z for a basis of the hermitean 2×2 matrices. Show explicitly that the pure states are those that come from rays in $\mathbb{C}^2$.

E10.2 Let ρ be a density matrix for $Mat(2 \times 2, \mathbb{C})$. Work out explicitly the GNS construction for $\omega_\rho(A) = \mathrm{Tr}(\rho A)$. What is its dimension? You can assume that ρ is diagonal (why?) and you will have to distinguish two cases.

E10.3 Alice and Bob have each their separted labs each with their local Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_b$ respectively which we assume to be finite dimesional. The whole "bipartite" set up is described by the Hilbert space $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$ and observables $\mathcal{A} = \mathcal{B}(\mathscr{H}) = \mathcal{B}(\mathscr{H}_A \otimes \mathscr{H}_B)$. Let $\omega\colon \mathcal{A} \to \mathbb{C}$ be a state.

Alice can only make measurements in her own lab, thus she has only access to observables of the form $A \otimes \mathbb{I}$ for $A \in \mathcal{B}(\mathscr{H}_A)$. Show that those form a sub-C*-algebra of $\mathcal{A}$ and we have thuse an inclusion $\mathcal{B}(\mathscr{H}_A) \hookrightarrow \mathcal{B}(\mathscr{H}_A \otimes \mathscr{H}_B)$. Accordingly, we can define a state on Alices's subsystem $\mathcal{B}(\mathscr{H}_A)$ via

$$\omega_A := \omega|_{\mathcal{B}(\mathscr{H}_A)}.$$

Express the density matrix for ω_A in terms of the density matrix of ω!

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Homework Sheet 9

(Released 11.12.2024 – Discussed 17.12.2024)

E9.1 Instead of two types of generators, one can also write the CCR algebra in terms of operators $W(z)$ for $z \in \mathbb{C}$ and

$$W(s+it) = e^{-ist/2}U(s)V(t)$$

with relations $W(z_1)W(z_2) = e^{i\Im(z_1\bar{z}_2)/2}W(z_1+z_2)$. (In fact, this can be defined for any symplectic vector space, that is a real vector space with a non-degenerate anti-symmetric bi-linear form σ. In this case only replace $\Im(z_1\bar{z}_2)$ by the symplectic form σ). Show that all $W(z)$ are unitary. Thus their spectrum has to be contained in the unit circle.

Then compute $W(u)W(z)W(-u)$ and use this to argue that the spectrum of $W(z)$ is either empty (this is impossible) or it is all of the unit circle. You can use that the spectrum is invariant under unitary conjugation.

This tells you about the spectrum of $W(z) - \mathbb{I}$. Finally conclude that for $z \neq 0$

$$\|W(z) - \mathbb{I}\| = 2$$

.

E9.2 Prove the Stone-von Neumann theorem. Assume that you have a regular irreducible represenations $\pi\colon CCR \to \mathcal{B}(\mathcal{H})$. Show first that

$$P := \frac{1}{2\pi}\int d^2z\, \pi(W(z))e^{-|z|^2/4}$$

(defined in terms of matrix elements) is an orthogonal projection. Show also that

$$P\pi(W(z))P = e^{-|z|^2/4}P.$$

If $\Omega \in P\mathcal{H}$ normalized, show that the span of all $\pi(W(z))$ for $z \in \mathbb{C}$ is invariant under the action of all Weyl-operators W. From the irriducubility of the representation conclude that $P\mathcal{H}$ is one dimensional.

Use this to construct the unitary equivalence of two regular irreducible representations of CCR.

E9.3 Construct an irreducible representation of CCR on a Hilbert space of functions $\psi : \mathbb{R} \to \mathbb{C}$ where $\pi(U(s))$ is the multiplication operator e^{isx} and $(\pi(V(t))\psi)(x) = \psi(x+t)$, both act as unitary operators. Different from the Schrödinger representation, let $\mathcal{H}$ contain the function that is 1 everywhere (assume this to be normalized to $\|1\| = 1$). Show that this representation is inequivalten to the Schrödinger representation.

