Partial Differential Equations II

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Unofficial Lecture Notes

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Contents

Chapter 1

L^p-Spaces

Definition 1.1. Let Ω be a set, Σ a collection of subsets of Ω which is a σ-algebra and let $\mu : \Sigma \to [0, \infty]$ be a measure. We call (Ω, Σ, μ) a measure space. □

Example 1.2. Let $\Omega \subset \mathbb{R}^d$ be open, Σ the Borel-σ-algebra, $\mu := \lambda^d$ the Borel-Lebesgue measure, uniquely characterised by

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = \prod_{j=1}^{n} |b_j - a_j|.$$

Definition 1.3. Given a measure space (Ω, Σ, μ) and $f : \Omega \to \mathbb{R}$ and f measurable. Define $Sf(t) := f^{-1}\big((t, \infty)\big)$ and note that Sf is monotone and non-increasing. Then $Ff : \mathbb{R} \to [0, \infty]$, $Ff(t) = \mu(Sf(t))$, for $t \in \mathbb{R}$, is decreasing in t.
For $f \geqslant 0$ everywhere define

$$\int_\Omega f(x) \mathrm{d}\mu(x) := \int_0^\infty Ff(t) \mathrm{d}t$$

where the r.h.s. is a Riemann-integral.
If the integral is not infinite, we say that f is Lebesgue-integrable.
For $f : \Omega \to \mathbb{C}$, f is measurable iff $\Re f$ and $\Im f$ are. For all $x \in \mathbb{R}$ let $x_\pm := \max\{\pm x, 0\}$. Then

$$f = (\Re f)_+ - (\Re f)_- + i(\Im f)_+ - i(\Im f)_-$$

If $(\Re f)_\pm$ and $(\Im f)_\pm$ are integrable, we say that f is integrable and

$$\int_\Omega f \mathrm{d}\mu := \int_\Omega (\Re f)_+ \mathrm{d}\mu - \int_\Omega (\Re f)_- \mathrm{d}\mu + i \int_\Omega (\Im f)_+ \mathrm{d}\mu - i \int_\Omega (\Im f)_- \mathrm{d}\mu$$

An alternative construction: First define the Lebesgue integral on simple function and then pass to $f : \Omega \to [0, \infty)$ by approximation. □

Corollary 1.4. *For all $f : \Omega \to \mathbb{C}$ measurable and integrable for all $\varepsilon > 0$ there exists a $\varphi_\varepsilon \in \mathcal{S}$ such that*

$$\int_\Omega |f(x) - \varphi_\varepsilon(x)| \mathrm{d}\mu(x) < \varepsilon$$

□

Theorem 1.5 (Monotone Convergence). *Let $(f_j)_{j\in\mathbb{N}}$ a non-decreasing sequence of non-negative integrable functions on (Ω, Σ, μ) (i.e. μ-a.e. $(f_j(x))_j$ for $x \in \Omega$ is increasing), then*

$$\lim_{j\to\infty} f_j(x) = f(x)$$

is measurable and

$$\lim_{j\to\infty} \int_\Omega f_j(x) \mathrm{d}\mu(x) = \int_\Omega \lim_{j\to\infty} f_j(x) \mathrm{d}\mu(x).$$

□

Theorem 1.6 (Dominated Convergence). *Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of integrable complex-valued function on (Ω, Σ, μ) which converge to f pointwise μ-a.e. If there exists a $G \geqslant 0$ integrable on (Ω, Σ, μ) satisfying $|f_j(x)| \leqslant G(x)$ for all $j \in \mathbb{N}$ μ-a.e., then f is integrable and*

$$\lim_{j\to\infty} \int_\Omega f(x) \mathrm{d}\mu(x) = \int_\Omega f(x) \mathrm{d}\mu(x)$$

□

Theorem 1.7 (Fatou's Lemma). *Let $(f_j)_{j\in\mathbb{N}}$ be a non-negative, integrable on (Ω, Σ, μ). Then $f(x) := \liminf_{j\to\infty} f_j(x)$ is measurable and*

$$\liminf_{j\to\infty} \int_\Omega f_j(x)\mathrm{d}\mu(x) \geqslant \int_\Omega f(x)\mathrm{d}\mu(x).$$

□

Theorem 1.8 (Brezis-Lieb, refinement of Fatou's Lemma). *Let $(f_j)_{j\in\mathbb{N}} : \Omega \to \mathbb{C}$ be measurable and converging towards to $f : \Omega \to \mathbb{C}$ μ-a.e. and for $p \in (0, \infty)$ let there exist a $C > 0$ such that for all $j \in \mathbb{N}$ $\int_\Omega |f_j(x)|^p \mathrm{d}\mu(x) \leqslant C$. Then*

$$\lim_{j\to\infty} \int_\Omega ||f_j(x)|^p - |f_j(x) - f(x)|^p - |f(x)|^p|\mathrm{d}\mu(x) = 0$$

□

Corollary.

$$\int_\Omega |f_j(x)|^p \mathrm{d}\mu(x) = \int_\Omega |f|^p \mathrm{d}\mu + \int_\Omega |f - f_j|^p \mathrm{d}\mu + o(1)$$

□

Proof of Theorem 1.8. By Fatou's lemma $\int_\Omega |f|^p \mathrm{d}\mu \leqslant C$.

We claim that for all $p \in (0, \infty)$ and all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for all $a, b \in \mathbb{C}$

$$||a + b|^p - |b|^p| \leqslant \varepsilon|b|^p + c_\varepsilon|a|^p$$

the proof of which is an exercise.

For all $j \in \mathbb{N}$ let $g_j := f_j - f$, then $\lim_{j\to\infty} g_j(x) = 0$ μ-a.e. Now fix $\varepsilon > 0$.

$$0 \leqslant \int_\Omega ||f + g_j|^p - |g_j|^p - |f|^p|\mathrm{d}\mu \leqslant \varepsilon \int_\Omega |g_j|^p \mathrm{d}\mu + \int_\Omega G_{j,\varepsilon}\mathrm{d}\mu$$

with

$$G_{j,\varepsilon}(x) := \Big(\underbrace{||f+g_j|^p - |g_j|^p - |f|^p|}_{\leqslant ||f+g_j|^p - |g_j|^p| + |f|^p \leqslant \varepsilon |g_j|^p + (1+c_\varepsilon)|f|^p} - \varepsilon |g_j|^p \Big)_+ \leqslant (1+c_\varepsilon)|f|^p$$

and thus by dominated convergence $\int G_{j,\varepsilon} \mathrm{d}\mu \xrightarrow{j\to\infty} 0$, on the other hand

$$\int |g_j|^p \mathrm{d}\mu \leqslant \int (|f| + |f_j|)^p \mathrm{d}\mu 2^p \leqslant 2^{p+1} C,$$

taking lim sup and letting $\varepsilon \to 0$ and the claim follows. *q.e.d.*

For $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ σ-finite measure spaces and define the product σ-algebra, $\Sigma_1 \otimes \Sigma_2$ as the smallest σ-algebra containing all rectangles $\left\{ A_1 \times A_2 \,\middle|\, A_1 \in \Sigma_1, A_2 \in \Sigma_2 \right\}$. Then there exists a unique product measure $\mu_1 \otimes \mu_2$ on $\Sigma_1 \otimes \Sigma_2$ that satisfies

$$\forall A_j \in \Sigma_j, j = 1, 2 \qquad (\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$$

Theorem 1.9 (Fubini-Tonelli). *If $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$ is $\Sigma_1 \otimes \Sigma_2$ measurable, then for $g \in \left\{ (\Re f)_+, (\Re f)_-, (\Im f)_+, (\Im f)_- \right\}$ the maps*

$$x_1 \longmapsto \int_{\Omega_2} g(x_1, x_2) \mathrm{d}\mu_2(x_2)$$

$$x_2 \longmapsto \int_{\Omega_1} g(x_1, x_2) \mathrm{d}\mu_1(x_1)$$

are respectively μ_1 and μ_2 measurable.
If $f \geqslant 0$, $\mu_1 \otimes \mu_2$-a.e., then

$$\int_{\Omega_1 \times \Omega_2} f \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \int_{\Omega_2} f(x_1, x_2) \mathrm{d}\mu_2(x_2) \mathrm{d}\mu_1(x_1) = \int_{\Omega_2} \int_{\Omega_1} f(x_1, x_2) \mathrm{d}\mu_1(x_1) \mathrm{d}\mu_2(x_2)$$

The same holds for $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$ provided one the above integrals is finite for $|f|$. □

Let (Ω, Σ, μ) be a measure space.

Definition 1.10 (L^p-space)**.** For $p \in [1, \infty)$, let

$$\tilde{L}^p(\Omega, \mathrm{d}\mu) := \left\{ f : \Omega \to \mathbb{C} \,\middle|\, f \text{ is measurable and } |f|^p \text{ is integrable} \right\}.$$

Introducing the equivalence relation

$$f \sim g :\iff \exists N \in \Sigma : \mu(N) = 0 \wedge \forall x \in N^C : f(x) = g(x) \iff f = g \ \mu\text{-a.e.}$$

We define $L^p(\Omega, \mathrm{d}\mu) := \tilde{L}^p(\Omega, \mathrm{d}\mu)/\sim$. L^p is a vector space over $\mathbb{C}$ with pointwise linear operations on $\tilde{L}^p$. This follows from $|\alpha + \beta|^p \leqslant 2^{p-1}(|\alpha|^p + |\beta|^p)$ for all $\alpha, \beta \in \mathbb{C}$.
We define the norm

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \mathrm{d}\mu(x) \right)^{1/p}$$

on $L^p(\Omega, \mathrm{d}\mu)$, which is only a semi-norm on $\tilde{L}^p(\Omega, \mathrm{d}\mu)$.
Further

$$\tilde{L}^\infty(\Omega, \mathrm{d}\mu) := \left\{ f : \Omega \to \mathbb{C} \,\middle|\, f \text{ is measurable}, \exists K \geqslant 0 : |f(x)| \leqslant K \ \mu\text{-a.e.} \right\}$$

For $f \in L^\infty(\Omega, \mathrm{d}\mu)$ we define the norm

$$\|f\|_\infty := \inf\left\{ K \,\middle|\, |f(x)| \leqslant K \ \mu\text{-a.e.} \right\}.$$

□

Theorem 1.11 (Hölder's Inequality)**.** *Let $p, q \in [1, \infty]$ be daul indices, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(\Omega, \mathrm{d}\mu), g \in L^q(\Omega, \mathrm{d}\mu)$ then $fg \in L^1(\Omega, \mathrm{d}\mu)$ and*

$$\left| \int_\Omega fg \mathrm{d}\mu \right| \overset{(a)}{\leqslant} \int |f||g| \mathrm{d}\mu \overset{(b)}{\leqslant} \|f\| \|g\|_q.$$

Equality holds at (a) iff there exists a $\vartheta \in \mathbb{R}$ such that $f(x)g(x) = e^{i\vartheta}|f(x)||g(x)|$ μ-a.e. For $f \neq 0$, equality holds at (b) iff there exists a $\lambda \in \mathbb{R}$ such that for $p \in (1, \infty)$, $|g(x)| = \lambda |f(x)|^{p-1}$ μ-a.e. For $p = 1$, $|g(x)| \leqslant \lambda$ μ-a.e. and $|g(x)| = \lambda$ $\mu - a.e.$ when $f(x) \neq 0$. For $p = \infty$, $|f(x)| \leqslant \lambda$ μ-a.e. and $|f(x)| = \lambda$ $\mu - a.e.$ when $g(x) \neq 0$. □

Theorem 1.12 (Minkowski). *Let (Ω, Σ, μ) and (Γ, Ξ, ν) be measure spaces with σ-finite measures. Then if $p \in [1, \infty)$ and $f \geqslant 0$ $\mu \otimes \nu$ measurable*

$$\left(\int\limits_{\Omega} \left(\int\limits_{\Gamma} f(x,y) \mathrm{d}\nu(y) \right)^p \mathrm{d}\mu(x) \right)^{1/p} \leqslant \int\limits_{\Gamma} \left(\int\limits_{\Omega} f(x,y)^p \mathrm{d}\mu(x) \right)^{1/p} \mathrm{d}\nu(y)$$

holds. Equality and finiteness for $p \in (1, \infty)$ imply the existence of a μ-measurable $\alpha : \Omega \to [0, \infty)$ and a ν-measurable $\beta : \Gamma \to [0, \infty)$ such that $f(x, y) = \alpha(x)\beta(y)$ for $\mu \otimes \nu$-a.e. □

Corollary 1.13. *For all $p \in [1, \infty]$ and all $f, g \in L^p(\Omega, \mathrm{d}\mu)$*

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$$

If $f \neq 0$ and $p \in (1, \infty)$, equality holds iff there exists a $\lambda \geqslant 0$ with $g = \lambda f$ μ-a.e. □

Theorem 1.14 (Completeness of L^p). *For $p \in [1, \infty]$ let $(f_j)_{j \in \mathbb{N}} \subset L^p(\Omega)$ be a Cauchy sequence, i.e.*

$$\|f_j - f_k\| \xrightarrow{\min\{j,k\} \to \infty} 0.$$

Then there exists a $f \in L^p(\Omega)$ such that $f_j \xrightarrow[L^p]{j \to \infty}$ converges (strongly) in L^p. Moreover there exists a subsequence $(f_{j_k})_k$ and $F \geqslant 0 \in L^p(\Omega)$ such that for all $k \in \mathbb{N}$ $|f_{j_k}| \leqslant F$ μ-a.e. and

$$f_{j_k}(x) \xrightarrow{k \to \infty} f(x) \ \mu\text{-a.e.}$$

□

Definition 1.15 (Convolution). Let f, g be measurable on $\left(\mathbb{R}^d, \mathcal{B}^d, \lambda^d\right)$. The convolution is defined as

$$(f * g)(x) := \int\limits_{\mathbb{R}^n} f(x - y) g(y) \mathrm{d}y = (g * f)(x)$$

For $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$ with p, q dual $g * f$ is well-defined and bounded by Hölder's inequatity for all $x \in \mathbb{R}^d n$. It is also measurable by Fubini's theorem. □

Theorem 1.16 (Young's Inequality)**.** *For $f \in L^1(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$ with $p \in [1, \infty]$, then $f * g \in L^p(\mathbb{R}^d)$ and*

$$\|f * g\|_p \leqslant \|f\|_1 \|g\|_p$$

□

Proof.

$(p = \infty)$

$$\|(f * g)(x)\|_\infty \leqslant \|g\|_\infty \int\limits_{\mathbb{R}^n} |f(x-y)| \mathrm{d}y = \|g\|_\infty \|f\|_1.$$

$(p \in [1, \infty))$

$$\left(\int\limits_{\mathbb{R}^n} |(f * g)(x)|^p \mathrm{d}x \right)^{1/p} \leqslant \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} |g(x-y)||f(y)| \mathrm{d}y \right)^p \mathrm{d}x \right)^{1/p} \overset{Theorem\ 1.12}{\leqslant}$$

$$\leqslant \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} |g(x-y)|^p \mathrm{d}x \right)^{1/p} |f(y)| \mathrm{d}y = \|g\|_p \|f\|_1$$

q.e.d.

Theorem 1.17. *For all $\Omega \subset \mathbb{R}^d$ open, for all $f \in L^p\big(\Omega, d\lambda^d\big)$, $p \in [1, \infty)$ there exists $(f_j)_{j \in \mathbb{N}} \subset \mathscr{C}_c^\infty(\Omega)$ such that*

$$f_j \xrightarrow[L^p]{j \to \infty} f.$$

□

Theorem 1.18. *For $\Omega \subset \mathbb{R}^d$ and $p \in [1, \infty)$. $L^p\big(\Omega, \mathrm{d}\lambda^d\big)$ is separable, i.e. there exists $\mathcal{F} \subset L^p\big(\Omega, \mathrm{d}\lambda^d\big)$* ***countable*** *and dense, i.e. for all $f \in L^p(\Omega)$ for all $\varepsilon > 0$ there exists $g \in \mathcal{F}$, such that $\|f - g\|_p < \varepsilon$.* □

Proof. Given $f \in L^p(\Omega)$ there exists $h \in \mathscr{C}_c^\infty(\Omega)$ such that $\|f - h\| < \frac{\varepsilon}{2}$. Hence w.l.o.g. let us assume that $f \in \mathscr{C}_c^\infty$. For all $N \in \mathbb{N}$ we have

$$\mathbb{R}^d = \bigcup_{j \in \mathbb{Z}^d} \underbrace{\left([0, 2^{-N})^d + 2^{-N} j\right)}_{C_{j,N}}$$

The set of step functions with support in $C_{j,N}$ and $\mathbb{C}$-rational values is a countable set. Given N, j we can choose

$$c_{N,j} := \frac{1}{(2^{-N})^n} \int\limits_{C_{j,N}} f(x)\mathrm{d}x$$

Since $f \in \mathscr{C}_c^\infty$ it is uniformly continuous, i.e. we can find for all $\delta > 0$ an N big enough such that for all $x \in C_{j,N}$

$$|f(x) - c_{N,j}| < \frac{\delta}{2}.$$

Further we can choose a $\tilde{c}_{N,j}$ in the rational complex numbers such that $|c_{N,j} - \tilde{c}_{N,j}| < \frac{\delta}{2}$, therefore

$$\left\| h - \sum_{j \in \mathbb{Z}^d} \tilde{c}_{N,j} \chi_{C_{j,N}}(x) \right\|_\infty < \delta$$

By construction the sum of step functions is compactly supported as f is therefore there exists some compact set K such that

$$\left\| h - \sum_{j \in \mathbb{Z}^d} \tilde{c}_{N,j} \chi_{C_{j,N}}(x) \right\|_p \leqslant \left\| h - \sum_{j \in \mathbb{Z}^d} \tilde{c}_{N,j} \chi_{C_{j,N}}(x) \right\|_\infty \mu(K)$$

thus by choosing $\delta < \frac{\varepsilon}{2\mu(K)}$, we have found the approximating function. *q.e.d.*

