Mathematisches Institut LMU
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Mathematical Quantum Mechanics
Summer 2025

Homework Sheet 2

(Released 2.5.2025 – Discussed 9.5.2025)

E2.1 Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that H is bounded from below and $\mu_\infty = \inf \sigma_{\rm ess}(H)$ is finite.

(a) Prove that for every $N \in \mathbb{N}$, there exists an orthonormal family $\{u_n\}_{n=1}^N$ such that

$$\max_{1\le n\le N} \langle u_n, Hu_n\rangle \le \mu_\infty.$$

(b) Prove that there exists an orthonormal family $\{u_n\}_{n=1}^\infty$ such that $\langle u_n, Hu_n\rangle \to \mu_\infty$.

E2.2 Let $A \ge 0$ be a self-adjoint compact operator on a Hilbert space $\mathcal{H}$. Prove that the following statements are equivalent:

(a) A is trace class;

(b) $\sum_{n\ge1}\langle u_n, Au_n\rangle$ is finite for some orthonormal basis $\{u_n\}_{n\ge1}$ for $\mathcal{H}$.

Prove also that in this case, $\operatorname{Tr} A = \sum_{n\ge1}\langle u_n, Au_n\rangle$ for every orthonormal basis $\{u_n\}_{n\ge1}$.

Note: If $A \ge 0$ is self-adjoint and $\sum_{n\ge1}\langle u_n, Au_n\rangle < \infty$ for some orthonormal basis $\{u_n\}_{n\ge1}$, then we can still deduce that A is trace class. But the proof is more difficult.

E2.3 Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ which is bounded from below, namely $\mathcal{H} \ge -C$. Let Γ be a (mixed) state, namely $\Gamma \ge 0$ and $\operatorname{Tr}\Gamma = 1$. Prove that

$$\operatorname{Tr}[\Gamma^{1/2}H\Gamma^{1/2}] = \operatorname{Tr}[(H+C)^{1/2}\Gamma(H+C)^{1/2}] - C$$

where the both sides make sense as a number in $[-C,\infty]$. We denote this by $\operatorname{Tr}[H\Gamma]$.

E2.4 Let Γ be a (mixed) state on a Hilbert space $\mathcal{H}$. Let $S(\Gamma) = -\operatorname{Tr}[\Gamma\log\Gamma]$.

(a) Prove that $S(\Gamma) = 0$ if and only if Γ is a pure state (i.e. a rank-1 operator).

(b) Prove that if Γ is a rank-N operator (namely $\dim \Gamma\mathcal{H} = N$), then

$$0 \le \Gamma(\Gamma) \le N\log N.$$

Deduce that if $S(\Gamma) = \infty$, then Γ is an infinite-rank operator.

(c) Give an example where Γ is infinite-rank but $S(\Gamma) < \infty$.

E2.5 Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ such that $\operatorname{Tr}(e^{-H/T}) = \infty$ for some temperature $T > 0$. Prove that

$$F_T := \inf\{\operatorname{Tr}[H\Gamma] + T\operatorname{Tr}[\Gamma\log\Gamma] : \Gamma \ge 0, \operatorname{Tr}\Gamma = 1\} = -\infty.$$

Hint: If $\mathcal{H}$ has compact resolvent, you may construct a sequence of trial states Γ using eigenfunctions of $\mathcal{H}$. If H has essential spectrum, then E2.1 is helpful.

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Homework Sheet 1

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E1.1 Let $f \in L^2(\mathbb{R})$. Prove that if both f and $\hat{f}$ are compactly supported, then $f = 0$. This is an illustration for the uncertainty principle.

E1.2 Let Ω be a measurable subset of $\mathbb{R}^d$. Let $N \geq 2$. Consider the two operators $P_\pm$ on $\mathcal{H}^N = L^2(\Omega^N)$ defined by

$$(P_+\Psi)(x_1, ..., x_N) = (N!)^{-1} \sum_{\sigma \in S_N} \Psi(x_{\sigma(1)}, ..., x_{\sigma(N)}), \quad x_j \in \Omega$$

$$(P_-\Psi)(x_1, ..., x_N) = (N!)^{-1} \sum_{\sigma \in S_N} (-1)^\sigma \Psi(x_{\sigma(1)}, ..., x_{\sigma(N)}).$$

Here S_N is the permutation group of $\{1, 2, ..., N\}$ and $(-1)^\sigma$ is the sign of $\sigma \in S_N$.

(a) Prove that P_+ and P_- are two orthogonal projections on $\mathcal{H}^N$ (i.e. $P = P^*$, $P^2 = P$).

(b) Prove that the subspaces $\mathcal{H}_{\mathrm{s}}^N = P_+\mathcal{H}^N$ and $\mathcal{H}_{\mathrm{a}}^N = P_-\mathcal{H}^N$ satisfy $\mathcal{H}_{\mathrm{s}}^N \cap \mathcal{H}_{\mathrm{a}}^N = \{0\}$.

The symmetric space $\mathcal{H}_{\mathrm{s}}^N$ corresponds to bosons, while the antisymmetric space $\mathcal{H}_{\mathrm{a}}^N$ corresponds to fermions.

E1.3 Let Ω be a measurable subset of $\mathbb{R}^d$. Let $u_1, u_2, ..., u_N$ be orthonormal functions in $L^2(\Omega)$. Consider the *Slater determinant*

$$(u_1 \wedge \cdots \wedge u_N)(x_1, \ldots, x_N) = (N!)^{-1/2} \det \begin{pmatrix} u_1(x_1) & \cdots & u_N(x_1) \\ u_1(x_2) & \cdots & u_N(x_2) \\ & \vdots & \\ u_1(x_N) & \cdots & u_N(x_N) \end{pmatrix}.$$

Prove that $u_1 \wedge \cdots \wedge u_N$ is a normalized vector in the antisymmetric space $\mathcal{H}_{\mathrm{a}}^N$.

E1.4 Let Ω be a measurable subset of $\mathbb{R}^d$. Let $w : \mathbb{R}^d \to \mathbb{R}$ be a regular even function.

(a) Consider the Hamiltonian $H_N = \sum_{j=1}(-\Delta_{x_j}) + \sum_{j<k} w(x_j - x_k)$ and a function $u \in H^1(\Omega)$, $\|u\|_{L^2(\Omega)} = 1$. Compute the expectation

$$\mathcal{E}(u) = N^{-1}\langle u^{\otimes N}, H_N u^{\otimes N}\rangle, \quad u^{\otimes N}(x_1, ..., x_N) = u(x_1)...u(x_N).$$

(b) Prove that if u_0 is a minimizer for $\mathcal{E}(u)$ under the constraint $\|u\|_{L^2(\Omega)} = 1$, then

$$-\Delta u_0 + (N-1)(w * |u_0|)u_0 = \mu u_0$$

in the distributional sense $\mathcal{D}'(\Omega)$, with a parameter $\mu \in \mathbb{R}$.