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Homework Sheet 8

(Released 4.12.2024 – Discussed 11.12.2024)

E8.1 Let $d \geq 1$, $s > d/2$ and $f \in H^s(\mathbb{R}^d)$. Prove that f is Hölder's continuous, namely

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\varepsilon} \leq C \|f\|_{H^s(\mathbb{R}^d)}$$

for some constants $\varepsilon \in (0, 1)$ and $C = C(d, s, \varepsilon)$ independent of f.

Hint: You can use the inverse Fourier and $|1 - e^{ia}| \leq \min(2, |a|)$ for all $a \in \mathbb{R}$.

In the following we always consider the Schrödinger operator $A = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$ with domain $D(A) = C_c^2(\mathbb{R}^d)$. Here $V : \mathbb{R}^d \to \mathbb{R}$ is a real-valued potential.

E8.2 Let $d = 2$.

(a) Prove that for all $q \in [2, \infty)$ we have the Sobolev inequality

$$\|u\|_{L^q(\mathbb{R}^2)} \leq C_q \|u\|_{H^1(\mathbb{R}^2)}, \quad \forall u \in H^1(\mathbb{R}^2).$$

Hint: You can mimic the proof of the case $d \geq 3$.

(b) Prove that $-\Delta + V$ is bounded from below if $V \in L^p(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ for $p > 1$.

E8.3 Let $d \geq 3$.

(a) Prove that the Sobolev inequality $\|u\|_{L^{2^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^d)}$, with $2^* = 2d/(d-2)$, is *equivalent* to the fact that $-\Delta + V \geq 0$ if $\|V\|_{L^{d/2}}$ small enough.

(b) Prove that $-\Delta + V$ is bounded from below if $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$.

E8.4 Prove that for all dimensions $d \geq 1$ and $V \in L^{1+d/2}(\mathbb{R}^d)$ we have

$$-\Delta + V \geq -C \int_{\mathbb{R}^d} |V(x)|^{1+d/2}(x).$$

E8.5 Let $\{u_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^d)$ be orthonormal functions in $L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |x|^2 |u_n(x)|^2 \mathrm{d}x \leq C, \quad \forall n \in \mathbb{N}.$$

Prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} |\nabla u_n(x)|^2 \mathrm{d}x = \infty.$$

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Homework Sheet 7

(Released 27.11.2024 – Discussed 4.12.2024)

E7.1 Let $s, d \in \mathbb{N}$ and $f \in H^s(\mathbb{R}^d)$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi(x) = 1$ if $|x| \le 1$ and $\chi(x) = 0$ if $|x| \ge 2$. Define

$$f_n(x) = \chi(x/n) \int_{\mathbb{R}^d} n^d \hat{\chi}(n(x-y)) f(y) \mathrm{d}y.$$

Prove that $f_n \in C_c^\infty(\mathbb{R}^d)$ and $f_n \to f$ in $H^s(\mathbb{R}^d)$.

E7.2 Prove that if $f, g \in H^1(\mathbb{R}^d)$, then we have the integration by part

$$\int_{\mathbb{R}^d} (\partial_j f)(x) g(x) \mathrm{d}x = -\int_{\mathbb{R}^d} f(x)(\partial_j g)(x) \mathrm{d}x, \quad j = 1, 2, ..., d.$$

Hint: You first consider f, g smooth, and then use a density argument (thanks to E7.1).

E7.3 (a) Let $d \ge 1$. Prove that the equation

$$\Big(-\Delta - \frac{(d-2)^2}{4|x|^2} \Big) f(x) = 0$$

has a solution $f(x) = |x|^{-\alpha}$ for a suitable $\alpha > 0$. Deduce that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \mathrm{d}x \ge \int_{\mathbb{R}^d} \frac{(d-2)^2}{4|x|^2} |u(x)|^2 \mathrm{d}x$$

for all $u \in C_c^\infty(\mathbb{R}^d)$ such that $u = 0$ in a neighborhood of 0.