Definition 1.19. $L : L^p(\Omega, \mathrm{d}\mu) \to \mathbb{C}$ is a linear function iff for all $f_1, f_2 \in L^p$, $\alpha \in \mathbb{C}$

$$L(\alpha f_1 + f_2) = \alpha L(f_1) + L(f_2).$$

L is bounded iff there exists a $K > 0$ such that $|L(f)| \leqslant K\|f\|_p$ for all $f \in L^p$.
L is (sequentially) continuous iff for all $(f_j)_{j \in \mathbb{N}} \subset L^p$ with $f_j {}_{L^p}\!\xrightarrow{j \to \infty} f$ implies that $L(f_j) \xrightarrow{j \to \infty} L(f)$.
In the case of linear functionals/maps the latter two properties are equivalent.
The space of bounded linear functionals on $L^p(\Omega)$, denoted by $(L^p(\Omega))^*$ is a complete

vector space with norm

$$\|L\| := \sup_{f \in L^p(\Omega)\setminus\{0\}} \frac{|Lf|}{\|f\|_p}$$

A sequence $(f_j)_{j\in\mathbb{N}} \subset L^p(\Omega)$ converges weakly to $f \in L^p(\Omega)$ iff for all $L \in (L^p(\Omega))^*$, $Lf_j \xrightarrow{j\to\infty} Lf$. This is written as

$$f_j \overset{j\to\infty}{\rightharpoonup} f$$

By Hölder's inequality $L^{p'}(\Omega) \to (L^p(\Omega))^*$ (injectively) for all $p \in [1,\infty]$ via

$$g \mapsto L_g$$

with

$$L_g(f) := \int_\Omega f(x)g(x)\mathrm{d}\mu(x)$$

with $\|Lg\| \leqslant \|g\|_{p'}$. □

Theorem 1.20 (Linear Functionals Separate)**.** *Let $p \in [1,\infty]$ (for $p = \infty$, (Ω, Σ, μ) must be σ-finite). Let $f \in L^p(\Omega)$ such that for all $L \in L^p(\Omega)^*$ $L(f) = 0$ holds then $f = 0$. Consequently, if $f_j \overset{j\to\infty}{\rightharpoonup} k$ and $f_j \overset{j\to\infty}{\rightharpoonup} l$, then $k = l$, i.e. weak limits are unique.* □

Proof. For $p \in [1,\infty)$, take

$$g(x) := \begin{cases} \overline{f(x)}|f(x)|^{p-2}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

and

$$L_g h := \int g(x)h(x)\mathrm{d}\mu(x).$$

Since, by Hölder's inequality

$$\infty > \int |f(x)|^p \mathrm{d}x = \int |g(x)|^{p'} \mathrm{d}x$$

it follows that $g \in L^{p'}(\Omega)$ and $L_g \in L^p(\Omega)^*$, where $\frac{1}{p} + \frac{1}{p'} = 1$. For this functional we have

$$L_g(f) = \int_\Omega \overline{f(x)}|f(x)|^{p-2} f(x)\mathrm{d}\mu(x) = \int_\Omega |f|^p \mathrm{d}\mu(x) = \|f\|_p^p.$$

For $p=\infty$, for $\varepsilon>0$ choose Ω_ε with $\mu(\Omega_\varepsilon)<\infty$ such that $|f(x)|>\|f\|_\infty-\varepsilon$ for all $x\in\Omega_\varepsilon$. Choosing

$$g(x):=\frac{\overline{f(x)}}{|f(x)|}\chi_{\Omega_\varepsilon}(x)\in L^1(\Omega)\implies L_g\in L^\infty(\Omega)^*$$

One finds that

$$L_g(f)=\int\limits_{\Omega_\varepsilon}\frac{\overline{f(x)}}{|f(x)|}f(x)\mathrm{d}\mu(x)=\int\limits_{\Omega_\varepsilon}|f(x)|\mathrm{d}\mu(x)\leqslant\|f\|_\infty\mu(\Omega_\varepsilon)$$

and on other hand using the definition of Ω_ε

$$L_g(f)\geqslant(\|f\|_\infty-\varepsilon)\int\limits_{\Omega_\varepsilon}\mathrm{d}\mu(x)=(\|f\|_\infty-\varepsilon)\mu(\Omega_\varepsilon)$$

q.e.d.

Theorem 1.21 (Hanner's Inequality)**.** *Let $f,g\in L^p(\Omega)$, $p\in[1,2]$. Then*

$$(\|f\|_p+\|g\|_p)^p+|\|f\|_p-\|g\|_p|^p\leqslant\|f+g\|_p^p+\|f-g\|_p^p \tag{1}$$

and

$$(\|f+g\|_p+\|f-g\|_p)^p+|\|f+g\|_p-\|f-g\|_p|^p\leqslant 2^p\big(\|f\|_p^p+\|g\|_p^p\big) \tag{2}$$

For $p\in[2,\infty)$ the inequalities are reversed.

□

Remark. For $\|f-g\|_p\leqslant\|f+g\|_p$, $p\in[1,2]$, then the

$$\mathrm{LHS}(2)\geqslant 2\|f+g\|_p^p+p(p-1)\|f+g\|_p^{p-2}\|\|f-g\|_p^2$$

which follows from the inequality for $a,b\geqslant$

$$(a+b)^p+|a-b|^p\geqslant 2a^p+p(p-1)a^{p-2}b^2.$$

To prove it we may assume w.l.o.g. that $a\neq 0$ (since otherwise the inequality holds trivially) and devide by b to get the inequality

$$(1+x)^p + |1-x|^p \geqslant 2 + p(p-1)x^2$$

Noting that by assumption $1 \geqslant x$ hence $|1-x| = (1-x)$ Since by differentiating twice this expression

$$(p-1)\big((1+x)^{p-2} + (1-x)^{p-2}\big) \geqslant 2(p-1)$$

which indeed holds. Then by integration one finds the asserted inequality.

□

Theorem 1.22 (Uniform Convexity)**.** *For all $p \in (1,\infty)$*

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall f,g \in L^p(\Omega) : \|f\|_p = \|g\|_p = 1, \left\|\frac{f+g}{2}\right\|_p^p \geqslant 1-\delta \implies \left\|\frac{f-g}{2}\right\|_p < \varepsilon$$

□

Lemma. *Let $\alpha(r) := (1+r)^{p-1} + (1-r)^{p-1}$, and $\beta(r) := ((1+r)^{p-1} - (1-r)^{p-1})r^{1-p}$ for $r \in [0,1]$ with $\beta(0) := 0$ ($\beta(0) := \infty$ for $p \in [2,\infty)$). Then for all $A, B \in \mathbb{C}$*

$$\alpha(r)|A|^p + \beta(r)|B|^p \leqslant |A+B|^p + |A-B|^p \tag{$*$}$$

for $p \in [1,2)$. Equality holds iff $r = \frac{|B|}{|A|} \in [0,1]$.

□

Proof. It is sufficient to assume $A, B \geqslant 0$. Otherwise $a := |A|$, $b := |B|$ satisfy

$$|A+B|^p + |A-B|^p = \big(a^2+b^2+2ab\cos(\vartheta)\big)^{p/2} + \big(a^2+b^2-2ab\cos(\vartheta)\big)^{p/2} \geqslant (a+b)^p + (a-b)^p$$

Let $R := \frac{B}{A}$, and rewrite the asserted inequality as

$$\alpha(r) + R^p\beta(r) \leqslant (1+R)^p + (1-R)^p$$

differentiating both sides

$$\frac{d}{dr}(\alpha(r) + R^p\beta(r)) = (p-1)(1+r)^{p-2} - (p-1)(1+r)^{p-2} + R^p(p-1)\big((1+r)^{p-2} + (1-r)^{p-2}\big) +$$
$$+ R^p(1-p)\big((1+r)^{p-2}(1+r) - (1-r)^{p-2}(1-r)\big)r^{-p} =$$
$$= (p-1)\big((1+r)^{p-2} - (1-r)^{p-2}\big)\left(1 - \left(\frac{R}{r}\right)^p\right)$$

which vanishes only for $r = R$. Further since the derivative for $R \leqslant 1$ is positive for $r < R$ and negative for $r > R$, this is indeed the maximum. *q.e.d.*

Proof of Theorem 1.21. Noting that $R \leqslant 1$ can always be attained by exchanging f and g if necessary one finds that for all $r \in [0, 1]$

$$|f+g|^p + |f-g|^p \geqslant \alpha(r)|f|^p + \beta(r)|g|^p = \alpha(R)|f|^p + \beta(R)|g|^p$$

for $R := \frac{\|g\|_p}{\|f\|_p}$. Integrating one finds that

$$\|f+g\|_p^p + \|f-g\|_p^p \geqslant \alpha(R)\|f\|_p^p + \beta(R)\|g\|_p^p = (\|f+g\|_p + \|f-g\|_p)^p + |\|f+g\|_p - \|f-g\|_p|^p$$

(2) follows immediately from (1) by substituting $f \to f + g$ and $g \to f - g$.
For $p = 2$ this is just the standard parallelogram identity. For $p \in [1, 2)$, otherwise reverse all the inequalities. *q.e.d.*

Theorem 1.23 (Lower Semi-Continuity of Norms)**.** *For $p \in [1, \infty]$ if*

$$f_j \xrightarrow{j\to\infty} f \implies \liminf_{j\to\infty} \|f_j\|_p \geqslant \|f\|_p$$

(For $p = \infty$, μ needs to σ-finite). If $p \in (1, \infty)$ and $\lim_{j\to\infty} \|f_j\|_p = \|f\|_p$ then

$$f_j \xrightarrow[L^p]{j\to\infty} f.$$

□

Theorem 1.24 (Uniform Boundedness Principle)**.** *Let $p \in [1, \infty]$ (for $p = \infty$, (Ω, Σ, μ) need be σ-finite). Let $(f_j)_{j\in\mathbb{N}} \subset L^p(\Omega)$ such that for all $L \in L^p(\Omega)^*$ there exists a $C_L > 0$ such that $|L(f_j)| \leqslant C_L$ for all $j \in \mathbb{N}$. Then there exists a $C > 0$ such that $\|f_j\| \leqslant C$ for* *all* $j \in \mathbb{N}$. □

Theorem 1.25 (The Dual of $L^p(\Omega)$)**.** *For* $p \in [1, \infty)$, $L^p(\Omega)^* = L^q(\Omega)$ *for* $\frac{1}{p} + \frac{1}{q} = 1$, *i.e for all* $L \in L^p(\Omega)^*$ *there exists a* $v \in L^q(\Omega)$ *such that for all* $g \in L^p(\Omega)$

$$L(g) = L_v(g) := \int vg\mathrm{d}\mu$$

with $\|L\| = \|v\|$.

□

Theorem 1.26 (Banach-Alaoglu)**.** *For* $p \in (1, \infty)$ *let* $(f_j)_{j\in\mathbb{N}}$ *be bounded in* $L^p(\Omega)$. *Then there exists a subsequence* $(f_{j_n})_{n\in\mathbb{N}}$ *and* $f \in L^p(\Omega)$ *such that*

$$f_{j_n} \xrightharpoonup[L^p]{n\to\infty} f$$

□

Chapter 2

Distributions

Remark 2.1. $\left(L^p\left(\mathbb{R}^d\right)\right)^* = L^q\left(\mathbb{R}^d\right)$ for $\frac{1}{p} + \frac{1}{q}$, $1 \leqslant p < \infty$. □

Definition 2.2 (Test Functions)**.** Let $\Omega \subset \mathbb{R}^d$ be open. We define the set of test functions to be $\mathscr{D}(\Omega) = \mathscr{C}_c^\infty\left(\mathbb{R}^d\right)$. We define a topology on this space by requiring that a sequence $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$ converges iff

$$\begin{cases} \exists \text{ compact set } K \subset \Omega : \operatorname{supp} \varphi_n \subset K \\ \forall \alpha \in \mathbb{N}^n : \sup\limits_{x \in \Omega} |D^\alpha \varphi_n - D^\alpha \varphi| \xrightarrow{n \to \infty} 0 \end{cases}$$

□

Definition 2.3 (Distributions)**.** We define the space of distributions to be dual space to the space of test functions, i.e. $\mathscr{D}'(\Omega)$

$$T \in \mathscr{D}'(\Omega) :\iff T : \mathscr{D}(\Omega) \to \mathbb{C}, \text{ linear \& continuous.}$$

We define the weak-$*$ topology on this space, i.e. a sequence $T_n \to T$ converges in $\mathscr{D}'(\Omega)$ iff for all $\varphi \in \mathscr{D}(\Omega)$, $T_n(\varphi) \xrightarrow{n \to \infty} T(\varphi)$. □

Example 2.4. If $f \in L^1_{\text{loc}}(\Omega)$, then

$$T_f : \begin{aligned} \mathscr{D}(\Omega) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \int_\Omega f(x)\varphi(x)\mathrm{d}x \end{aligned}$$

is a distribution.

Example 2.5 (Dirac delta function)**.** The linear functional

$$\delta : \begin{aligned} \mathscr{D}(\mathbb{R}^n) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \varphi(0) \end{aligned}.$$

Informally one may one may say that $\delta(x) = 0$ for all $x \neq 0$ and $\delta(0) = \infty$ such that $\int_{\mathbb{R}^n} \delta = 1$.

One might now ask the question whether if for $f, g \in L^1_{\text{loc}}(\Omega)$ with $T_f = T_g$ does imply that $f = g$.

Theorem 2.6 (Fundamental Theorem of the Calculus of Variations)**.** *If $f \in L^1_{loc}(\Omega)$ such that for all $\varphi \in \mathscr{C}_c^\infty(\Omega)$*

$$\int_\Omega f\varphi = 0$$

then $f = 0$. □

Proof. Assume that $f \in L^1\left(\mathbb{R}^d\right)$. Then

$$\int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x = 0$$

for all $\varphi \in \mathscr{C}_c^\infty\left(\mathbb{R}^d\right)$ implies that

$$0 = \int_{\mathbb{R}^d} f(x)\varphi(y - x)\mathrm{d}x = (f * \varphi)(y)$$

for all $y \in \mathbb{R}^d$.

Recall now that if $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$, with $\int \varphi \mathrm{d}\lambda = 1$, and $\varphi_n(x) = n^d \varphi(nx)$ then $\varphi_n * f \to f$ in $L^1(\mathbb{R}^d)$, since for all $y \in \mathbb{R}^d$

$$(f * \varphi_n)(y) = 0$$

it follows that $f = 0$ in $L^1(\mathbb{R}^d)$, i.e. $f(x) = 0$ a.e.
Now let us consider the general case, let $\Omega \subset \mathbb{R}^d$ be open, and $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\int f(x)\varphi(y-x)\mathrm{d}x = 0$$

for all $\varphi \in \mathscr{C}_c^\infty(\Omega)$. We need $x \in \Omega_2$, $\bar{\Omega}_2 \subset\subset \Omega$ such that $y - x \in \operatorname{supp} \varphi$, then

$$y = x + (y-x) \in \Omega_2 + \operatorname{supp} \varphi.$$

We choose $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} \varphi \subset B(0,1)$. Define $\varphi_n(x) = n^d \varphi(nx)$. Then $\operatorname{supp} \varphi_n \subset B\left(0, \frac{1}{n}\right)$. Then we have

$$\int f(x)\varphi_n(y-x)\mathrm{d}x = 0$$

for all $y \in \Omega_2$, with $\Omega_2 \subset\subset \Omega$. Then

$$x = y - (y-x) \in \Omega_2 \setminus \operatorname{supp} \varphi_n \subset \Omega_2 + B_{\frac{1}{n}}(0) \subset \Omega_3$$

when n is large enough. Thus we have

$$\int_\Omega f(x)\varphi_n(y-x)\mathrm{d}x = \int_\Omega \mathbf{1}_{\Omega_3} f(x)\varphi_n(y-x)\mathrm{d}x = \int_{\mathbb{R}^n} \mathbf{1}_{\Omega_3} f(x)\varphi_n(y-x)\mathrm{d}x = (\varphi_n * \mathbf{1}_{\Omega_3} f)(y)$$

Since $\mathbf{1}_{\Omega_3} f \in L^1(\mathbb{R}^d)$, we have that $\varphi_n * \mathbf{1}_{\Omega_3} f \to \mathbf{1}_{\Omega_3} f$. Thus $f\big|_{\Omega_3} = 0$ which implies that $f(x) = 0$ a.e. $x \in \Omega_3$ and thus also $x \in \Omega$.

q.e.d.

Definition 2.7 (Derivative of Distributions)**.** For a $T \in \mathscr{D}'(\Omega)$ we define its α-derivative to be the distribution $D^\alpha T \in \mathscr{D}'(\Omega)$ such that

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$$

for all $\varphi \in \mathscr{D}$. □

Remark 2.8. This definition is motivated by the fact that for $f \in \mathscr{C}^\infty(\mathbb{R}^d)$

$$\int (D^\alpha f)\varphi = (-1)^{|\alpha|} \int f(D^\alpha \varphi).$$

□

In particular we have that if $T_n \to T$ in $\mathscr{D}'(\Omega)$, then $D^\alpha T_n \to D^\alpha T$ for any $\alpha \in \mathbb{N}^n$.

Proof. For all $\varphi \in \mathscr{D}(\Omega)$ we have

$$(D^\alpha T_n)(\varphi) = (-1)^{|\alpha|} T_n(D^\alpha \varphi) \xrightarrow{n\to\infty} (-1)^{|\alpha|} T(D^\alpha \varphi) = (D^\alpha T)(\varphi).$$

q.e.d.

