Mathematisches Institut LMU — Mathematical Quantum Mechanics

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Homework Sheet 2

(Released 29.4.2026)

E2.1 Consider the operator $A = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$ with a real-valued potential $V \in C_c(\mathbb{R}^d)$.

(a) Prove that A is a self-adjoint operator with $D(A) = H^2(\mathbb{R}^d)$.

(b) Use Weyl's criterion to prove that $\sigma_{\text{ess}}(A) = [0, \infty)$.

E2.2 Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that A is bounded from below and $\lambda_1 \le \lambda_2 \le ...$ are the min-max values of A, with $\lim_{n\to\infty\infty} \lambda_n = \lambda_\infty \in (-\infty, \infty]$.

(a) Prove that if $\{u_n\}_{n=1}^\infty \subset D(A)$ is an orthonormal family, then

$$\liminf_{n\to\infty} \langle u_n, Au_n\rangle \ge \lambda_\infty.$$

(b) Prove that for every $N \in \mathbb{N}$ we have

$$\sum_{i=1}^N \lambda_i = \inf \left\{ \sum_{i=1}^N \langle u_i, Au_i\rangle : \{u_i\}_{i=1}^N \subset D(A) \text{ an orthonormal family} \right\}.$$

Hint: You may use the min-max principle.

E2.3 Recall the Lieb–Thirring inequality: if the operator $A = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$ has N negative eigenvalues $\lambda_1 \le ... \le \lambda_N$ (with multiplicity), then

$$\sum_{i=1}^N \lambda_i \ge -L_d \int_{\mathbb{R}^d} |V_-|^{1+d/2}.$$

Use this result to prove the kinetic inequality: for every $N \in \mathbb{N}$ and for every orthonormal family $\{u_n\}_{n=1}^N$, we have

$$\sum_{n=1}^N \|\nabla u_n\|^2_{L^2(\mathbb{R}^d)} \ge K_d \int_{\mathbb{R}^d} \rho_N^{1+2/d}, \quad \rho_N(x) = \sum_{n=1}^N |u_n(x)|^2.$$

Here $K_d > 0$ and $L_d > 0$ are constants depending only on $d \ge 1$.

Hint: In the lecture we proved the opposite direction by a duality argument.

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Homework Sheet 1

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E1.1 Let $\mathcal{H} = L^2(\mathbb{R}^M)$ and $\mathcal{K} = L^2(\mathbb{R}^N)$ with orthonormal bases $\{f_m\}_m$ and $\{g_n\}_n$. Use $\mathcal{H} \otimes \mathcal{K} = \overline{\mathrm{Span}(f \otimes g : f \in \mathcal{H}, g \in \mathcal{K})}$ with $(f \otimes g)(x,y) = f(x)g(y)$ to prove that $\{f_m \otimes g_n\}_{m,n}$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{K}$.

E1.2 Let Ω be a measurable subset of $\mathbb{R}^d$. Let $u_1, u_2, ..., u_N$ be orthonormal functions in $L^2(\Omega)$. Consider the *Slater determinant*

$$(u_1 \wedge \cdots \wedge u_N)(x_1, \ldots, x_N) = (N!)^{-1/2} \det \begin{pmatrix} u_1(x_1) & \cdots & u_1(x_N) \\ u_2(x_1) & \cdots & u_2(x_N) \\ & \vdots & \\ u_N(x_1) & \cdots & u_N(x_N) \end{pmatrix}.$$

Prove that $\Psi = u_1 \wedge \cdots \wedge u_N$ is a normalized function in antisymmetric space $L^2_a(\Omega^N)$.

E1.3 For any normalized wave function $\Psi \in L^2_a(\Omega^N)$, the one-body density matrix γ_Ψ is an operator on $L^2(\Omega)$ with kernel

$$\gamma_\Psi(x,y) = N \int_{\Omega^{N-1}} \Psi(x, x_2, ..., x_N)\overline{\Psi(y, x_2, ..., x_N)} \mathrm{d}x_2 ... \mathrm{d}x_N$$

(a) Prove that $\gamma_\Psi \geq 0$ and $\operatorname{Tr} \gamma_\Psi = N$.

(b) Prove that if $\Psi = u_1 \wedge \cdots \wedge u_N$ as in E1.2, then

$$\gamma_\Psi = \sum_{i=1}^N |u_i\rangle\langle u_i|.$$

E1.4 Let $\mathcal{H}$ be a Hilbert space. Let U be a linear operator on $\mathcal{H}^{\otimes N}$ defined by

$$U(f_1 \otimes f_2 \otimes ... \otimes f_N) = \frac{1}{N!} \sum_{\tau \in P_N} (-1)^\sigma f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes ... \otimes f_{\sigma(N)}.$$

(a) Prove that $U\Psi$ is anti-symmetric for every $\Psi \in \mathcal{H}^{\otimes N}$.

(b) Prove that $\Psi \in \mathcal{H}^{\otimes N}$ is anti-symmetric if and only if $U\Psi = \Psi$.

(c) Prove that if $\dim \mathcal{H} < N$, and if $\Psi \in \mathcal{H}^{\otimes N}$ is anti-symmetric, then $\Psi = 0$.