(b) Let $d \ge 3$. Prove that the above Hardy inequality holds true for all $u \in H^1(\mathbb{R}^d)$.

Hint: You can use the Perron-Frobenius principle for (a) and a density argument for (b).

E7.4 (a) Use the hydrogen atom lower bound to prove the following inequality

$$\int_{\mathbb{R}^3} |\nabla u(x)|^2 \mathrm{d}x \ge \left(\int_{\mathbb{R}^3} \frac{1}{|x|} |u(x)|^2 \mathrm{d}x \right)^2, \quad \forall u \in H^1(\mathbb{R}^3), \|u\|_{L^2(\mathbb{R}^3)} = 1.$$

(b) Use the above inequality to prove the Heisenberg uncertainty principle

$$\left(\int_{\mathbb{R}^3} |\nabla u(x)|^2 \mathrm{d}x \right) \left(\int_{\mathbb{R}^3} |x|^2 |u(x)|^2 \mathrm{d}x \right) \ge 1, \quad \forall u \in H^1(\mathbb{R}^3), \|u\|_{L^2(\mathbb{R}^3)} = 1.$$

Note: The optimal constant in the Heisenberg uncertainty principle is $d^2/4 = 9/4$.

E7.5 Let $d \ge 3$ and $1 \le p \le \infty$. We denote by $C_{d,p}$ a constant depending only on d, p.

(a) Prove that if $\|u\|_{L^p(\mathbb{R}^d)} \le C_{d,p} \|\nabla u\|_{L^2(\mathbb{R}^d)}$ for all $u \in H^1(\mathbb{R}^d)$, then $p = 2^* = \frac{2d}{d-2}$.

(b) Prove that if $\|u\|_{L^p(\mathbb{R}^d)} \le C_{d,p} \|u\|_{H^1(\mathbb{R}^d)}$ for all $u \in H^1(\mathbb{R}^d)$, then $2 \le p \le 2^*$.

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Homework Sheet 6

(Released 20.11.2024 – Discussed 27.11.2024)

E6.1 Let $A : D(A) \to \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$. Let $\overline{A} : D(\overline{A})$ be the closure of A.

(a) Prove that $\overline{A}$ is a closed operator, namely the graph $\{(x, \overline{A}x) : x \in D(\overline{A})\}$ is close in $\mathcal{H} \oplus \mathcal{H}$. Here $(x, y)_{\mathcal{H}\oplus\mathcal{H}} = \|x\| + \|y\|$ with the norm $\|.\|$ in $\mathcal{H}$.

(b) Prove that if A is a closed operator, namely if the graph $\{(x, Ax) : x \in D(A)\}$ is close in $\mathcal{H} \oplus \mathcal{H}$, then $A = \overline{A}$.

E6.2 Let $A : D(A) \to \mathcal{H}$ be an operator on a Hilbert space $\mathcal{H}$ and $A \geq 1$. Prove that

$$\langle x, y \rangle_Q = \langle x, Ay \rangle_{\mathcal{H}}, \quad \forall x, y \in D(A)$$

is an inner product on $D(A)$. (We used this to defined $Q(A) = \overline{D(A)}^{\|.\|_Q}$.)

E6.3 Let $A : D(A) \to \mathcal{H}$ be a symmetric operator on a Hilbert space $(\mathcal{H}, \|.\|)$ with $A \geq 1$. Let A_F be the Friedrichs extension of A.

(a) Prove that the quadratic form domain $Q(A)$ is the same with the domain $D(\sqrt{A_F})$.

(b) Prove that

$$\inf_{x \in D(A), \|x\|=1} \langle x, Ax \rangle = \inf_{x \in Q(A), \|x\|=1} \|x\|^2_{Q(A)} = \inf_{x \in D(A_F), \|x\|=1} \langle x, A_F x \rangle.$$

E6.4 Let $A : D(A) \to \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ such that $A \geq 1$. Prove that if A is essentially self-adjoint, namely $\overline{A}$ is self-adjoint, then $\overline{A}$ coincides with the Friedrichs extension of A.

