LMU | LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN | MATHEMATISCHES INSTITUT

Mathematical Quantum Mechanics (Summer 2025)

Midterm Exam

First name: . . . . Family name: . . . .

You have **120 minutes** to solve **3 problems**. You can use your notes from the lectures, homeworks and tutorials. Electronic devices are not allowed.

Problem 1. (1+1 points) Let $\mathcal{H}$ be a finite-dimensional Hilbert space.

(a) Prove that if $\dim \mathcal{H} = N$, then the fermionic space $\mathcal{H}_a^N$ is one-dimensional.

(b) If $\dim \mathcal{H} = M > N$, what is the dimension of $\mathcal{H}_a^N$?

Problem 2. (1+1+2 points) Let u_0, u_1 be two orthonormal vectors in a Hilbert space $\mathcal{H}$. Let Ω be the vacuum of the bosonic Fock space $\mathcal{F}(\mathcal{H})$. For $N \geq 2$, consider the N-body vector

$$\Psi_N = \frac{a^*(u_0)^{N-1}}{\sqrt{(N-1)!}} a^*(u_1)\Omega \in \mathcal{H}_s^N.$$

(a) Prove that Ψ_N is a normalized vector.

(b) Compute the one-body density matrix $\gamma_{\Psi_N}^{(1)}$ and determine $\lim_{N\to\infty} N^{-1}\gamma_{\Psi_N}^{(1)}$.

(c) Prove or disprove that $\|\Psi_N - u_0^{\otimes N}\|_{\mathcal{H}_s^N} = 0$ when $N \to \infty$.

Problem 3. (1+1+2 points) Let Ψ_N be a normalized vector in the bosonic space $\mathcal{H}_s^N$ with $\mathcal{H} = L^2(\mathbb{R}^d)$. Assume that the one-body density matrix $\gamma_{\Psi_N}^{(1)}$ has the spectral decomposition

$$\gamma_{\Psi_N}^{(1)} = \sum_{n\geq 1} \lambda_n |u_n\rangle\langle u_n|$$

with $\lambda_n \geq 0$, $\sum_n \lambda_n = N$, $\{u_n\}$ orthonormal, and denote $\rho_{\Psi_N}(x) = \sum_{n\geq 1} \lambda_n |u_n(x)|^2$.

(a) Prove that

$$\rho_{\Psi_N}(x) = N \int_{\mathbb{R}^{d(N-1)}} |\Psi_N(x, x_2, ..., x_N)|^2 \mathrm{d}x_2 ... \mathrm{d}x_N.$$

(b) Prove that

$$\mathrm{Tr}(-\Delta\gamma_{\Psi_N}^{(1)}) \geq \int_{\mathbb{R}^d} |\nabla\sqrt{\rho_{\Psi_N}}(x)|^2 \mathrm{d}x.$$

Hint: You may use the diamagnetic inequality $|\nabla f|^2 + |\nabla g|^2 \geq |\nabla\sqrt{|f|^2 + |g|^2}|^2$.

(c) Let $0 \leq \hat{w} \in L^1(\mathbb{R}^d)$ and let $V : \mathbb{R}^d \to \mathbb{R}$ be sufficiently regular. Prove that

$$\frac{\langle \Psi_N, H_N\Psi_N\rangle}{N} \geq \mathcal{E}^{\mathrm{H}}(\sqrt{N^{-1}\rho_{\Psi_N}}) - \frac{w(0)}{2(N-1)}$$

where

$$H_N = \sum_{n=1}^{N}(-\Delta_{x_n} + V(x_n)) + \frac{1}{N-1}\sum_{1\leq m<n\leq N} w(x_m - x_n)$$

and

$$\mathcal{E}^{\mathrm{H}}(u) = \langle u, (-\Delta + V)u\rangle + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(x)|^2|u(y)|^2 w(x-y)\mathrm{d}x\mathrm{d}y.$$