Example 2.9. Let $f(x) = |x|$. Then its distributional derivative is

$$f'(x) = \begin{cases} -1, & y < 0 \\ +1, & y > 0 \end{cases}$$

and its second distributional derivative is

$$f'' = 2\delta.$$

Theorem 2.10 (Equivalence of Classical and Distributional Derivatives)**.** *1) If $f \in \mathscr{C}^1(\Omega) \subset L^1_{loc}(\Omega)$, then $g_i = \partial_{x_i} f \in \mathscr{C}(\Omega)$ and $\partial_i(T_f) = T_{g_i}$.*

2) Let $T \in \mathscr{D}'(\Omega)$ and assume that $T_{g_i} = \partial_{x_i} T$ and $g_i \in \mathscr{C}(\Omega)$, for all $i = 1, \ldots, n$. Then there exists a $f \in \mathscr{C}^1(\Omega)$ such that $T = T_f$ and $\partial_{x_i} f = g_i$. □

Proof. Let $\Omega = \mathbb{R}^d$.

1) If $f \in \mathscr{C}^1(\mathbb{R}^d)$ and $g_i = \partial_i f \in \mathscr{C}(\Omega)$. Then for all $\varphi \in \mathscr{D}'(\Omega)$

$$(\partial_i(T_f))(\varphi) = -T_f(\partial_i \varphi) = -\int f(x)\partial_i\varphi(x)\mathrm{d}x = \int \partial_i f(x)\varphi(x)\mathrm{d}x = T_{\partial_i f}(\varphi)$$

   i.e. $\partial_i T_f = T_{\partial_i f}$ in $\mathscr{D}'(\mathbb{R}^d)$.

2) Assume that $T \in \mathscr{D}'()$

q.e.d.

Chapter 3

Fourier Transform

Definition 3.1. For $f \in L^1(\mathbb{R}^d)$ one defines its Fourier transform to be

$$\hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x} \mathrm{d}x$$

□

Remark (Motivation). 1) For nice enough functions one has

$$\widehat{\partial_{x_i} f}(k) = 2\pi i k_i \hat{f}(k).$$

Formally we have

$$\widehat{\partial_{x_i} f}(k) = \int_{\mathbb{R}^d} (\partial_{x_i} f)(x) e^{-2\pi i k \cdot x} \mathrm{d}x = -\int_{\mathbb{R}^d} f(x) \partial_{x_i} e^{-2\pi i k \cdot x} \mathrm{d}x = 2\pi i k_i \hat{f}(k).$$

More generally one has

$$\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k).$$

2) Further we have for nice enough functions that

$$\widehat{f * g}(k) = \hat{f}(k) \hat{g}(k)$$

because formally

$$\widehat{f*g}(k) = \int\int f(x-y)g(y)e^{-2\pi ik\cdot x}\mathrm{d}y\mathrm{d}x = \int\int f(x-y)g(y)e^{-2\pi ik\cdot(x-y)}e^{-2\pi ik\cdot y}\mathrm{d}x\mathrm{d}y =$$
$$= \int\int f(x-y)e^{-2\pi ik\cdot(x-y)}\mathrm{d}x g(y)e^{-2\pi ik\cdot y}\mathrm{d}y = \hat{f}(k)\hat{g}(k)$$

□

Theorem 3.2 (Plancherl). *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then*

$$\|f\|_2 = \|\hat{f}\|_2$$

Consequently, $f \mapsto \hat{f}$ can be extended into an isometry on $L^2(\mathbb{R}^d)$, as $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. Moreover for all $f, g \in L^2(\mathbb{R}^d)$

$$\langle f, g\rangle = \left\langle \hat{f}, \hat{g}\right\rangle$$ [1].

□

Theorem 3.3 (Inverse Formula). *Define $\check{f}(k) = \int f(x)e^{2\pi ik\cdot x}\mathrm{d}x = \hat{f}(-k)$. Then for all $f \in L^2(\mathbb{R}^d)$*

$$\check{\hat{f}} = f.$$

□

We know that $f \mapsto \hat{f}$ is a bounded map from $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ as

$$\left|\hat{f}(k)\right| = \left|\int_{\mathbb{R}^n} f(x)e^{-2\pi ik\cdot x}\mathrm{d}x\right| \leqslant \int_{\mathbb{R}^n} |f(x)|\mathrm{d}x = \|f\|_{L^1}$$

and $L^2 \to L^2$ with $\|\hat{f}\|_2 = \|f\|_2$.

Theorem 3.4 (Hausdorff-Young inequality). *If $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $1 < p \leqslant 2$, then*

$$\|\hat{f}\|_{p'} \leqslant \|f\|_p$$

[1] Here we shall use the convention $\langle f, g\rangle = \int \bar{f}(x)g(x)\mathrm{d}x$

Consequently, $f \mapsto \hat{f}$ is a bounded mapping from $L^p \to L^{p'} = (L^p)^*$.

□

Theorem 3.5 (Riesz-Thorin Interpolation inequality)**.** *Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. If*

$$\mathscr{L} : L^{p_0} \longrightarrow L^{q_0}, \qquad \text{with } \|\mathscr{L}\|_{p_0,q_0} \leqslant 1$$
$$\mathscr{L} : L^{p_1} \longrightarrow L^{q_1}, \qquad \text{with } \|\mathscr{L}\|_{p_1,q_1} \leqslant 1$$

Then $\|\mathscr{L}u\|_{p_s,q_s}$ for all $s \in (0,1)$ where

$$\frac{1}{p_s} = \frac{1-s}{p_0} + \frac{s}{p_1}, \qquad \frac{1}{q_s} = \frac{1-s}{q_0} + \frac{s}{q_1}$$

□

The proof this theorem is based on Hadamard's 3-line Theorem.

Theorem (Hadamard 3-lines theorem)**.** *Let $\mathbb{C} \ni z = x + iy$, and let f be holomorphic on $\Omega = \{z = x + iy, 0 < x < 1\}$. Define $M(x) = \sup_{y\in\mathbb{R}} |f(x+iy)|$, then*

$$M(x) \leqslant M(0)^{1-x} M(1)^x$$

□

Sketch of Proof. Assume that $M(0) = 1 = M(1)$. We need to prove that $|f(x+iy)| \leqslant 1$ in Ω. Define now $F_n(x) = f(z)e^{\frac{z^2-1}{n}}$ for $n \in \mathbb{N}$. Then $|F_n(z)| \leqslant 1$, for all $z \in \partial\Omega$, and $|F_n(z)| \to 0$ as $|z| \to \infty$. Applying the maximum principle we find that $|F_n(z)| \leqslant 1$ for all $z \in \Omega$. *q.e.d.*

Proof of Theorem 3.5. To prove this, we neeed the duality

$$\|\mathscr{L}\|_{q_s} = \sup_{\|\varphi\|_{q_s'} \leqslant 1} |\int (\mathscr{L}u)\varphi|.$$

Then define u_z and φ_z in an appropriate way

$$\sup\left|\int (\mathscr{L}u_z)\varphi_z\right| \leqslant \|u\|_{p_s'}$$

q.e.d.

Proof of Theorem 3.4. Define $\mathscr{L}u = \hat{u}$. Then

$$\mathscr{L} : L^1 \longrightarrow L^\infty, \qquad \text{with } \|\mathscr{L}\|_{1,\infty} \leqslant 1$$
$$\mathscr{L} : L^2 \longrightarrow L^2, \qquad \text{with } \|\mathscr{L}\|_{2,2} = 1$$

By Riesz-Thorin we have that $\|\hat{u}\|_{q_s} \leqslant \|u\|_{p_s}$ for all $s \in (0,1)$

$$\frac{1}{p_s} = \frac{1-s}{1} + \frac{s}{2}, \quad \frac{1}{q_s} = \frac{1-s}{\infty} + \frac{s}{2}$$

which implies that $\frac{1}{p_s} = 1 - \frac{s}{2}$ and $\frac{1}{q_s} = \frac{s}{2}$ and thus

$$\frac{1}{p_s} + \frac{1}{q_s}, \quad 1 \leqslant p_s \leqslant 2 \leqslant q_s.$$

This means that $q_s = (p_s)'$. *q.e.d.*

Theorem 3.6. *If $f \in L^p$, $g \in L^q$, then $f * g \in L^r$ for $\frac{1}{q} + \frac{1}{p} = 1 + \frac{1}{r}$ and $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$.* □

Proof. Take $f \in L^p$ fixed and define

$$\mathscr{L}g = f * g$$

We know that

$$\|f * g\| \leqslant \|f\|_p \|g\|_{p'}$$
$$\|f * g\|_p \leqslant \|f\|_p \|g\|_1$$

By Riesz-Thorin,

$$\|f * g\|_{q_s} \leqslant \|f\|_p \|g\|_{p_s}$$

for all $s \in (0,1)$. In particular

$$\frac{1}{p_s} = \frac{1-s}{p'} + \frac{s}{1}, \quad \frac{1}{q_s} = \frac{1-s}{\infty} + \frac{s}{p}$$

from which follows that for $q_s = r$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. *q.e.d.*

Corollary 3.7. *If $f \in L^p$, $g \in L^q$, $1 \leqslant q, p \leqslant 2$ then $f * g \in L^r$, for $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then*

$$\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$$

□

Proof. Do it for $f, g \in \mathscr{D}$, and then approximate. *q.e.d.*

Theorem 3.8 (Fourier Transform of Gaussian)**.**

$$\widehat{e^{-\pi|\cdot|^2}}(k) = e^{-\pi|k|^2}$$

More generally

$$\widehat{e^{-\pi\lambda|\cdot|^2}}(k) = \lambda^{-\frac{n}{2}} e^{-\frac{\pi|k|}{\lambda}}$$

for all $\lambda > 0$. □

Proof. For $\lambda = 1$, and $n = 1$ we have

$$\widehat{e^{-\pi|\cdot|^2}}(k) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i k \cdot x} \mathrm{d}x = \int_{\mathbb{R}} e^{-\pi k^2} e^{-\pi(x+ik)^2} \mathrm{d}x = e^{-\pi k^2} \int_{\mathbb{R}} e^{-\pi x^2} \mathrm{d}x = e^{-\pi k^2}$$

where the penultimate equality follows from the Cauchy formula. *q.e.d.*

Theorem 3.9 (Heat Equation)**.** *Consider for $t \geqslant 0$*

$$\begin{aligned} \partial_t u - \Delta u &= 0 \\ u(0, x) &= f(x) \in L^2(\mathbb{R}^d) \end{aligned}$$

The unique L^2 solution is given by

$$u(t, x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) \mathrm{d}y$$

□

Proof. Via the Fourier transform we find the equivalent equation

$$\partial_t \hat{u} - (2\pi|k|)^2 \hat{u} = 0$$
$$\hat{u}(0,k) = \hat{f}(k) \in L^2\left(\mathbb{R}^d\right)$$

which can be rewritten as

$$\partial_t \left(\hat{u} e^{(2\pi|k|)^2 t} \right) = 0$$
$$\hat{u}(t,k) e^{(2\pi|k|)^2 t} \Big|_{t=0} = \hat{f}(k) \in L^2\left(\mathbb{R}^d\right)$$

which implies that $\hat{u}(t,k)e^{(2\pi|k|)^2 t} = \hat{f}(k)$ for all $t \geqslant 0$ and therefore $\hat{u}(t,k) = e^{-(2\pi|k|)^2} \hat{f}(k) = \hat{G}_t(k)\hat{f}(k) = \widehat{G_t * f}(k)$. Thus $u(t,x) = (G_t * f)(x)$.
What is $G_t(x)$. We need $\hat{G}_t(k) = e^{-(2\pi|k|)^2 t}$. Using the formula for the Fourier transform of a Gaussian

$$\widehat{e^{-\pi\lambda|\cdot|^2}}(k) = \lambda^{-\frac{n}{2}} e^{-\frac{\pi|k|^2}{\lambda}}$$

Choosing $(2\pi|k|)^2 t = \frac{\pi|k|^2}{\lambda}$ which implies that $\lambda = \frac{1}{4\pi t}$, from wich the assertoin follows.

q.e.d.

Remark 3.10. If K is a linear operator $L^2 \to L^2$ such that

$$(Ku)(x) = \int K(x,y)u(y)\mathrm{d}y$$

for all $u \in L^2$, then $K(x,y)$ is called the ***kernel*** of K. In particular

$$G(t,x,y) = \frac{1}{(4\pi t)^{-\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$$

is called the ***heat kernel***. □

Theorem 3.11 (Heat Kernel)**.** *Let* $G(t,x) = \frac{1}{(4\pi t)^{-\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$. *Then for* $t > 0$

$$\partial_t G - \Delta G = 0$$

and

$$\lim_{t\to 0^+} G(t,x) \xrightarrow{\mathscr{D}'(\mathbb{R}^n)} \delta_x$$

□

Proof. For all $\varphi \in \mathscr{D}(\mathbb{R}^d)$

$$\int (\partial_t G(t,x) - \Delta G(t,x))\varphi(y-x)\mathrm{d}x = \partial_t(G*\varphi)(y) - (\Delta G*\varphi)(y) = \partial_t(G*\varphi)(y) - \Delta(G*\varphi)(y) = 0.$$

Because $u = G*\varphi$ solves the heat equation. Thus $\partial_t G - \Delta G = 0$.
Moreover, formally we find that

$$\int G(t,x)\varphi(x)\mathrm{d}x = (G_t*\varphi)(0) = u(t,0) \xrightarrow{t\to 0} u(0) = \varphi(0) = \delta(\varphi)$$

$$\lim_{t\downarrow 0} G(t,x) = \delta(x) \quad \text{in } \mathscr{D}'(\mathbb{R}^d)$$

The last step can be made rigorous by using the fact that

$$u(t,x) = G_t * f \xrightarrow{L^2} f$$

strongly, since from Theorem 3.9 we have

$$\|u(t,\cdot) - f\|_{L^2} = \left\|\hat{u}(t,\cdot) - \hat{f}\right\| = \left\|\left(e^{-(2\pi|k|)^2 t} - 1\right)\hat{f}(k)\right\|_2 \xrightarrow{\text{Dom Conv}} 0.$$

q.e.d.

Now let us consider the Poisson equation

$$-\Delta u = f, \quad f \in L^2(\mathbb{R}^d)$$

Formally we find that

$$(2\pi|k|)^2\hat{u}(k) = \hat{f}(k)$$

which implies that

$$\hat{u}(k) = (2\pi|k|)^{-2}\hat{f}(k) = \hat{G}(k)\hat{f}(k)$$

with $\hat{G}(k) = \frac{1}{(2\pi|k|)^2}$. Then $\hat{u}(k) = \widehat{G*f}(k)$, i.e. $u = G*f$.
What is G? $\hat{G}(k) = \frac{1}{(2\pi|k|)^2}$. More generally what is the Fourier transform of $\frac{1}{|x|^s}$.

Theorem 3.12. *For $0 < s < d$, then*

$$c_s \frac{\hat{1}}{|x|^s} = c_{d-s} \frac{1}{|k|^{d-s}}$$

in the sense of distributions and $c_s = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$. This means that for all $\varphi \in \mathscr{D}\left(\mathbb{R}^d\right)$, then

$$\left(c_s \widehat{\frac{1}{|\cdot|^s} \check{\varphi}}\right)(k) = c_{d-s}\left(\frac{1}{|k|^{d-s}} * \varphi\right)(k)$$

The latter formula serves as a definition of a convolution of distribution and a test function and is well-defined since for $0 < s < n$, $\frac{1}{|x|^s}\check{\varphi}(x) \in L^1\left(\mathbb{R}^d\right)$. □

Proof. Formally we have

$$c_s = \pi^{-\frac{s}{2}} \int_0^\infty \lambda^{\frac{s}{2}-1} e^{-\lambda} \mathrm{d}\lambda = \pi^{-\frac{s}{2}} \int_0^\infty (\pi |x|^2 t)^{\frac{s}{2}-1} e^{-\pi|x|^2 t} \mathrm{d}t = |x|^s \int_0^\infty t^{\frac{s}{2}-1} e^{-\pi |x|^2 t} \mathrm{d}t$$

which implies that $\frac{c_s}{|x|^s} = \int_0^\infty t^{\frac{s}{2}-1} e^{-\pi|x|^2 t} \mathrm{d}t$ and thus

$$\widehat{\frac{c_s}{|\cdot|^s}}(k) \text{``}=\text{''} \int_0^\infty t^{\frac{s}{2}-1} \widehat{e^{-\pi|\cdot|^2 t}}(k) \mathrm{d}t = \int_0^\infty t^{\frac{s}{2}-1} t^{-\frac{d}{2}} e^{-\frac{\pi|k|^2}{t}} \mathrm{d}t = \int_0^\infty \left(\frac{\pi |k|^2}{\lambda}\right)^{\frac{s}{2}-\frac{d}{2}-1} e^{-\lambda} \pi |k|^2 \frac{\mathrm{d}\lambda}{\lambda^2} =$$

$$= |k|^{s-d} \pi^{-\frac{d-2}{2}} \int_0^\infty \lambda^{\frac{d-s}{2}-1} e^{-\lambda} \mathrm{d}\lambda = \frac{c_{d-s}}{|k|^{d-s}}$$

Rigorously we have

$$\left(\widehat{\frac{c_s}{|\cdot|^s} \check{\varphi}}\right)(k) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c_s}{|x|^s} \varphi(p) e^{2\pi i p \cdot x} e^{-2\pi i k \cdot x} \mathrm{d}p \mathrm{d}x \overset{\text{Fubini}}{=\!=\!=} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} t^{\frac{s}{2}-1} e^{-\pi|x|^2 t} \varphi(p) e^{2\pi i p \cdot x} e^{-2\pi i k \cdot x} \mathrm{d}p \mathrm{d}x \mathrm{d}t =$$

$$= \int_0^\infty t^{\frac{s}{2}-1} (\widehat{e^{-\pi|\cdot|^2 t} \check{\varphi}})(k) \mathrm{d}t = \int_0^\infty t^{\frac{s}{2}-1} c\left(e^{-\frac{\pi|\cdot|^2}{t}} * \varphi\right)(k) \mathrm{d}t$$

q.e.d.