E6.5 Let $A : D(A) \to \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$. Prove that A is self-adjoint if and only if $C = (A + i)(A - i)^{-1}$ is well-defined as a unitary operator on $\mathcal{H}$. (This is called the Cayley transform.)

Homework Sheet 5

(Released 13.11.2024 – Discussed 20.11.2024)

E5.1 Let $A : D(A) \to \mathcal{H}$ be a symmetric operator on a Hilbert space. Let $i^2 = -1$.

(a) Prove that $\|(A \pm i)x\| \geq \|x\|$ for all $x \in D(A)$.

(b) Prove that if $\mathrm{Ran}(A + i) = \mathcal{H}$, then $(A + i)^{-1}$ is a bounded operator.

(c) Prove that A is self-adjoint if and only if $\mathrm{Ran}(A \pm i) = \mathcal{H}$.

E5.2 Let $A : D(A) \to \mathcal{H}$ be a symmetric operator on a Hilbert space. Prove that A is self-adjoint if and only if its spectrum is real, namely $\sigma(A) \subset \mathbb{R}$.

E5.3 Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a Hilbert space. Then $S = (A+i)^{-1}$ is a bounded operator by Problem E5.1. Prove that S is a normal operator, namely $SS^* = S^*S$. (Note: You are not allowed to use the spectral theorem here, since we used this fact to prove the spectral theorem for self-adjoint unbounded operators.)

E5.4 Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Define the spectral projection

$$A_n = \mathbb{1}(A \leq n), \quad n \in \mathbb{N}$$

by functional calculus (i.e. via the spectral theorem). Prove that A_n converges to the identity $\mathbb{1}$ strongly, namely $A_n u \to u$ strongly for every $u \in \mathcal{H}$ as $n \to \infty$.

E5.5 (a) Let $(X, \|.\|_X)$ and $(X, \|.\|_Y)$ be two Banach spaces (the same vector space X with two different norms). Assume that $\|u\|_X \leq \|u\|_Y$ for all $u \in X$. Prove that there exists a constant $C > 0$ such that $\|u\|_Y \leq C\|u\|_X$ for all $u \in X$.

(b) Let (Ω, μ) be a sigma-finite measure space. Let $a : \Omega \to \mathbb{R}$ be a measurable function such that $af \in L^2(\Omega, \mu)$ for all $f \in L^2(\Omega, \mu)$. Prove that $a \in L^\infty(\Omega, \mu)$.

(c) Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that if $D(A) = \mathcal{H}$, then A is a bounded operator. (This is called the Hellinger–Toeplitz theorem.)

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Homework Sheet 4

(Released 8.11.2024 – Discussed 13.11.2024)

E4.1 Let A be a self-adjoint compact operator on a Hilbert space with the representation

$$A = \sum_{n\geq 1} \lambda_n |u_n\rangle\langle u_n|$$

where $\{u_n\}$ is an orthonormal family and $\lambda_n \in \mathbb{R}$, $\lambda_n \to 0$. Prove that

$$\sigma(A) = \{0\} \bigcup \{\lambda_n : n \geq 1\}.$$

E4.2 Let B be a bounded operator on a Hilbert space $\mathcal{H}$ such that

$$\|u\| \leq C\|Bu\|, \quad \|u\| \leq C\|B^* u\|, \quad \forall u \in \mathcal{H}$$

with a constant $C > 0$ independent of u. Prove that B^{-1} is a bounded operator.

E4.3 In the lecture, we have proved that if A is a self-adjoint bounded operator on a Hilbert space $\mathcal{H}$, then

$$\sup_{\|u\|\leq 1} |\langle u, Au\rangle| \leq \sup |\sigma(A)| \leq \|A\|.$$

Use these inequalities to prove that $\sup|\sigma(A)| = \|A\|$ (this is what we actually need to prove the Spectral theorem). Hint: You can apply the above inequalities to A and A^2.