Corollary 3.13. *If* $0 < 2s < d$ *and* $f \in L^p$, $p = \frac{2d}{d+2s}$, *then, since* $1 \leqslant p \leqslant 2$, $\hat{f}(k)$ *makes sense and*

$$\frac{c_{2s}}{|k|^{2s}}\hat{f}(k) = \left(\widehat{\frac{c_{d-2s}}{|\cdot|^{d-2s}} * f}\right)(k).$$

Moreover

$$c_{d-2s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\overline{f(x)}f(y)}{|x-y|^{d-2s}} = c_{2s}\int_{\mathbb{R}^d}\frac{|\hat{f}(k)|^2}{|k|^{2s}}\mathrm{d}k \geqslant 0.$$

□

Proof. First formula, take $\varphi_n \in \mathscr{D}$ such that $\varphi_n \to f$ in L^p. Using the formula for φ_n and passing to $n \to \infty$ we find that

$$\left\|\hat{\varphi}_n - \hat{f}\right\|_{p'} \leqslant C\|\varphi_n - f\| \to 0.$$

The first formula combined with Plancherl's theorem yields the second formula as

$$c_{d-2s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\overline{f(x)}f(y)}{|x-y|^{d-2s}}\mathrm{d}x\mathrm{d}y = c_{d-2s}\int_{\mathbb{R}^d}\overline{f(x)}\left(f * \frac{1}{|x|^{d-2s}}\right)(x)\mathrm{d}x = \left\langle f, f * \frac{c_{d-2s}}{|\cdot|^{d-2s}}\right\rangle =$$

$$= \left\langle \hat{f}, \widehat{f * \frac{c_{d-2s}}{|x|^{d-2s}}}\right\rangle = \left\langle \hat{f}, \frac{c_{2s}}{|\cdot|^{2s}}\hat{f}\right\rangle = c_{2s}\int_{\mathbb{R}^d}\frac{|\hat{f}(k)|^2}{|k|^{2s}}\mathrm{d}k$$

q.e.d.

Returning to the Poisson equation we find that

$$G(x) = \frac{1}{4\pi^2}\frac{\check{1}}{|\cdot|^2} = \frac{1}{4\pi^2}\frac{c_{d-2}}{c_n}\frac{1}{|x|^{d-2}} = \begin{cases}\frac{1}{4\pi|x|}, & d = 3\\ \frac{1}{(d-2)|\mathbb{S}^{d-1}|}\frac{1}{|x|^{d-2}}, & d \geqslant 3\end{cases}$$

for $d \geqslant 3$.

Remark 3.14.

$$G(x) = \begin{cases}\frac{1}{(d-2)|\mathbb{S}^{n-2}|}\frac{1}{|x|^{n-2}}, & d \geqslant 3\\ -\frac{1}{2\pi}\ln(x), & d = 2\\ -|x|, & d = 1\end{cases}$$

is called the Greens function of the Laplacian $(-\Delta)$ in $\mathbb{R}^d$. In particular $G(x-y)$ is the kernel of the operator $(-\Delta)^{-1}$ in $L^2(\mathbb{R}^d)$, i.e.

$$(-\Delta)^{-1}f(x) = \int_{\mathbb{R}^d} G(x-y)f(y)\mathrm{d}y$$

□

Theorem 3.15 (Poisson Equation)**.** *If $f \in L^2(\mathbb{R}^d)$, then $u = G * f \in L^1_{loc}(\mathbb{R}^d)$ and $-\Delta u = f$ in $\mathscr{D}'(\mathbb{R}^d)$. Consequently, $-\Delta G = \delta$ in $\mathscr{D}'(\mathbb{R}^d)$.* □

Proof. For $n \geqslant 3$ Take $\varphi_n \in \mathscr{D}$, $\varphi_n \to f$ in $L^2(\mathbb{R}^d)$. Then

$$-\widehat{\Delta(G*\varphi_n)} = G*\widehat{(-\Delta\varphi_n)} = \hat{G}\widehat{-\Delta\varphi_n} = \frac{1}{(2\pi|k|^2)}(2\pi|k|^2)\hat{\varphi}_n(k) = \hat{\varphi}_n(k).$$

Thus $-\Delta(G*\varphi_n) = \varphi_n$. Since $G*\varphi_n \to G*f$ in $\mathscr{D}'$ it follows that $-\Delta(G*\varphi_n) \to -\Delta(G*f)$ in $\mathscr{D}'$. We conclude that $-\Delta(G*f) = f$ in $\mathscr{D}'(\mathbb{R}^d)$. Moreover

$$\int G(-\Delta\varphi) = \int \hat{G}\widehat{-\Delta\varphi} = \int \hat{\varphi} = \varphi(0)$$

for all $\varphi \in \mathscr{D}$, thus $-\Delta G = \delta$ in $\mathscr{D}'(\mathbb{R}^d)$. *q.e.d.*

We now turn to the Yukawa equation

$$\boxed{\mu u - \Delta u = f}$$

for $\mu > 0$. By taking the Fourier transform we find that

$$\left(\mu + (2\pi|k|)^2\right)\hat{u} = \hat{f}$$

which implies that $\hat{u} = \hat{G}\hat{f}$ with

$$\hat{G}(k) = \frac{1}{\mu + (2\pi|k|)^2}$$

which belong to $L^2(\mathbb{R}^d)$ for $n \geqslant 3$. Thus we find that the Green's function of the Yukawa equation is

$$G(x) = \begin{cases} \frac{1}{2\mu}e^{-\mu|x|}, & d=1 \\ \frac{1}{4\pi|x|}e^{-\mu|x|}, & d=3 \end{cases}$$

Chapter 4

Sobolev Space $H^m(\mathbb{R}^d)$

Definition 4.1. We define the Sobelev spaces to be

$$H^1(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^n) \mid \partial_{x_i} f \in L^2(\mathbb{R}^d), i = 1, \dots, d\}$$
$$H^m(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid D^\alpha f \in L^2(\mathbb{R}^d), |\alpha| \leqslant m\}$$

where the derivatives are taken in the distributional sense. □

Theorem 4.2. *$H^m(\mathbb{R}^d)$ is a Hilbert space with inner product,*

$$\langle f, g \rangle_{H^m} = \sum_{|\alpha| \leqslant m} \langle D^\alpha f, D^\alpha g \rangle_2$$

□

Proof. For H^1 it is easy to see that $\langle \cdot, \cdot \rangle_{H_1}$ is a well-defined inner product. Concerning completeness, if $\{\varphi_n\}$ is a Cauchy sequence in H^1, then both $\{f_n\}$ and $\{\partial_{x_i} f_n\}$ are Cauchy sequences in $L^2(\mathbb{R}^d)$. Hence there exist $f, g_i \in L^2$ such that $f_n \xrightarrow{L^2} f$ and $\partial_{x_i} f_n \xrightarrow{L^2} g_i$. We need to prove that $\partial_{x_i} f = g_i$ for all $i = 1, \dots, n$ from which follows that $f \in H^1$. Take any test function $\varphi \in \mathscr{D}'$, then per definitionem we have

$$\int \partial_{x_i} f_n \varphi = -\int f_n \partial_{x_i} \varphi \xrightarrow{n\to\infty} -\int f \partial_{x_i} \varphi = \int \partial_{x_i} f \varphi$$

thus $\partial_{x_i} f_n \xrightarrow{n\to\infty} g_i$ from which follows that $\partial_{x_i} f = g_i$ and therefore $f \in H^1(\mathbb{R}^d)$. *q.e.d.*

Theorem 4.3. *$\mathscr{D}(\mathbb{R}^d)$ is dense in $H^m(\mathbb{R}^d)$.* □

Proof. We shall only prove the case of H^1. Take $f \in H^1(\mathbb{R}^d)$. We need to find $f_\varepsilon \in \mathscr{D}$, such that $f_\varepsilon \to f$ in H^1.

Step 1. Find a sequence $g_\varepsilon \in H^1$, such that g_ε has compact support such that $g_\varepsilon \to f$ in H^1. Choose $h \in \mathscr{D}$ such that $h(x) = 1$ for all $|x| \leqslant 1$ and choose $g_\varepsilon(x) = f(x)h(\varepsilon x)$ has compact support and $g_\varepsilon(x) = f(x)$, when $|x| \leqslant \frac{1}{\varepsilon}$. We have

$$\|g_\varepsilon - f\|_2^2 = \int |1 - h(\varepsilon x)|^2 |f(x)|^2 \mathrm{d}x \longrightarrow 0$$

by dominated convergence. Similarly

$$\|\partial_{x_i} g_\varepsilon - \partial_{x_i} f\|_2^2 = \int |\partial_{x_i} f(h(\varepsilon x) - 1) + f(x)\partial_{x_i} h(\varepsilon x)|^2 \mathrm{d}x \leqslant$$
$$\leqslant 2\int |\partial_{x_i} f(x)(h(\varepsilon x) - 1)|^2 \mathrm{d}x + 2\int |f(x)|^2 |\partial_{x_i} h(\varepsilon x)|^2 \mathrm{d}x$$

Here $\int |\partial_{x_i} f(x)(h(\varepsilon x) - 1)|^2 \mathrm{d}x \to 0$ and since $\partial_{x_i} h = 0$ in $|x| \leqslant \frac{1}{\varepsilon}$

$$\int |f(x)|^2 |\partial_{x_i} h(\varepsilon x)|^2 \mathrm{d}x = \int_{B_{\frac{1}{\varepsilon}}(x)^C} |f(x)|^2 |\partial_{x_i} h(\varepsilon x)|^2 \mathrm{d}x \longrightarrow 0$$

by dominated convergence.

Step 2. Consider $g_\varepsilon \in H^1$ with compact support. Take $\varphi \in \mathscr{D}$ with $\int \varphi = 1$ and define $\varphi_k(x) = k^n \varphi(kx)$. We know that $\varphi_n * g_\varepsilon \in \mathscr{C}^\infty c$ and $D^\alpha(\varphi_n * g_\varepsilon) \to D^\alpha g_\varepsilon$ in $L^2(\mathbb{R}^n)$ for $|\alpha| \leqslant 1$

We conclude by noting that

$$\|\varphi_k * g_\varepsilon - f\|_{H^1} \leqslant \|\varphi_n * g_\varepsilon - g_\varepsilon\|_{H^1} + \|g_\varepsilon - f\|_{H^1} \xrightarrow[k\to\infty]{\varepsilon\to 0} 0$$

q.e.d.

Theorem 4.4. *$\mathscr{C}_c^\infty(\mathbb{R}^d)$ is dense in $H^m(\mathbb{R}^d)$.* □

Remark 4.5. If Ω is a bounded set of $\mathbb{R}^d$, then

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, \partial_{x_i} f \in L^2(\Omega), i = 1, \ldots, n \right\}$$

Then $\mathscr{C}_c^\infty(\Omega)$ is not dense in $H^1(\Omega)$. In fact $H_0^1(\Omega) = \overline{\mathscr{C}_c^\infty}^{H^1(\Omega)} \neq H^1(\Omega)$. We well come back to this (boundary value problems). □

Theorem 4.6 (Chain Rule)**.** *If $G \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$, $|G'| \leqslant C$, $G(0) = 0$. Then for all $f \in H^1\left(\mathbb{R}^d\right)$, $G(f) \in H^1\left(\mathbb{R}^d\right)$ and*

$$\partial_{x^i} G(f) = G'(f) \partial_{x_i} f$$

in $\mathscr{D}'\left(\mathbb{R}^d\right)$. □

Proof. Since $f \in H^1\left(\mathbb{R}^d\right)$, we can find a sequence $\{\varphi_n\} \subset \mathscr{C}_c^\infty\left(\mathbb{R}^d\right)$ such that $\varphi_n \to f$ in $H^1\left(\mathbb{R}^d\right)$. We can also assume that

$$\begin{aligned} \varphi_n(x) &\longrightarrow f(x) \quad \text{a.e.} \\ \partial_{x_i} \varphi_n(x) &\longrightarrow \partial_{x_i} f(x) \quad \text{a.e.} \\ |\varphi_n| + \sum_{i=1}^n |\partial_{x_i} \varphi_n| &\leqslant F \in L^2\left(\mathbb{R}^d\right). \end{aligned}$$

We can do this by Theorem 1.14. We have (by the usual chain rule)

$$\partial_{x_i} G(\varphi_n(x)) = G'(\varphi_n(x)) \partial_{x_i} \varphi_n(x)$$

and

$$\begin{aligned} G'(\varphi_n(x)) \partial_{x_i} \varphi_n(x) &\longrightarrow G'(f(x)) \partial_{x_i} f(x), \quad \text{a.e.} \\ |G'(\varphi_n(x)) \partial_{x_i} \varphi_n(x)| &\leqslant |G'| |\partial_{x_i} \varphi_n(x)| \leqslant C F(x) \in L^2\left(\mathbb{R}^d\right) \end{aligned}$$

which implies that

$$\partial_{x_i} G(\varphi_n(x)) = G'(\varphi_n(x)) \partial_{x_i} \varphi_n(x) \xrightarrow{L^2} G'(f(x)) \partial_{x_i} f(x)$$

Moreover, we have $G(\varphi_n(x)) \to G(f(x))$ a.e. since

$$|g(\varphi_n(x)) - G(f(x))| \leqslant (\sup |G'|)|\varphi_n(x) - f(x)| \leqslant C|\varphi_n(x) - f(x)| \xrightarrow{n\to 0} 0$$

and thus $G(\varphi_n(x)) \to G(f(x))$ in L^2. The result follows from a general fact. *q.e.d.*

Lemma 4.7. *If $f_n \to f$ in $L^2(\mathbb{R}^d)$ and $\partial_{x_i} f_n \to g_i$ in $L^2(\mathbb{R}^d)$ for $i = 1, \dots, d$, then $f \in H^1(\mathbb{R}^d)$ and $\partial_{x_i} f = g_i$ for $i = 1, \dots, n$.* □

Proof. Take $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Compute

$$\int g_i\varphi \longleftarrow \int (\partial_{x_i} f_n)\varphi = -\int f_n(\partial_{x_i}\varphi) \longrightarrow -\int f(\partial_{x_i}\varphi)$$

and thus $-\int f(\partial_{x_i}\varphi) = \int g_i\varphi$ for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and therefore $\partial_{x_i} f = g_i$ in $\mathscr{D}'(\mathbb{R}^d)$, i.e. $f \in H^1(\mathbb{R}^d)$. *q.e.d.*

Theorem 4.8 (Derivative of $|f|$)**.** *If $f \in H^1(\mathbb{R}^d)$ then $|f| \in H^1(\mathbb{R}^d)$ and*

$$\partial_{x_j}|f(x)| = \begin{cases} \frac{u\partial_j u + v\partial_j v}{|f(x)|}, & \text{if } f(x) \neq 0 \\ 0, & \text{if } f(x) = 0. \end{cases}$$

where $f(x) = u(x) + iv(x)$, where $u, v : \mathbb{R}^d \to \mathbb{R}$. Consequently we have the diamagnetic inequality

$$|\nabla f(x)| \geqslant |\nabla|f|(x)| \quad a.e.$$

□

Proof. Let $\varepsilon > 0$ and define $G_\varepsilon(t) = \sqrt{\varepsilon^2 + |t|^2} - \varepsilon$
Then $G \in \mathscr{C}^1$, $G_\varepsilon(0) = 0$ and

$$|G'_\varepsilon(t)| = \left|\frac{t}{\sqrt{\varepsilon^2 + |t|^2}}\right| \leqslant 1$$

By the chain rule $G_\varepsilon(f(x)) \in H^1(\mathbb{R}^d)$ and

$$\partial_{x_j} G_\varepsilon(f(x)) = \frac{(|f(x)|^2)'}{2\sqrt{\varepsilon^2 + |f(x)|^2}}\partial_{x_i} f(x) = \frac{u(x)\partial_j u(x) + v(x)\partial_j v(x)}{2\sqrt{\varepsilon^2 + |f(x)|^2}}\partial_{x_i} f(x), \quad \text{a.e.}$$

Passing to $\varepsilon \to 0$ we obtain

$$G_\varepsilon(f) = \sqrt{\varepsilon^2 + |f|^2} - \varepsilon \longrightarrow |f| \quad \text{in } L^2\big(\mathbb{R}^d\big)$$
$$\partial_{x_j} G_\varepsilon(f) \longrightarrow g_j(x)$$

From

$$\partial_{x_j}|f(x)| = \begin{cases} \frac{u\partial_j u + v\partial_j v}{|f(x)|}, & \text{if } f(x) \neq 0 \\ 0, & \text{if } f(x) = 0. \end{cases}$$

it follows that

$$\partial_{x_j}|f(x)| \leqslant \frac{|u\partial_j u + v\partial_j v|}{|f|} \leqslant \frac{\sqrt{|u|^2 + |v|^2}\sqrt{|\partial_j u|^2 + |\partial)jv|^2}}{|f|} = \frac{|f||\partial_j f|}{|f|} = |\partial_j f|$$

Thus $|\nabla|f|(x)| \leqslant |\nabla f(x)|$.

q.e.d.

Theorem 4.9 (Fourier Characterisation of $H^m\big(\mathbb{R}^d\big)$)**.** *If $f \in L^2\big(\mathbb{R}^d\big)$, then $f \in H^m\big(\mathbb{R}^d\big)$ if and only if*

$$\int (1 + 2\pi|k|^2)^m \left|\hat{f}(k)\right|^2 \mathrm{d}k < \infty.$$

□

Proof. For $m = 1$. Let $f \in H^1\big(\mathbb{R}^d\big)$, then

$$\|f\|_{H^1}^2 = \|f\|_2^2 + \sum_{i=1}^n \|\partial_{x_i} f\|_2^2 = \int |\hat{f}(k)|^2 \mathrm{d}k + \sum_{i=1}^n \int (2\pi k_i)^2 |\hat{f}(k)|^2 \mathrm{d}k = \int \big(1 + (2\pi|k|)^2\big)|\hat{f}(k)|^2 \mathrm{d}k.$$

For $m > 1$

$$\|f\|_{H^m}^2 = \sum_{|\alpha| \leqslant m} \|D^\alpha f\|_2^2 = \sum_{|\alpha| \leqslant m} \int |(2\pi k)^\alpha \hat{f}(k)|^2 \mathrm{d}k.$$

q.e.d.