E4.4 Let A be a self-adjoint bounded operator on a Hilbert space $\mathcal{H}$. For any polynomial $f(t) = \sum_{j=1}^N \alpha_j t^j$, $\alpha_j \in \mathbb{C}$, we defined $f(A) = \sum_{j=1}^N \alpha_j A^j$.

(a) Prove that for any polynomial, $\sigma(f(A)) = f(\sigma(A))$, and then deduce that

$$||f(A)|| = \sup_{t\in\sigma(A)} |f(t)|.$$

Hint: You can consider the factorized form of $\lambda - f(t)$ (fundamental theorem of algebra).

(b) Prove that we can extend the definition $f(A)$ for any $f \in \mathcal{C}(\sigma(A), \mathbb{C})$ by Weierstrass theorem. Moreover, we have

$$f(A)g(A) = (fg)(A), \quad \forall f, g \in \mathcal{C}(\sigma(A), \mathbb{C}).$$

This means that $f \mapsto f(A)$ is a C^*- isomorphism from $\mathcal{C}(\sigma(A), \mathbb{C})$ to $\mathcal{B}(\mathcal{H})$.

E4.5 Let A be a self-adjoint bounded operator on a Hilbert space $\mathcal{H}$. Let $u \in \mathcal{H}$ and let μ_u be the spectral measure associated with A. Prove that

$$\mu_u(\sigma(A)) = \|u\|^2.$$

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Homework Sheet 3

(Released 30.10.2024 – Discussed 6.11.2024)

E3.1 Let $A : \mathcal{H} \to \mathcal{H}$ be a trace class operator on a Hilbert space. Let $\{x_n\}_{n\ge1}$ be an orthonormal basis for $\mathcal{H}$ (not necessary the eigenfunctions of A). Prove that

$$\mathrm{Tr}(A) = \sum_{n\ge1} \langle x_n, Ax_n\rangle.$$

E3.2 Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator on a Hilbert space.

(a) Assume that $A \ge 0$ and that there exists an orthonormal basis $\{x_n\}_{n\ge1}$ such that

$$\sum_{n\ge1} \langle x_n, Ax_n\rangle < \infty.$$

Prove that A is trace class.

(b) Can we relax the condition $A \ge 0$ in (a)? Namely, if we only assume that A is bounded self-adjoint and that there exists an orthonormal basis $\{x_n\}_{n\ge1}$ satisfying $\sum_n |\langle x_n, Ax_n\rangle| < \infty$, then can we conclude that A is trace class?

E3.3 Let $A \le 0$ be a compact operator on a Hilbert space $\mathcal{H}$. Prove that its eigenvalues $\lambda_1 \le \lambda_2 \le ...$ (counting multiplicity) satisfy the *min-max principle*

$$\lambda_n = \min_{\substack{M\subset\mathcal{H}\\ \dim M=n}} \max_{\substack{u\in M\\ \|u\|=1}} \langle u, Au\rangle, \quad \forall n = 1, 2, ...$$

Hint: The minimum is attained when M is the space spanned by the first n eigenfunctions.

E3.4 Prove that if A is a bounded operator on a Hilbert space and $B \in S^p$, the Schatten space with $1 \le p \le \infty$, then both AB and BA belong to S^p.
Hint: You can estimate the singular values of X via those of X^*X or XX^* and use E3.3.

E3.5 Let (Ω, μ) be a σ-finite measure space. Let $a : \Omega \to \mathbb{C}$ be a measurable. Define the multiplication operator M_a on $L^2(\Omega, \mu)$ by

$$(M_a f)(x) = f(x)a(x), \quad D(M_a) = \{f \in L^2 : af \in L^2\}.$$

Prove the following statements:

(a) The spectrum of M_a is equal to the essential range of a.

(b) M_a is bounded if and only if a is bounded. Moreover, in this case $\|M_a\| = \|a\|_{L^\infty}$.

(c) M_a is self-adjoint if and only if a is real-valued almost everywhere.