Corollary 4.10. *If $f \in L^2(\mathbb{R}^d)$, then $f \in H^m(\mathbb{R}^d)$ iff*

$$(-\Delta)^{\frac{m}{2}} f \in L^2(\mathbb{R}^d) \iff \int (2\pi|k|)^{2m} |\hat{f}(k)|^2 \mathrm{d}k < \infty$$

□

Proof. For $m = 2$, let $f \in L^2$, then $f \in H^2(\mathbb{R}^d)$ iff $\Delta f \in L^2(\mathbb{R}^d)$ for all $|\alpha| \leqslant 2$, e.g. $\partial_{x_1}\partial_{x_2} f \in L^2$, while $\Delta f \in L^2$ only iff $\left(\partial_{x_1}^2 + \partial_{x_2}^2\right) f \in L^2$. But this follow easily from the Fourier characterisation. Indeed if $\Delta f \in L^2$ iff

$$\int (2\pi|k|)^4 |\hat{f}(k)|^2 \mathrm{d}k < \infty.$$

So if $f, \Delta f \in L^2$ then

$$\int \left(1 + (2\pi|k|)^4\right) \left|\hat{f}(k_j)\right|^2 \mathrm{d}k < \infty$$

hence by $1+(2\pi|k|)^4 \geqslant \frac{1}{2}(1 + |2\pi k|^2)^2$ (which follows from $A^2+B^2 \geqslant \frac{1}{2}(A+B)^2$ for $A, B \geqslant 0$)

$$\int \left(1 + |2\pi k|^2\right)^2 |\hat{f}(k)|^2 < \infty$$

which implies that $f \in H^2(\mathbb{R}^d)$ by the last theorem. *q.e.d.*

Chapter 5

Sobolev Inequalities

These inqualities find great practical application in physics for example. Consider in the context of quantum mechanics the energy functional of a wave function ψ

$$\mathcal{E}(\psi) := \int |\nabla\psi(x)|^2 \mathrm{d}x + \int V(x)|\psi(x)|^2 \mathrm{d}x.$$

An important question concerns the stability of such a system, i.e. when does

$$\inf_{\|\psi\|_2} \mathcal{E}(\psi) \geqslant -C$$

for some $C \geqslant 0$ hold. A particular example of this would be an atom with the Coloumb potential

$$\mathcal{E}(\psi) = \int |\nabla\psi(x)|^2 \mathrm{d}x - \int \frac{|\psi(x)|^2}{|x|} \mathrm{d}x.$$

To prove the stability of this system one can use an uncertainty principle,

$$\text{‘} \int |\nabla\psi|^2 \geqslant G \left| \int V(x)|\psi(x)|^2 \mathrm{d}x \right|$$

An example would be the Heisenberg uncertainty principle which states that

$$\left(\int |\nabla\psi(x)|^2 \right) \left(\int |x|^2 |\psi(x)|^2 \mathrm{d}x \right) \geqslant \frac{n^2}{4}$$

for all $n \geqslant 1$ and all $\psi \in H^1(\mathbb{R}^d)$. This can be proven using the commutation relation

$$\nabla \cdot x - x \cdot \nabla = n$$

and the Cauchy Schwarz inequality. Note that for all $f \in H^1$ there exists a $\varphi_n \in H^1(\mathbb{R}^d)$ such that $\varphi_n \to f$ in H^1 and

$$\int |x|^2 |\varphi_n(x)|^2 \mathrm{d}x \to \infty,$$

i.e. the Heisenberg principle becomes "trivial" for φ_n. Hence we need a stronger inequality

Sobolev Inequality For all $\psi \in H^1(\mathbb{R}^d)$

$$\|\nabla\psi\|_2 \geqslant C\|\psi\|_p$$

holds. Now what is p? Let us assume that the Sobolev inequality holds and let $\psi_l(x) = \psi(lx)$ for some $\psi \in H^1$. Then

$$\begin{aligned}
\|\nabla\psi_l\|_1 &= \left(\int \|\nabla\psi_l\|^2\right)^{1/2} = \left(\int |l\nabla\psi(lx)|^2\right)^{1/2} = \left(\int l^2|\nabla\psi(lx)|^2\right)^{1/2} = \left(l^{2-d}\int |\nabla\psi(y)|\mathrm{d}y\right)^{1/2} = \\
&= l^{\frac{2-d}{2}}\|\nabla\psi\|_2 \\
\|\psi_l\|_p &= \left(\int |\psi(lx)|^p \mathrm{d}x\right)^{1/p} = \left(l^{-d}\int |\psi(y)|\mathrm{d}y\right)^{1/p} = l^{-\frac{d}{p}}\|\psi\|_p
\end{aligned}$$

Thus the Sobolev inequality $\|\nabla\psi_l\|_2 \geqslant C\|\psi_l\|_2$ implies that

$$l^{\frac{2-n}{2}}\|\nabla\psi\|_2 \geqslant l^{-\frac{n}{p}}\|\psi\|_p$$

for all $l > 0$. This can be possible iff $\frac{2-d}{2} = -\frac{d}{p}$, i.e.

$$\boxed{p = \frac{2d}{d-2}}, \qquad (n \geqslant 3).$$

Theorem 5.1. *For all $d \geqslant 3$*

$$\|\nabla f\|_2 \geqslant C\|f\|_p$$

for all $f \in H^1(\mathbb{R}^d)$ and $p = \frac{2d}{d-2}$. The constant $C > 0$ is independent of f in particular this implies that if $f \in H^1$ then $f \in L^p$. □

Lemma. *For* $\varphi \in \mathscr{D}(\mathbb{R}^d)$*, then*

$$\|\nabla\varphi\|_1 \geqslant \|\varphi\|_{\frac{d}{d-1}}.$$

□

Proof. Let us focus on $d = 3$. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then

$$\varphi(x) = \varphi(x_1, x_2, x_3) = \int_{-\infty}^{x_1} \partial_{x_1}\varphi(x_1', x_2, x_3)\mathrm{d}x_1'$$

which implies that

$$|\varphi(x)| \leqslant \int_{-\infty}^{x_1} |\partial_{x_1}\varphi(x_1', x_2, x_3)|\mathrm{d}x_1' \leqslant \int_{\mathbb{R}} |\partial_{x_1}\varphi(x_1', x_2, x_3)|\mathrm{d}x_1' \leqslant \int_{\mathbb{R}} |\nabla\varphi(x_1', x_2, x_3)|\mathrm{d}x_1' =: g_1(x_2, x_3)$$

Similarly, one finds that

$$|\varphi(x)|^{3/2} \leqslant \sqrt{g_1(x_2, x_3)}\sqrt{g_2(x_1, x_3)}\sqrt{g_3(x_1, x_2)}$$

which implies that

$$\int_{\mathbb{R}} |\varphi(x)|^{3/2}\mathrm{d}x_1 \leqslant \sqrt{g_1}\int_{\mathbb{R}} \sqrt{g_2}\sqrt{g_3}\mathrm{d}x_1 \leqslant \sqrt{g_1}\sqrt{\int_{\mathbb{R}} g_2\mathrm{d}x_1}\sqrt{\int_{\mathbb{R}} g_3\mathrm{d}x_1}$$

and thus

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\varphi(x)|^{3/2}\mathrm{d}x_1\mathrm{d}x_2 \leqslant \sqrt{\int_{\mathbb{R}} g_2\mathrm{d}x_1}\int_{\mathbb{R}}\left(\sqrt{g_1}\sqrt{\int_{\mathbb{R}} g_3\mathrm{d}x_1}\right)\mathrm{d}x_2 \leqslant \sqrt{\int_{\mathbb{R}} g_2\mathrm{d}x_1}\sqrt{\int_{\mathbb{R}} g_1\mathrm{d}x_2}\sqrt{\int_{\mathbb{R}}\int_{\mathbb{R}} g_3\mathrm{d}x_1\mathrm{d}x_2}$$

and analogously

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |\varphi(x)|^{3/2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \leqslant \sqrt{\int_{\mathbb{R}}\int_{\mathbb{R}} g_1\mathrm{d}x_2\mathrm{d}x_3}\sqrt{\int_{\mathbb{R}}\int_{\mathbb{R}} g_2\mathrm{d}x_1\mathrm{d}x_3}\sqrt{\int_{\mathbb{R}}\int_{\mathbb{R}} g_3\mathrm{d}x_1\mathrm{d}x_2} = \|\nabla\varphi\|_1^{3/2}$$

q.e.d.

Proof of Theorem 5.1. Consider $f \in \mathscr{D}(\mathbb{R}^3)$ and $n = 3$. Choose $\varphi = |f|^4$ and applying the above lemma one finds that

$$\|\varphi\|_{3/2} = \left(\int |\varphi|^{3/2}\right)^{2/3} = \left(\int |f|^6\right)^{2/3}$$
$$\|\nabla\varphi\|_1 \leqslant \int 4f^3|\nabla f| \leqslant 4\left(\int |f|^6\right)^{1/2}\|\nabla f\|_2$$

Then from the lemma

$$\left(\int |f|^6\right)^{2/4} \leqslant 4\left(\int |f|^6\right)^{1/2}\|\nabla f\|_2$$

and thus $\|f\|_6 \leqslant 4\|\nabla f\|_2$. For $n \geqslant 3$ choose $\varphi = |f|^{\frac{2(d-1)}{d-2}}$ and use

$$\int |\nabla f|^2 \geqslant \int |\nabla |f||^2.$$

q.e.d.

Theorem 5.2 (Sobolev Inequality in low dimensions)**.**

($d = 2$) *For all* $f \in H^1(\mathbb{R}^2)$ *and* $2 \leqslant p < \infty$

$$\|f\|_p \leqslant C\|\nabla f\|_2^{\frac{p-2}{p}}\|f\|_2^{\frac{2}{p}}$$

($d = 1$) *For all* $f \in H^1(\mathbb{R})$

$$\|f\|_\infty^2 \leqslant \|f'\|_2\|f\|_2$$

(General fact the Sobolev inequality becomes "weaker" in higher dimensions)

□

Proof.

($d = 2$) From the above lemma it follows that for all $\varphi \in \mathscr{D}(\mathbb{R}^2)$, $\|\varphi\|_2 \leqslant \|\nabla\varphi\|_1$. Choose $\varphi = f^\alpha$ for $\alpha > 0$, $f \in \mathscr{D}(\mathbb{R}^2)$ and $f \geqslant 0$. We have

$$\left(\int f^{2\alpha}\right)^{1/2} \leqslant \int \alpha f^{\alpha-1}|\nabla f| \leqslant \alpha\left(\int f^{2(\alpha-1)}\right)^{1/2}\|\nabla f\|_2.$$

Using Hölder's inequality we find

$$\int f^{2(\alpha-1)} \leqslant \left(\int f^{2\alpha}\right)^{1/q'} \int \left(\int f^2\right)^{1/q}$$

with $\frac{1}{q'} + \frac{1}{q} = 1$, $2(\alpha - 1) = \frac{2\alpha}{q'} + \frac{2}{q}$, hence

$$2((\alpha-1) = \frac{2\alpha}{q'} + \frac{2}{q} = \frac{2\alpha}{q'} + \frac{2\alpha}{q} + \frac{2-2\alpha}{q} = 2\alpha + \frac{2-2\alpha}{q} \implies -2 = \frac{2-2\alpha}{q} \implies q = \alpha - 1$$

Thus

$$\int f^{2\alpha} \leqslant C\Big(\int f^{2(\alpha-1)}\Big)\|\nabla f\|_2^2 \leqslant C\Big(\int f^{2\alpha}\Big)^{1/q'}\Big(\int f^2\Big)^{1/q}\|\nabla f\|_2^2$$

hence

$$\Big(\int f^{2\alpha}\Big)^{1/q} \leqslant C\Big(\int f^2\Big)^{1/q}\|\nabla f\|_2^2 \implies \int f^{2\alpha} \leqslant C\Big(\int f^2\Big)\|\nabla f\|_2^{2(\alpha-1)} \implies$$
$$\implies \|f\|_{2\alpha} \leqslant \|f\|_2^{1/\alpha}\|\nabla f\|_2^{\frac{\alpha-1}{\alpha}}$$

for all $\alpha > 1$. Thus we have $\|f\|_p \leqslant C\|f\|_2^{2/p}\|\nabla f\|_2^{\frac{p-2}{p}}$ for all $p \geqslant 2$. Thus the inequality holds for all $f \in \mathscr{D}$, $f \geqslant 0$, and therefore in can be extended to all $f \in H^1(\mathbb{R}^2)$ by density and the diamagnetic inequality Theorem 4.8.

$(d = 1)$ For every $f \in \mathscr{D}$,

$$f(x) = \int_{-\infty}^{x} f'(t)\mathrm{d}t \implies |f(x)| \leqslant \int_{-\infty}^{x} |f'(t)|\mathrm{d}t$$
$$f(x) = -\int_{x}^{\infty} f'(t)\mathrm{d}t \implies |f(x)| \leqslant \int_{x}^{\infty} |f'(t)|\mathrm{d}t$$

hence

$$|f(x)| \leqslant \frac{1}{2}\int_{\mathbb{R}} |f'(t)|\mathrm{d}t$$

i.e. $\|f\|_\infty \leqslant \frac{1}{2}\|f'\|_1$. Now we can replace f by f^2 to find that

$$\|f\|_\infty^2 \leqslant \frac{1}{2}\int |(f^2)'| \leqslant \int |f||f'| \leqslant \|f\|_2\|f'\|_2$$

for all $f \in \mathscr{D}$. Then by density we get the inequality for all $f \in H^1\big(\mathbb{R}^d\big)$.

q.e.d.

Additional Proof of $d=2$. Recall that we have the Hausdorff-Young inequality, that

$$\|\hat{f}\|_{p'} \leqslant \|f\|_p$$

for all $f \in L^p(\mathbb{R}^n)$ and all $1 \leqslant p \leqslant 2 \leqslant p' \leqslant \infty$ with $\frac{1}{p}+\frac{1}{p'}=1$. This inequality is equivalent to

$$\|f\|_p \leqslant \|\hat{f}\|_{p'}$$

for all $f \in \mathscr{D}(\mathbb{R}^n)$ with $p \geqslant 2 \geqslant p'$, $\frac{1}{p}+\frac{1}{p'}=1$. We have

$$\begin{aligned}\|f\|_p \leqslant \left(\int |\hat{f}(k)|^{p'}\right)^{1/p'} &= \left(\int |\hat{f}(k)|^{p'}(1+2\pi|k|)^{p'}\frac{1}{(1+2\pi|k|)^{p'}}\mathrm{d}k\right)^{1/p'} \leqslant \\ &\leqslant \left(\int |\hat{f}(k)|^2(1+2\pi|k|)^2\mathrm{d}k\right)^{\alpha/p'}\left(\int \frac{1}{(1+2\pi|k|)^{pp'}}\mathrm{d}k\right)^{1-\alpha/p'}\end{aligned}$$

when $pp'>2$ we have

$$\int \frac{1}{(1+2\pi|k|)^{pp'}}\mathrm{d}k \leqslant C < \infty$$

Thus $\|f\|_p \leqslant C_p\|f\|_{H_1}$ for all $p \geqslant 2$ and all $f \in \mathscr{D}$. This implies the Sobolev inequality $\|f\|_p \leqslant C\|\nabla f\|^{\frac{p-2}{p}}\|f\|_2^{\frac{2}{p}}$, by a scaling argument, i.e. use $\|f\|_p \leqslant C\|f\|_{H^1}$, for $f \mapsto f_l(x) = f(lx)$ for $l>0$ and optimise over $l>0$ *q.e.d.*

Theorem 5.3 (Sobolev Continuous Embedding)**.**

$$H^1(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \qquad \textit{for all} \begin{cases} 2 \leqslant p \leqslant \frac{2d}{d-2}, & \textit{if } d \geqslant 3 \\ 2 \leqslant p < \infty, & \textit{if } d = 2 \\ 2 \leqslant p \leqslant \infty, & \textit{if } d = 1 \end{cases}$$

and the inclusion is continuous, i.e.

$$\|f\|_p \leqslant C\|f\|_{H^1}.$$

Moreover, when $d=1$, $H^1(\mathbb{R}) \subset \mathscr{C}(\mathbb{R})$, i.e. for all $f \in H^1(\mathbb{R})$, there exists exactly one $\tilde{f} \in \mathscr{C}(\mathbb{R})$, such that $f=\tilde{f}$ almost everywhere. □

Proof.

$(d \geqslant 3)$ We know that

$$\|f\|_{\frac{2d}{d-2}} \leqslant C\|\nabla f\|_2 \leqslant C\|f\|_{H^1}$$

By Hölder's inequality for all $2 \leqslant p \leqslant \frac{2d}{d-2}$,

$$\|f\|_p \leqslant C\|f\|_{H^1}.$$

$(d = 2)$

$$\|f\|_p \leqslant C\|\nabla f\|_2^{\frac{p-2}{p}} \|f\|_2^{\frac{2}{p}} \leqslant C\|f\|_{H^1}.$$

$(d = 1)$

$$\begin{aligned}\|f\|_\infty &\leqslant \|f'\|_2^{1/2}\|f\|_2^{1/2} \leqslant \|f\|_{H^1}\\ \|f\|_2 &\leqslant \|f\|_{H^1}\end{aligned}$$

hence by Hölder's inequality for all $2 \leqslant p \leqslant \infty$, $\|f\|_p \leqslant \|f\|_{H^1}$

We now have to prove that $H^1 \subset \mathscr{C}(\mathbb{R})$. Take $f \in H^1$. Then we can find a sequence φ_n such that $\varphi_n \in \mathscr{D}$, $\varphi_n \to f$ in H^1 and $\varphi_n(x) \to f(x)$ a.e. $x \in \mathbb{R}$. We know that

$$\varphi_n(x) - \varphi_n(y) = \int_x^y \varphi'(t)\mathrm{d}t$$

and thus for $x \leqslant y$

$$|\varphi_n(x) - \varphi_n(y)| \leqslant \left|\int_x^y \varphi_n'(t)\mathrm{d}t\right| \leqslant \left(\int_x^y \mathrm{d}t\right)^{1/2} \left(\int_x^y |\varphi'(t)|^2\mathrm{d}t\right)^{1/2} \leqslant \sqrt{|y-x|}\|\varphi_n'\|_2$$

for all $x, y \in \mathbb{R}$. Since $\varphi_n \to f$ in H^1 and $\varphi_n(x) \to f(x)$ for all $x \in \mathbb{R} \setminus A$ with $|A| = 0$. Then for all $x, y \in \mathbb{R} \setminus A$ we have

$$|f(x) - f(y)| = \lim_{n\to\infty} |\varphi_n(x) - \varphi_n(y)| \leqslant \sqrt{|y-x|} \lim_{n\to\infty} \|\varphi_n'\|_2 = \sqrt{|x-y|}\|f'\|_2$$

Define $\tilde{f}(x) = f(x)$ for all $x \in \mathbb{R} \setminus A$. Then we can extend $\tilde{f}$ to be a continuous function on all of $\mathbb{R}$ such that $|\tilde{f}(x) - \tilde{f}(y)| \leqslant \sqrt{|x-y|}\|f'\|_2$ for all $x, y \in \mathbb{R}$.

q.e.d.