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Homework Sheet 2

(Released 25.10.2024 – Discussed 30.10.2024)

E2.1 Let $A : D(A) \to \mathcal{H}$ be a densely defined operator on a Hilbert space. Prove that the following statements are equivalent:

(a) A is a symmetric operator.

(b) $\langle x, Ax \rangle \in \mathbb{R}$ for all $x \in D(A)$.

(c) $A \subset A^*$.

E2.2 Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space.

(a) Prove that A is bounded if and only if A^* is bounded, and in this case $\|A\| = \|A^*\|$.

(b) Prove that A is compact if and only if A^* is compact.

E2.3 Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space. Prove that A is bounded if and only if A maps weak convergence to weak convergence, namely if $x_n \rightharpoonup x$ weakly then $Ax_n \rightharpoonup Ax$ weakly.

E2.4 Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator on a Hilbert space. Prove that A^*A and AA^* have the same non-zero eigenvalues with the same multiplicities. Moreover, A maps each eigenspace of A^*A with a non-zero eigenvalue to the corresponding eigenspace of AA^*.

E2.5 Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator on a Hilbert space such that for every orthonormal sequence $\{x_n\}$, we have $Ax_n \to 0$ strongly. Can we conclude that A is a compact operator?

Homework Sheet 1

(Released 18.10.2024 – Discussed 23.10.2024)

E1.1 Let $(\mathcal{H}, \|.\|)$ be a Banach space. Prove that it is a Hilbert space if and only if the norm satisfies

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2), \quad \forall u, v \in \mathcal{H}.$$

How can we write the inner product $\langle u, v\rangle$ in terms of the norm of suitable vectors?

E1.2 Let V be a (complex) vector space and let $Q : V \times V \to \mathbb{C}$ be a bilinear form (in particular $Q(u,v)$ is linear in v and anti-linear in u). Prove that if $Q(u,u) \geq 0$ for all $v \in \mathcal{H}$, then we have the Cauchy–Schwarz inequality

$$|Q(u,v)| \leq \sqrt{Q(u,u)}\sqrt{Q(v,v)}, \quad \forall u, v \in V.$$

E1.3 Let V be a dense subspace of a separable Hilbert space $\mathcal{H}$. Prove that there exists an orthonormal basis $\{u_n\}_{n\geq 1}$ for $\mathcal{H}$ such that $u_n \in V$ for all $n \geq 1$.

E1.4 (a) Argue that for symmetric operators A, B on a Hilbert space $\mathcal{H}$, we have

$$\langle \Delta A\rangle \cdot \langle \Delta B\rangle \geq \frac{1}{4}\langle \mathrm{Im}[A,B]\rangle^2$$

where $\langle \Delta A\rangle = \langle \psi, A^2\psi\rangle - \langle \psi, A\psi\rangle^2$, $\langle \Delta B\rangle = \langle \psi, B^2\psi\rangle - \langle \psi, B\psi\rangle^2$ and $\langle \mathrm{Im}[A,B]\rangle = \mathrm{Im}\langle \psi, [A,B]\psi\rangle$ with a suitable vector $\psi \in \mathcal{H}$ which belongs to the domain of all relevant operators. When does the equality occur?

(b) For $A = -\mathbf{i}\partial_x$ (momentum operator) and $B = x$ (multiplication operator) on $\mathcal{H} = L^2(\mathbb{R})$, what do we get from (a)? Determine all functions $\Psi \in L^2(\mathbb{R})$ satisfying the corresponding equality.

E1.5 Let $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$.

(a) Prove Young's identity

$$\frac{a^p}{p} = \sup_{b\in[0,\infty)} \left(ab - \frac{b^q}{q}\right), \quad \forall a \in [0,\infty).$$

(b) Deduce Hölder's inequality

$$\|f\|_{L^p} = \sup_{\|g\|_{L^q}\leq 1} \left| \int_{\mathbb{R}^d} fg \right|, \quad \forall f \in L^p(\mathbb{R}^d)$$

and the triangle inequality

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}, \quad \forall f, g \in L^p(\mathbb{R}^d).$$