Theorem 5.4 (Sobolev Compact Embedding). *Let B be a bounded set of $H^1(\mathbb{R}^d)$ and A a bounded set of $\mathbb{R}^d$. Then we have*

$$\mathbf{1}_A B \subset\subset L^p(A), \qquad \text{with} \begin{cases} 2 \leqslant p < \frac{2n}{n-2}, & \text{if } n \geqslant 3 \\ 2 \leqslant p < \infty, & \text{if } n = 2 \\ 2 \leqslant p \leqslant \infty, & \text{if } n = 1 \end{cases}$$

□

Remark. By $\mathbf{1}_A$ we denote the indicator/characteristic function of the set A.

$$\mathbf{1}_A B \subset\subset L^p(A)$$

means that if $(f_n)_n \subset \mathbf{1}_A B$, i.e. $f_n = \mathbf{1}_A g_n$ with $g_n \in B$, then there exists a subsequence f_{n_k} such that f_{n_k} converges strongly in $L^p(A)$. □

Corollary. *If f_n is bounded in $H^1(\mathbb{R}^d)$, there exists a subsequence such that $f_n(x) \to f(x)$ a.e. $x \in \mathbb{R}^d$.* □

Proof. A subsequence of $\mathbf{1}_{B_R(0)} f_n(x)$ converges strongly in $L^p(\mathbb{R}^d)$. Since L^p convergence implies that pointwise convergence of a subsequence we find that there exists a subsequence

$$f_{n_{k_l}}(x) \longrightarrow f(x) \quad \text{a.e.}$$

for $x \in B_R(0)$. Renaming this subsequence f_n and taking $R \to \infty$ using Cantor's diagonal argument one finds a subsequence of f_n such that it converges pointwise on almost all of $\mathbb{R}^d = \bigcup_{R\uparrow\infty} B_R(0)$. *q.e.d.*

Proof of Theorem 5.4.

($d \geqslant 3$) Take a sequence $(f_n)_n \subset B$, with $(f_n)_n$ bounded in $H^1(\mathbb{R}^d)$. By Banach-Alaoglu Theorem 1.26, we can find a subsequence

$$f_{j_n} \xrightharpoonup[H^1]{n\to\infty} f$$

We have to prove that $\mathbf{1}_A f_n \to \mathbf{1}_A f$ strongly in $L^p(\mathbb{R}^n)$. By linearity, we can assume that $f = 0$ (i.e. we consider $f_n - f$ instead of f_n). Thus we need to prove that if $f_n \rightharpoonup 0$ in $H^1(\mathbb{R}^d)$, then $\mathbf{1}_A f_n \to 0$ strongly in $L^p(\mathbb{R}^d)$. Now we write

$$\mathbf{1}_A f_n = \mathbf{1}_A e^{t\Delta} f_n + \mathbf{1}_A\big(f_n - e^{t\Delta} f_n\big).$$

Recall that

$$\widehat{e^{t\Delta} f}(k) = e^{-t4\pi^2k^2}\hat{f}(k)$$

where $\big(e^{t\Delta} f\big)(x) = \int G(x-y)f(y)\mathrm{d}y$, where G is the heat kernel. We have

$$\|\mathbf{1}_A f_n\|_2 \leqslant \|\mathbf{1}_A e^{t\Delta} f_n\|_2 + \|\mathbf{1}_A(f_n - e^{t\Delta} f_n)\|_2$$

By the Fourier transform and the Plancherl theorem we have

$$\|\mathbf{1}_A(f_n - e^{t\Delta} f_n)\|_2 \leqslant \|f_n - e^{t\Delta} f_n\|_2 = \|\hat{f}_n - \widehat{e^{t\Delta} f}_n\|_2 = \left(\int \left(1 - e^{-t4\pi^2k^2}\right)^2 \left|\hat{f}_n(k)\right|^2 \mathrm{d}k\right)^{1/2} \overset{1}{\leqslant}$$

$$\leqslant \left(\int (t4\pi^2k^2)^2 \left|\hat{f}_n(k)\right|^2 \mathrm{d}k\right)^{1/2} = \sqrt{t}\|\nabla f_n\|_2 \leqslant \sqrt{t}C$$

We have $\mathbf{1}_A e^{t\Delta} f_n \to 0$ strongly since, for every $x \in \mathbb{R}^d$

$$e^{t\Delta} f_n(x) = \langle G(x-\cdot), f_n\rangle \to 0$$

as $G(x-\cdot) \in L^2$ and f_n converges weakly and for all $x \in \mathbb{R}^d$

$$\left|\big(e^{t\Delta} f_n\big)(x)\right| \leqslant \left(\int_{\mathbb{R}^d} |G(x-y)|^2 \mathrm{d}y\right)^{1/2} \left(\int_{\mathbb{R}^d} |f_n(y)|^2 \mathrm{d}y\right)^{1/2} \leqslant C_t,$$

i.e. $\mathbf{1}_A e^{t\Delta} f_n$ is dominated by $C_t \mathbf{1}_A$ and thus as $e^{t\Delta} f_n$ converges pointwise it also converges strongly by the dominated convergence theorem.

[1] $1 - e^{-s} \leqslant \min\{1, s\}$

Concluding we find that

$$\|\mathbf{1}_A f_n\|_2 \leqslant \left\|\mathbf{1}_A e^{t\Delta} f_n\right\|_2 + C\sqrt{t}.$$

Taking $n \to \infty$ we have

$$\limsup_{n\to\infty} \|\mathbf{1}_A f_n\|_2 \leqslant 0 + C\sqrt{t}$$

and taking $t \to 0$ we find that

$$\limsup_{n\to\infty} \|\mathbf{1}_A f_n\|_2 \leqslant 0$$

i.e. $\mathbf{1}_A f_n \to 0$ converges strongly in $L^2(\mathbb{R}^d)$.

Moreover, we know that

$$\|\mathbf{1}_A f_n\|_q \leqslant \|f_n\|_q \leqslant C\|f_n\|_{H^1}$$

for all

$$\begin{cases} q \leqslant \frac{2d}{d-2}, & \text{if } d \geqslant 3 \\ q < \infty, & \text{if } d = 2 \\ q \leqslant \infty, & \text{if } dn = 1 \end{cases}$$

Then by interpolation (Hölder's inequality) we find that $\mathbf{1}_A f_n \to 0$ converges strongly in L^p for

$$\begin{cases} 2 \leqslant p < \frac{2d}{d-2}, & \text{if } d \geqslant 3 \\ 2 \leqslant p < \infty, & \text{if } d \leqslant 2 \end{cases}$$

$(d = 1)$ As in $n \geqslant 3$ we can prove $\mathbf{1}_A B \subset\subset L^p(\mathbb{R}^n)$, $2 \leqslant p \leqslant \infty$.

Why can we include $p = \infty$? Let $f_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R})$. We need to prove

$$\sup_{x\in A} |f_n(x)| \xrightarrow{n\to\infty} 0$$

Indeed, we can write

$$f_n(x) = f_n(y) + f_n(x) - f_n(y) \implies f_n(x) = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f_n(y)\mathrm{d}y + \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} (f_n(x) - f_n(y))\mathrm{d}y$$

By the triangle inequality and Sobolev inequality we have

$$|f_n(x)| \leqslant \frac{1}{2\varepsilon}\left|\int\limits_{x-\varepsilon}^{x+\varepsilon} f_n(y)\mathrm{d}y\right| + \frac{1}{2\varepsilon}\int\limits_{x-\varepsilon}^{x+\varepsilon}\sqrt{|x-y|}\|f_n'\|_2\mathrm{d}y \leqslant \frac{1}{2\varepsilon}\left|\int\limits_{x-\varepsilon}^{x+\varepsilon} f_n(y)\mathrm{d}y\right| + \sqrt{\varepsilon}\|f_n'\|_2$$

Take $n \to \infty$, then

$$\limsup_{n\to\infty} |f_n(x)| \leqslant \sqrt{\varepsilon}\|f'\|_2$$

since $f_n \rightharpoonup L^2$. Take $\varepsilon \to 0$ to see that $f_n(x) \to 0$ or all $x \in \mathbb{R}$. Now we assume that $\sup_{x\in A} |f_n(x)| \not\to 0$, then there must exists a subsequence f_n, and a sequence $(x_n)_n \subset A$ such that

$$\liminf_{n\to\infty} |f_n(x_n)| > 0.$$

Because A is bounded, there must exists a subsequence such that $x_n \to x_0$. Then

$$f_n(x_n) = f_n(x_0) + f_n(x_n) - f_(x_0) \implies |f_n(x_n)| \leqslant |f_n(x_0)| + \sqrt{|x_n - x_0|}\|f_n'\|_2 \xrightarrow{n\to\infty} 0$$

which is a contradiction. ↯

q.e.d.

Sobolev Spaces $W^{m,p}(\mathbb{R}^d)$

Definition 5.5.

$$W^{m,p}(\mathbb{R}^d) := \left\{ f \in L^p \,\middle|\, \forall |\alpha| \leqslant m : D^\alpha f \in L^p \right\}$$

□

Theorem 5.6. *For all $m \in \mathbb{N}$, $p \in [1,\infty]$ $W^{m,p}(\mathbb{R}^d)$ is a Banach space with the norm*

$$\|f\|_{W^{m,p}} = \left(\sum_{|\alpha|\leqslant m|} \|D^\alpha f\|_p^p \right)^{1/p}$$

(In particular $W^{m,2} = H^m$ is a Hilbert space). □

Proof. Analogous to H^m. *q.e.d.*

Theorem 5.7 (Weak Convergence)**.** *For $m \in \mathbb{N}$, $1 < p < \infty$, then $f_n \rightharpoonup f$ weakly in $W^{m,p}$ iff $D^\alpha f_n \rightharpoonup D^\alpha f$ weakly in $L^p(\mathbb{R}^d)$.* □

Proof. Analogous to H^m *q.e.d.*

Theorem 5.8 (Sobolev Inequalities)**.** *Let $m \in \mathbb{N}$, $1 < p < \infty$. Then have a continuous embedding*

$$W^{m,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \qquad \text{with} \begin{cases} p \leqslant q \leqslant \frac{dp}{d-mp}, & \text{if } d > mp \\ p \leqslant q < \infty, & \text{if } d = mp \\ p \leqslant q \leqslant \infty, & \text{if } n < mp \end{cases}$$

In particular if $n < mp$, then $W^{m,p}(\mathbb{R}^n) \subset \mathscr{C}(\mathbb{R}^n)$ and for $m = 1$

$$W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \qquad \text{with} \begin{cases} p \leqslant q \leqslant \frac{dp}{d-p}, & \text{if } d > mp \\ p \leqslant q < \infty, & \text{if } d = p \\ p \leqslant q \leqslant \infty, & \text{if } d < p \end{cases}$$

□

Proof.

$(m = 1)$ We consider $n > p$. We want to prove that

$$\|f\|_{W^{1,p}} \geqslant c\|f\|_q, \qquad p \leqslant q \leqslant \frac{dp}{d-p}$$

Using the inequality $\|u\|_{\frac{d}{d-1}} \leqslant \|\nabla u\|_1$, for all $u \in \mathscr{D}(\mathbb{R}^d)$, $d \geqslant 2$ with $u = f^\alpha$, $f \in \mathscr{D}$, $f \geqslant 0$. Then

$$\left(\int f^{\alpha\frac{d}{d-1}}\right)^{\frac{n-1}{n}} \leqslant \alpha \int f^{\alpha-1}|\nabla f| \leqslant \alpha\left(\int f^{p'(\alpha-1)}\right)^{1/p'} \|\nabla f\|_p, \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

We need $\alpha\frac{n}{n-1} = p'(\alpha - 1)$ which is equivalent to

$$\frac{d}{(d-1)p'} = \frac{\alpha - 1}{\alpha} = 1 - \frac{1}{\alpha} \implies \frac{1}{\alpha} = 1 - \frac{d(p-1)}{(d-1)p} = \frac{d-p}{(d-1)p}$$

i.e.

$$\boxed{\alpha = \frac{(d-1)p}{d-p}}$$

Hence,

$$\alpha \frac{d}{d-1} = \frac{(d-1)p}{d-p} \frac{d}{d-1} = \frac{dp}{d-p}$$

Thus

$$\|f\|_{\frac{dp}{d-p}} \leqslant C \|\nabla f\|_p$$

for all $f \in \mathscr{D}$, $f \geqslant 0$ and thus this holds for all $f \in W^{1,p}$ by density and the diagmagnetic inequality.

The case $p = d$ is similar to H^1. Let $p > d$. Why $W^{1,p} \subset L^\infty(\mathbb{R}^d) \cap \mathscr{C}(\mathbb{R}^d)$. Take $f \in \mathscr{C}_c^\infty(\mathbb{R}^d)$. Write

$$f(x) - f(y) = \int_0^1 \nabla f(y + t(x-y)) \cdot (x-y) \mathrm{d}t.$$

Integrating over $B_r(y)$ we find that

$$\begin{aligned}
\int_{B_r(y)} |f(x) - f(y)| \mathrm{d}x &\leqslant \int_0^1 \int_{B_r(y)} |\nabla f(y + t(x-y))||x-y| \mathrm{d}x \mathrm{d}t \overset{z=t(x-y)}{=\!=\!=} \\
&= \int_0^1 \int_{|z|<tr} |\nabla f(y+z)| \frac{|z|}{t} \frac{dz}{t^d} \mathrm{d}t \leqslant \\
&\leqslant \int_0^1 \frac{1}{t^d} \left(\int_{|z|<tr} \mathrm{d}z \right)^{1/p'} \left(\int_{|z|<tr} |\nabla f(y+z)|^p \mathrm{d}z \right)^{1/p} \mathrm{d}t \leqslant \\
&\leqslant Cr \int_0^1 \frac{(tr)^{\frac{d}{p'}}}{t^d} \|\nabla f\| \mathrm{d}t = \\
&= Cr^{1+\frac{d}{p'}} \left(\int_0^1 t^{\frac{d}{p'}-d} \mathrm{d}t \right) \|\nabla f\|_p
\end{aligned}$$

Here

$$\int_0^1 t^{\frac{d}{p'}-d}\mathrm{d}t < \infty \iff \frac{d}{p'} - d > 1 \iff d - 1 < \frac{p}{p'} = p - 1 \iff p > d.$$

Thus

$$\int_{|x-y|<r} |f(x) - f(y)|\mathrm{d}x \leqslant C r^{1+\frac{d(p-1)}{p}} \|\nabla f\|_p.$$

note that

$$1 + \frac{d(p-1)}{p} > d \iff d(p-1) > (d-1)p \iff p > d$$

Thus for some $s > 0$

$$\int_{|x-y|<r} |f(x) - f(y)|\mathrm{d}x \leqslant C r^{d+s} \|\nabla f\|_p$$

Take $z \in \mathbb{R}^d$, we write

$$f(y) - f(z) = f(y) - f(x) + f(x) - f(z) \implies |f(y) - f(z)| \leqslant |f(y) - f(x)| + |f(x) - f(z)|$$

integrating over x we find that $|x - y| \leqslant |y - z| = r$.

$$C|y-z|^d |f(y) - f(z)| \overset{2}{\leqslant} \int_{|x-y|\leqslant|y-z|} |f(x) - f(y)|\mathrm{d}x + \int_{|x-z|\leqslant 2|y-z|} |f(x) - f(z)|\mathrm{d}x \leqslant$$
$$\leqslant C'|y-z|^{d+s} \|\nabla f\|_p \implies |f(x) - f(y)| \leqslant C|y-z|^s \|\nabla f\|_p$$

for some $s > 0$. This implies that $W^{1,p}(\mathbb{R}^n) \subset\subset \mathscr{C}(\mathbb{R}^n)$. We still need to prove that $W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. Write $f(y) = f(x) + f(y) - f(x)$ and thus $|f(y)| \leqslant |f(x)| + |f(y) - f(x)|$. Integrating over $|x - y| < 1$

$$C|f(y)| \leqslant \int_{|x-y|<1} |f(x)|\mathrm{d}x + \int_{|x-y|<1} |f(y) - f(x)|\mathrm{d}x \leqslant \left(\int_{|x-y|<1} \mathrm{d}x \right)^{1/p'} \|f\|_p + C'\|\nabla f\|_p \leqslant C'\|f\|_{W^{1,p}}$$

Thus $\sup_{y\in\mathbb{R}^n} |f(y)| \leqslant C\|f\|_{W^{1,p}}$.

For higher m, use that $f \in W^{m,p}(\mathbb{R}^d)$ implies that $\partial_{x_i} f \in W^{m-1,p}(\mathbb{R}^d)$. By induction and Sobolev inequality for $W^{1,p}$ implies that $\|\partial_{x_i} f\|_q \leqslant \|f\|_{W^{m,p}}$. Thus $f \in L^p$ and

$\partial_{x_i} f \in L^q$.

q.e.d.

Example 5.9. This proof yields that $H^1(\mathbb{R}^1) \subset \mathscr{C}(\mathbb{R}^1)$, but $H^1(\mathbb{R}^2) \not\subset \mathscr{C}(\mathbb{R}^2)$, $H^1(\mathbb{R}^3) \not\subset \mathscr{C}(\mathbb{R}^3)$. However,

$$H^2\big(\mathbb{R}^2\big) \subset \mathscr{C}\big(\mathbb{R}^2\big), \qquad \text{and } H^2\big(\mathbb{R}^3\big) \subset \mathscr{C}\big(\mathbb{R}^3\big)$$

Chapter 6

Ground States for Schrödinger Operators

Definition. A Schrödinger operator is operator of the form

$$-\Delta + V$$

for $V : \mathbb{R}^d \to \mathbb{R}$ some external potential. The corresponding Schrödinger equation is

$$(-\Delta + V)\psi = E\psi$$

for some $E \in \mathbb{R}$ (the energy of the system). □

Remark (Physical Interpretation)**.** Let $\psi \in L^2(\mathbb{R}^d)$, $\|\psi\|_2 = 1$ be the wave function of a quantum particle, then the ground state energy is given

$$E = \inf\left\{ \int_{\mathbb{R}^d} |\nabla\psi|^2 + \int_{\mathbb{R}^d} V|\psi|^2 \,\middle|\, \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1 \right\}$$

□

Theorem 6.1 (Minimisers are Solutions)**.** *If $V \in L^p_{loc}(\mathbb{R}^d)$ where*

$$\begin{cases} p \geqslant \frac{d}{2}, & \text{if } d \geqslant 3 \\ p > 1, & \text{if } d = 2 \\ p = 1, & \text{if } d = 1 \end{cases}$$

and ψ_0 is a minimiser for E, then

$$-\Delta\psi_0 + V\psi_0 = E\psi_0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^d)$$

(in particular, $V\psi_0 \in L^1_{loc}$.) □

Example. Let $f \in \mathscr{C}^1(\mathbb{R})$. Then $f'(x_0) = 0$ if x_0 is a minimiser of f, i.e. $f(x_0 + t) \geqslant f(x_0)$, hence for $t > 0$

$$\frac{f(x_0 + t) - f(x_0)}{t} \geqslant 0 \implies f'(x_0) \geqslant 0$$

and for $t < 0$

$$\frac{f(x_0 + t) - f(x_0)}{t} \leqslant 0 \implies f'(x_0) \leqslant 0$$

i.e. $f'(x_0) = 0$.

Proof. Let $\mathcal{E}(u) = \int |\nabla u|^2 + \int V|u|^2$, then per definitionem of ψ_0

$$\mathcal{E}(u) \geqslant \mathcal{E}(\psi_0)$$

for all $u \in H^1$ with $\|u\|_2 = 1$. Thus for all $\varphi \in \mathscr{C}_c^\infty$ and $|t|$ small enough

$$\mathcal{E}\left(\frac{\psi_0 + t\varphi}{\|\psi_0 + t\varphi\|_2}\right) \geqslant \mathcal{E}(\psi_0)$$

i.e. $t \mapsto \mathcal{E}\left(\frac{\psi_0 + t\varphi}{\|\psi_0 + t\varphi\|_2}\right)$ attains its minimum, when $t = 0$. Hence

$$0 = \frac{d}{dt}\mathcal{E}\left(\frac{\psi_0 + t\varphi}{\|\psi_0 + t\varphi\|_2}\right) = \frac{d}{dt}\frac{\mathcal{E}(\psi_0 + t\varphi)}{\|\psi_0 + t\varphi\|_2^2}.$$

Noting that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(\psi_0+t\varphi) &= 2\,\Re\int\overline{\nabla u_0}\nabla\varphi + 2\,\Re\int V\overline{\psi_0}\varphi \\
\mathcal{E}(\psi_0+t\varphi)\big|_{t=0} &= E \\
\frac{d}{dt}\|\psi_0+t\varphi\|_2^2 &= 2\,\Re\int\overline{u_0}\varphi \\
\|\psi_0+t\varphi\|_2^2\big|_{t=0} &= 1
\end{aligned}$$

one finds that

$$\begin{aligned}
0 = \frac{d}{dt}\frac{\mathcal{E}(\psi_0+t\varphi)}{\|\psi_0+t\varphi\|_2^2} &= 2\,\Re\int\overline{\nabla u_0}\nabla\varphi + 2\,\Re\int V\overline{\psi_0}\varphi - 2E\,\Re\int\overline{u_0}\varphi = \\
&= 2\,\Re\Big(-\int\overline{u_0}\Delta\varphi + \int V\overline{\psi_0}\varphi - 2E\int\overline{u_0}\varphi\Big)
\end{aligned}$$

By changing from φ to $i\varphi$ we find that the same must hold for the imaginary part and therefore

$$0 = \int\overline{u_0}(-\Delta\varphi + V\varphi - E\varphi)$$

for all $\varphi\in\mathscr{D}$, i.e.

$$-\Delta u_0 + Vu_0 = Eu_0$$

in $\mathscr{D}'(\mathbb{R}^d)$. Here the condition $V\in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ ensures that $Vu_0\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ because $u_0\in H^1\subset L^q(\mathbb{R}^d)$ by the Sobolev embedding. *q.e.d.*

Two different types of behaviour of external potentials

1) Trapping potential: $V(x)\to\infty$ as $|x|\to\infty$, i.e. $\inf_{|x|\geqslant R}V(x)\to\infty$ as $R\to\infty$

2) Decaying potential: $V(x)\to 0$ as $|x|\to\infty$, i.e. $\sup_{|x|\geqslant R}|V(x)|\to 0$ as $R\to\infty$

3) There are also other potentials such as periodic ones.

Theorem 6.2 (Existence of Minimisers for Trapping Potentials)**.** *Assume that* $0\leqslant V$, $V(x)\to\infty$ *as* $|x|\to\infty$. *Then*

$$E = \inf\{\int|\nabla\psi|^2 + \int V|\psi|^2 \,\big|\, \psi\in H^1(\mathbb{R}^d), \|\psi\|_2 = 1$$

has at least one minimiser. □

Proof. Assume that $V \geqslant 0$, then $E = \inf(\cdots) \geqslant 0$, thus E is finite. By definition of E, we can find a sequence $(u_n)_n \subset H^1(\mathbb{R}^d)$ such that

$$\mathcal{E}(u_n) = \int |\nabla u_n|^2 + \int V|u_n| \xrightarrow{n\to\infty} E$$

Since $\mathcal{E}(u_n) \to E$ it follows that $\mathcal{E}(u_n)$ is bounded (as $n \to \infty$) and thus $\int |\nabla u_n|^2$ and $\int V|u_n|^2$ are bounded. Thus $(u_n)_n$ is bounded in H^1, hence we may choose a subsequence such that $u_{n_k} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^d)$ and $u_n(x) \to u_0(x)$ a.e. (by Theorem 5.4). Since $\nabla u_n \rightharpoonup \nabla u_0$ weakly in L^2

$$\liminf_{n\to\infty} \int |\nabla u_n|^2 \geqslant \int |\nabla u_0|^2$$

Since $V|u_n|^2 \to V|u_0|^2$ converges pointwise

$$\liminf_{n\to\infty} \int V|u_n|^2 \geqslant \int V|u_0|^2$$

By Fatou's lemma. Thus

$$E = \liminf_{n\to\infty} \mathcal{E}(u_n) \geqslant \mathcal{E}(u_0)$$

Thus u_0 is a minimiser iff $\|u_0\|_2 = 1$, which is an Exercise. *q.e.d.*

Now we shall turn to vanishing potentials, i.e. $V \uparrow 0$ as $|x| \uparrow 0$ and a singularity.

Example. The Hydrogen atom potential $-\Delta - \frac{1}{|x|}$ on $L^2(\mathbb{R}^3)$.
Why is this potential bounded, i.e.

$$E = \inf\left\{ \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} \mathrm{d}x \,\middle|\, u \in H^1, \|u\|_2 = 1 \right\}$$

This is due to the Sobolev inequality $\|\nabla u\|_2 \geqslant C\|u\|_6$. For $r > 0$ we have

$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} \mathrm{d}x = \int_{|x|\leqslant r} \frac{|u(x)|^2}{|x|} \mathrm{d}x + \int_{|x|>r} \frac{|u(x)|^2}{|x|} \mathrm{d}x \leqslant$$

$$\leqslant \left(\int_{|x|\leqslant r} |u(x)|^6 \right)^{1/3} \left(\int_{|x|\leqslant r} \frac{1}{|x|^{3/2}} \right)^{2/3} + \int_{|x|>r} \frac{|u(x)|^2}{r} \mathrm{d}x \leqslant C_s \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right) r + \frac{1}{r}$$

i.e.

$$\mathcal{E}(u) = \int |\nabla u|^2 - \int \frac{|u(x)|^2}{|x|}\mathrm{d}x \geqslant \int |\nabla u|^2(1 - C_s r) - \frac{1}{r}$$

for all $r > 0$. Choosing $r > 0$ small enough one finds that

$$\mathcal{E}(u) \geqslant \frac{1}{2}\int |\nabla u|^2 - C > -\infty$$

i.e. $E > -\infty$.

Lemma 6.3. *If $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ where*

$$\begin{cases} p \geqslant \frac{d}{2}, & \text{if } d \geqslant 3 \\ p > 1, & \text{if } d = 2 \\ p \geqslant 1, & \text{if } d = 1 \end{cases}$$

then $E > -\infty$ and

$$\mathcal{E}(u) \geqslant \frac{1}{2}\int |\nabla u|^2 - C$$

for all $u \in H^1(\mathbb{R}^d)$, $\|u\|_2 = 1$. □

Remark 6.4.

- $L^p + L^\infty = \{f + g \mid f \in L^p, g \in L^\infty\}$, for example

$$\frac{1}{|x|} = \underbrace{\frac{1}{|x|}\mathbf{1}_{\{|x|\leqslant 1\}}}_{\in L^{3-\varepsilon}} + \underbrace{\frac{1}{|x|}\mathbf{1}_{\{|x|>1\}}}_{\in L^\infty}$$

- IF $p < q < \infty$ then $L^q(\mathbb{R}^n) \subset L^p + L^\infty$.

□

Proof.

($d \geqslant 3$) Let $V \in L^{d/2} + L^\infty$. Write $V = V_1 + V_2$, where $V_1 = V\mathbf{1}_{|V(x)|>\frac{1}{\varepsilon}}$, $V_2 = V\mathbf{1}_{|V(x)|\leqslant\frac{1}{\varepsilon}}$. Then for $\varepsilon > 0$ small, we have

$$V_2 \in L^\infty, \qquad \|V_2\|_\infty \leqslant \frac{1}{\varepsilon}$$

$$V_1 \in L^{d/2}, \qquad \|V_1\|_{d/2} = \left(\int |V(x)|^{d/2}\mathbf{1}_{|V(x)|\geqslant\frac{1}{\varepsilon}}\mathrm{d}x\right)^{2/d} \xrightarrow{\varepsilon\searrow 0} 0$$

by dominated convergence. We have

$$\left|\int V|u|^2\right| \leqslant \int |V_1||u|^2 + \int |V_2||u|^2 \leqslant \|V_1\|_{d/2}\|u\|_{d/d-2} + \|V_2\|_\infty \|u\|_2 \leqslant$$
$$\leqslant C_S \|V_1\|_{d/2} \int\limits_{\mathbb{R}^d} |\nabla u|^2 + \frac{1}{\varepsilon}$$

for all $u \in H^1$, $\|u\|_2 = 1$. Then

$$\mathcal{E}(u) = \int |\nabla u|^2 \big(1 - C_S \|V_1\|_{d/2}\big) - \frac{1}{\varepsilon} \geqslant \frac{1}{2} \int |\nabla u|^2 - C$$

if we choose $\varepsilon > 0$ small enough.

q.e.d.

Definition 6.5 (Vanishing in the Weak Sense)**.** We say that $V : \mathbb{R}^d \to \mathbb{R}$ vanishes at ∞ in the weak sense if for all $\varepsilon > 0$

$$\lambda(\{|V(x)| > \varepsilon\}) < \infty$$

□

Example. $V(x) \to 0$ as $|x| \to \infty$ in the strong sense, i.e.

$$\sup_{|x| \geqslant R} |V(x)| \xrightarrow{R \to \infty} 0$$

Remark 6.6. If $V \in L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, the V vanishes at ∞ in the weak sense. □

Theorem 6.7. *Assume that $V \in L^p(\mathbb{R}^d) + L_0^\infty(\mathbb{R}^d)$, where*

$$\begin{cases} p \geqslant \frac{d}{2}, & \textit{if } d \geqslant 3 \\ p > 1, & \textit{if } d = 2 \\ p \geqslant 1, & \textit{if } d = 1 \end{cases}$$

and L_0^∞ is the set of L^∞ which vanish weakly at infinity. Assume that $E < 0$ then E

has a minimiser $u_0 \in H^1(\mathbb{R}^d)$ and

$$-\Delta u_0 + V u_0 = E u_0 \quad \text{in } \mathscr{D}'(\mathbb{R}^d)$$

Moreover, we can choose $u_0 \geqslant 0$. □

Remark 6.8. Under certain conditions on V, then actually $u_0 > 0$ and it is unique. But we will prove this much later. □

Lemma 6.9. *Assume that $V \in L^p + L_0^\infty$. Assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^d)$. Then*

$$\int V|u_n|^2 \xrightarrow{n\to\infty} \int V|u_0|^2.$$

□

Proof.

Case 1 $V \in L^p$, $p = \frac{d}{2}$, $d \geqslant 3$. Then

$$V = V_1 + V_2 + V_3 = V\mathbf{1}_{\{\varepsilon<|V(x)|<\frac{1}{\varepsilon}\}} + V\mathbf{1}_{|V(x)|\leqslant\varepsilon} + V\mathbf{1}_{|V(x)|\geqslant\frac{1}{\varepsilon}}$$

Then $V_1 \in L^\infty$, $\lambda(\{V_1(x) \neq 0\}) < \infty$ and by the Sobolev embedding

$$\int_{\{V_1\neq 0\}} V_1|u_n|^2 \xrightarrow{n\to\infty} \int V_1|u_0|^2$$

strongly in L^2.

$V_2 \in L^\infty$ and $\|V_2\|_\infty \leqslant \varepsilon$, then for all $n \in \mathbb{N}$

$$\left|\int V_2|u_n|^2\right| \leqslant \varepsilon \implies \left|\int V_2|u|^2\right| \leqslant \varepsilon$$

$V_3 \in L^{d/2}$, $\|V_3\|_{d/2} \to 0$ as $\varepsilon \to 0$ and therefore

$$\left|\int V_3|u_n|^2\right| \leqslant \|V_3\|_{d/2}\big\||u_n|^2\big\|_{\frac{d}{d-2}} \leqslant C\|V_3\|_{d/2}$$

Then

$$\left|\int V|u_n|^2 - \int V|u_0|^2\right| \leqslant \left|\int V_1|u_n|^2 - \int V_1|u_0|^2\right| + \varepsilon + C\|V_3\|_{d/2}$$

and therefore

$$\limsup_{n\to\infty}\left|\int V|u_n|^2 - \int V|u_0|^2\right| \leqslant \varepsilon + C\|V_3\|_{d/2} \longrightarrow 0$$

Case 2 $V \in L_0^\infty$, then

$$V = V_1 + V_2 = V\mathbf{1}_{\{\varepsilon<|V(x)|<\frac{1}{\varepsilon}+V(x)\mathbf{1}_{|V(x)|\leqslant\varepsilon}\}}$$

The rest of the proof works analogously to the above.

q.e.d.

Proof of Theorem 6.7.

$(p = \frac{d}{2}, d \geqslant 3)$ By the lemma

$$\mathcal{E}(u) \geqslant \frac{1}{2}\int |\nabla u|^2 - C$$

for all $u \in H^1(\mathbb{R}^d)$, $\|u\|_2 = 1$. In particular E is finite and we can find a minimising sequence $(u_n)_n \subset H^1$, $\|u_n\|_2 = 1$, such that $\mathcal{E}(u_n) \to E$. Since

$$E \longleftarrow \mathcal{E}(u_n) \geqslant \frac{1}{2}\int |\nabla u_n|^2 - C$$

hence $(u_n)_n$ is bounded in $H^1(\mathbb{R}^d)$. Thus by the Sobolev compact embedding theorem, there exists a subsequence $(u_{n_k})_k$, $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^d)$ and $\mathbf{1}_A u_n \to \mathbf{1}_A u_0$ strongly in $L^2(\mathbb{R}^d)$ for any bounded set A. Because $\nabla u_n \rightharpoonup \nabla u$ weakly in L^2, and by Fatou's lemma

$$\liminf_{n\to\infty}\int |\nabla u_n|^2 \geqslant \int |\nabla u_0|^2$$

and by the previous lemma, $\int V|u_n|^2 \to \int V|u_0|^2$. Thus

$$E = \liminf \mathcal{E}(u_n) \geqslant \mathcal{E}(u_0).$$

It is not obvious that u_0 is a minimiser as we do not whether $\|u_0\|_2 = 1$, because

$$u_n \xrightharpoonup{n\to\infty} u_0 \implies \|u_0\|_2 \leqslant \liminf \|u_n\|_2 = 1$$

Now using the assumption $E<0$ we find that

$$0>E\geqslant \mathcal{E}(u_0)=\int|\nabla u_0|^2+\int V|u_0|^2=\|u_0\|_2^2\Big(\int|\nabla v|^2+\int V|v|^2\Big)\geqslant \underbrace{\|u_0\|_2^2}_{\leqslant 1}E \implies \|u_0\|_2=1,$$

where $v=\frac{u_0}{\|u_0\|_2}$, thus u_0 is a minimiser.

q.e.d.

Remark 6.10. If $E\geqslant 0$ then E might have no minimiser. For example if $V(x)=\frac{1}{|x|}$ in $\mathbb{R}^3$, then

$$E=\inf_{\substack{u\in H^1\\ \|u\|_2=1}}\Big(\int|\nabla u|^2\mathrm{d}x+\int\frac{|u(x)|^2}{|x|}\mathrm{d}x\Big)=0$$

but it has no minimiser. □

Theorem 6.11 (Hydrogen Atom). *Let*

$$E=\inf_{\substack{u\in H^1\\ \|u\|_2=1}}\Big(\int|\nabla u|^2\mathrm{d}x-\int\frac{|u(x)|^2}{|x|}\mathrm{d}x\Big)$$

then $E=-\frac{1}{4}$ *and* $u_0(x)=ce^{-\frac{|x|}{2}}$, $c\in\mathbb{R}$, *is a minimiser.* □

Theorem 6.12 (Perron-Frobenius Principle). *Take* $\Omega\subset\mathbb{R}^d$ *open,* $f\in\mathscr{C}^2(\Omega)$. *Assume that* $V\in L^1_{loc}(\mathbb{R}^d)$, $f>0$ *for all* $x\in\Omega$ *and*

$$-\Delta f+Vf=0$$

pointwise in Ω. *Then for all* $u\in\mathscr{C}^1_c(\Omega)$, *we have*

$$\int|\nabla u|^2\mathrm{d}x+\int V|u|^2\geqslant 0.$$

□

Proof. Since $u\in\mathscr{C}^1_c(\Omega)$ and $f>0$ we can write $u=f\varphi$ with $\varphi\in\mathscr{C}^1_c(\Omega)$ and

$$\int|\nabla u|^2=\int|\nabla(f\varphi)|^2=\int|\nabla f\varphi+f\nabla\varphi|^2=\int|\nabla f|^2|\nabla\varphi|^2+\int|f|^2|\nabla\varphi|^2+2\,\Re\int(\nabla f)f\bar{\varphi}\nabla\varphi.$$

Thus

$$\int |\partial_{x_i} f|^2 |\varphi|^2 = -\int f \partial_{x_i}\big((\partial_{x_i} f)|\varphi|^2\big) = -\int f\big(\partial_{x_i}^2 f\big)|\varphi|^2 - \int f\partial_{x_i} f \partial_{x_i}|\varphi|^2$$

hence

$$\int |\nabla f|^2|\varphi|^2 = -\int f\Delta f|\varphi|^2 - \int f\nabla f 2\,\Re(\bar{\varphi}\nabla\varphi).$$

Thus

$$\int |\nabla u|^2 = \int |f|^2|\nabla\varphi|^2 + \int f(-\Delta f)|\varphi|^2$$

and therefore

$$\int |\nabla u|^2 + \int V|u|^2 = \int |f|^2|\nabla\varphi|^2 + \int f\underbrace{(-\Delta f + Vf)}_{=0}|\varphi|^2 = \int |f|^2|\nabla\varphi|^2 \geqslant 0.$$

q.e.d.

Proof. Let $\Omega = \mathbb{R}^3 \setminus \{0\}$ and $f(x) = ce^{-\frac{|x|}{2}}$. Then $f \in \mathscr{C}^2(\Omega)$, $f > 0$ in Ω and

$$-\Delta f - \frac{f}{|x|} + \frac{1}{4}f = 0$$

on Ω. By the Perron-Frobenius principle

$$\int |\nabla u|^2 - \int \frac{|u(x)|^2}{|x|} + \frac{1}{4}\int |u(x)|^2 \geqslant 0$$

for all $u \in \mathscr{C}_c^1(\mathbb{R}^3 \setminus \{0\})$. As $\mathscr{C}_c^1(\mathbb{R}^3 \setminus \{0\})$ is dense in $H^1(\mathbb{R}^3)$ (the proof of which is left as an exercise)[1].

The for all $u \in H^1(\mathbb{R}^3)$, we can find a $u_n \in \mathscr{C}_c^1(\mathbb{R}^3 \setminus \{0\})$ such that $u_n \to u$ in $H^1(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$. Thus

$$\int |\nabla u_n|^2 \mathrm{d}x \xrightarrow{n\to\infty} \int |\nabla u|^2 \mathrm{d}x$$

$$\int |u_n|^2 \mathrm{d}x \xrightarrow{n\to\infty} \int |u|^2 \mathrm{d}x$$

$$\liminf_{n\to\infty} \int \frac{|u_n|^2}{|x|}\mathrm{d}x$$

[1] Since $\mathscr{C}_c^\infty$ is dense in $H^1\big(\mathbb{R}^3\big)$ one only needs to consider a $g \in \mathscr{C}_c^\infty\big(\mathbb{R}^3\big)$ and take $h \in \mathscr{C}_c^\infty$, with $0 \leqslant h \leqslant 1$, $h(x) = 1$ if $|x| \leqslant 1$, and define $g_n(1 - h(nx))g(x) \in \mathscr{C}_c^\infty\big(\mathbb{R}^3 \setminus \{0\}\big)$.

where the last inequality follows from Fatou's lemma and therefore we have

$$0 \leqslant \limsup_{n\to\infty}\Big(\int |\nabla u_n|^2 - \int \frac{|u_n|^2}{|x|} + \frac{1}{4}\int |u_n|^2\Big) \leqslant \Big(\int |\nabla u|^2 - \int \frac{|u|^2}{|x|} + \frac{1}{4}\int |u|^2\Big).$$

q.e.d.

Lemma 6.13. *For all $u \in H^1(\mathbb{R}^3)$ with $\|u\|_2 = 1$ holds*

$$\int |\nabla u|^2 \geqslant \Big(\int \frac{|u(x)|^2}{|x|}\Big)^2$$

□

Proof. Take $u \in H^1(\mathbb{R}^3)$, $\|u\|_2 = 1$. Let $u_l(x) = l^{3/2}u(lx)$ for which $\|u_l\|_2 = \|u\|_2 = 1$. We have

$$\int_{\mathbb{R}^3} |\nabla u_l|^2 = l^2 \int |\nabla u|^2, \qquad \int \frac{|u_l|^2}{|x|}\mathrm{d}x = l\int \frac{|u|^2}{|x|}\mathrm{d}x,$$

then we have by the above that for all $l > 0$

$$l^2 \int |\nabla u|^2 - l\int \frac{|u|^2}{|x|}\mathrm{d}x \geqslant -\frac{1}{4}$$

Noting that $l^2A - lB + C \geqslant 0$ for some $A, B, C \geqslant$ and $l \geqslant 0$ iff $4AC \geqslant B^2$, we find that the inequality implies

$$\int |\nabla u|^2 \geqslant \Big(\int \frac{|u|^2}{|x|}\Big)^2$$

for all $u \in H^1(\mathbb{R}^3)$ and $\|u\|_2 = 1$. *q.e.d.*

Remark 6.14. For all $u \in H^1(\mathbb{R}^3)$ and $\|u\|_2 = 1$ we have

$$\Big(\int |\nabla u|^2\Big)\Big(\int |x|^2|u(x)|^2\mathrm{d}x\Big) \geqslant \Big(\int \frac{|u(x)|^2}{|x|}\Big)^2\Big(\int |x|^2|u(x)|^2\mathrm{d}x\Big) \geqslant$$
$$\geqslant \Big(\int |u(x)|^2\mathrm{d}x\Big)^3 = 1$$

Comparing this to the Heisenberg uncertainty principle

$$\left(\int |\nabla u|^2\right)\left(\int |x|^2 |u(x)|^2 \mathrm{d}x\right) \geqslant \frac{g}{4}$$

we see that the Sobolev inequality is "stronger" that the Heisenberg-principle □

Theorem 6.15 (Hardy Inequality)**.**

$$\int |\nabla u|^2 \geqslant \frac{1}{4} \int \frac{|u(x)|^2}{|x|^2} \mathrm{d}x$$

for all $u \in H^1(\mathbb{R}^3)$*.* □

Proof. Homework. *q.e.d.*

Remark 6.16. Hardy's inequality implies

$$\int |\nabla u|^2 \geqslant \frac{1}{4} \int \frac{|u|^2}{|x|^2} \geqslant \frac{1}{4} \left(\int \frac{|u|^2}{|x|}\right)^2$$

if $\|u\|_2 = 1$. □

Chapter 7

Harmonic Functions

Definition 7.1. Let $f \in L^1_{\text{loc}}(\Omega)$, for $\Omega \subset \mathbb{R}^d$ open. Then f is harmonic iff

$$\Delta f = 0 \qquad \text{in } \mathscr{D}'(\Omega).$$

□

Theorem 7.2 (Equivalent Definition)**.** $f \in L^1_{loc}(\Omega)$. *The f is harmonic iff*

$$f(x) = \frac{1}{\lambda(B_r)} \int_{B_r(x)} f(y)\mathrm{d}y := \fint_{B_r(x)} f(y)\mathrm{d}y \qquad a.e.$$

for all $r > 0$ such that $B_r(x) \subset \Omega$. □

Proof.

Step 1. Let $f \in \mathscr{C}_c^\infty$ and assume that $\Delta f = 0$. Then

$$0 = \int_{B_r(x)} \Delta f(y)\mathrm{d}y = \int_{S_r(x)} \nabla f \cdot \nu \mathrm{d}S(y) = r^{d-1} \int_{\mathbb{S}^{d-1}} \nabla f(x + rw) \cdot w \mathrm{d}S(w)$$

where $\mathbb{S}^{d-1} = S_1(0)$. Thus we have

$$0 = \int_{\mathbb{S}^{d-1}} \nabla f(x + r\omega) \cdot \omega \mathrm{d}S(\omega) = \int_{\mathbb{S}^{d-1}} \frac{d}{dr} f(x + r\omega)\mathrm{d}\omega = \frac{d}{dr} \int_{\mathbb{S}^{d-1}} f(x + r\omega)\mathrm{d}S(\omega)$$

i.e. $r \mapsto \int_{\mathbb{S}^{d-1}} f(x+r\omega)\mathrm{d}S(\omega)$ is constant, i.e. for all $r>0$

$$f(x)\lambda\big(\mathbb{S}^{d-1}\big) = \int_{\mathbb{S}^{d-1}} f(x+r\omega)\cdot w\mathrm{d}S(\omega)$$

and therefore

$$|B_r(0)|f(x) = \int_0^R r^{d-1}\lambda\big(\mathbb{S}^{d-1}\big)f(x)\mathrm{d}r = \int_0^r r^{d-1}\int_{\mathbb{S}^{d-1}} f(x+r\omega)\cdot\omega\mathrm{d}S(\omega)\mathrm{d}r = \int_{B_r(x)} f(y)\mathrm{d}y$$

from which $f(x) = ⨍_{B_r(x)} f(y)\mathrm{d}y$ follows.

For the converse assume that $f(x) = ⨍_{B_r(x)} f(y)\mathrm{d}y$ holds for all $x\in\Omega$ and $r>0$. From the assumption we have

$$\lambda\big(\mathbb{S}^{d-1}\big)f(x) = \int_{\mathbb{S}^{d-1}} f(x+r\omega)\cdot\omega\mathrm{d}S(\omega)$$

Taking the derivative with respect to r we get

$$0 = \frac{d}{dr}\int_{\mathbb{S}^{d-1}} f(x+r\omega)\mathrm{d}S(\omega) = \int_{\mathbb{S}^{d-1}} \nabla f(x+r\omega)\cdot\omega\mathrm{d}S(\omega) = \int_{B_r(x)} \Delta f(y)\mathrm{d}y$$

Since this holds for all $r>0$ one finds that $\Delta f = 0$.

Step 2. Consider $f\in L^1_{\mathrm{loc}}(\Omega)$. Choosing $h\in\mathscr{C}_c^\infty\big(\mathbb{R}^d\big)$, with $0\leqslant h\leqslant 1$ and $\int h = 1$, $h(x)=0$ if $|x|>1$ and h is a radial function, i.e. $h(x) = f(|x|)$. Letting

$$h_n(x) = n^d h(nx)$$

for $n\in\mathbb{N}$. We know that $h_n * f \to f$ in $L^1_{\mathrm{loc}}(\Omega)$, $h_n * f\in\mathscr{C}^\infty$ and $D^\alpha(h_n*f) = (D^\alpha h_n)*f$.

Let $\Delta f = 0$ in $\mathscr{D}'(\Omega)$. Then

$$\Delta(h_n * f) = 0, \qquad \text{in } \mathscr{D}'(\Omega),$$

since for all $\varphi\in\mathscr{C}_c^\infty(\Omega)$, in particular also $h_n(\cdot - x)$,

$$\int \Delta\varphi(y)f(y)\mathrm{d}y = 0,$$

hence we have classically $\Delta(h_n * f) = (\Delta h_n * f) = 0$ and therefore also weakly. By step 1

$$(h_n * f)(x) = \fint_{B_r(x)} (h_n * f)(y)\mathrm{d}y = \frac{\mathbf{1}_{B_r(0)}}{\lambda(B_r(0))} * (h_n * f)(x) = h_n * \left(\frac{\mathbf{1}_{B_r(0)}}{\lambda(B_r(0))} * f\right)(x)$$

Taking the limit $n \to \infty$, the assertion follows.

For the converse assume that $f(x) = \left(\frac{\mathbf{1}_{B_r(0)}}{\lambda(B_r(0))} * f\right)(x)$ then

$$h_n * f(x) = h_n * \left(\frac{\mathbf{1}_{B_r(0)}}{\lambda(B_r(0))} * f\right)(x) = \frac{\mathbf{1}_{B_r(0)}}{\lambda(B_r(0))} * (h_n * f)(x)$$

and therefore by step 1., $\Delta(h_n * f) = 0$. Since $h_n * f \to f$ in L^1_{loc} it does also converge in $\mathscr{D}'(\Omega)$ and therefore $0 = \Delta(h_n * f) \to \Delta f$ in $\mathscr{D}'(\Omega)$.

q.e.d.

Corollary 7.3. *If f is harmonic, then $f \in \mathscr{C}^\infty(\Omega)$ and $f(x) = \fint_{\lambda(S_r(0))} f(y)\mathrm{d}y$.* □

Proof. The identity follows as in the case for smooth functions. For the smoothness we shall prove that $h_n * f = f$ everywhere.

$$(h_n * f)(x) = \int f(y)f(x-y)\mathrm{d}y = \int_0^\infty \int_{S^{d-1}} r^{d-1} h_n(r\omega) f(x - r\omega)\mathrm{d}S(\omega)\mathrm{d}r =$$

$$= \int_0^\infty h(r\omega) r^{d-1} \underbrace{\int_{\mathbb{S}^{d-1}} f(x - r\omega)\mathrm{d}S(\omega)\,\mathrm{d}r}_{= f(x)\lambda\left(\mathbb{S}^{d-1}\right)} = \left(\int_{\mathbb{R}^d} h_n(y)\mathrm{d}y\right) f(x) = f(x).$$

Thus since $h_n * f$ is smooth so must f. *q.e.d.*

Theorem 7.4 (Harnack's Inequality). *If f is harmonic on $B_r(0)$ and $f \geqslant 0$ then for all $x \in B_{\frac{R}{3}}(0)$, then*

$$\left(\frac{3}{2}\right)^d f(0) \geqslant f(x) \geqslant \frac{f(0)}{2^d}$$

□

Proof.

$$f(0) = \fint_{B_R(0)} f(y)\mathrm{d}y$$
$$f(x) = \fint_{B_{\frac{2}{3}R}(x)} f(y)\mathrm{d}y$$

for $x \in B_{\frac{R}{3}}(0)$. Thus we have

$$f(0) = \frac{\lambda(B_{\frac{2}{3}R}(x))}{\lambda(B_R(0))} \frac{1}{\lambda(B_{\frac{2}{3}R}(x))} \int\limits_{B_R(0)} f(y)\mathrm{d}y \geqslant \left(\frac{2}{3}\right)^d \fint_{B_{\frac{2}{3}R}(x)} f(y)\mathrm{d}y = \left(\frac{2}{3}\right)^d f(x)$$

The other inequality follows similarly using $B_{\frac{R}{3}}(0) \subset B_{\frac{2R}{3}}(x)$. *q.e.d.*

Corollary 7.5. *If f is harmonic on $\mathbb{R}^d$ and f is bounded from above $f \leqslant c$ for some $c \in \mathbb{R}$ (or bounded from below), then f is constant* □

Proof. Assuming that $f(x) \geqslant -C$ for all $x \in \mathbb{R}^d$. We want to prove that f is constant. Let $E = \inf_{x\in\mathbb{R}^d} f(x)$ and define $g = f - E$, then $g \geqslant 0$ and g is harmonic, $\inf_{x\in\mathbb{R}^d} g(x) = 0$. We want to prove that $g \equiv 0$. If not, then there must exist a x_0. If not then there exists a $x_0 \in \mathbb{R}^d$ such that $g(x_0) > 0$. By Harnack's inequality we find that

$$g(x) \geqslant \frac{g(x_0)}{2^d} > 0$$

for all $x \in \mathtt{R}^d$. Thus

$$\inf_{x\in\mathbb{R}^d} g(x) \geqslant \frac{g(x_0)}{2^d} > 0$$

which is a contradiction. *q.e.d.*

Theorem 7.6 (Newton's Theorem)**.** *Let μ be a positive Borel measure on $\mathbb{R}^n$ and let μ be radial, i.e. $\mu(RA) = \mu(A)$ for all $R \in SO(3)$. Then for all $x \in \mathbb{R}^3$*

$$\int\limits_{\mathbb{R}^3} \frac{\mathrm{d}\mu(y)}{|x-y|} = \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}\mu(y)}{\max\{|x|,|y|\}} = \frac{\int \mathrm{d}\mu}{|x|}$$

□

Proof. Using $-\Delta\frac{1}{4\pi|x|} = \delta$ in $\mathscr{D}'(\mathbb{R}^3)$, in particular $\Delta\frac{1}{|x|} = 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$, hence $\frac{1}{|x|}$ is harmonic on $\Omega = \mathbb{R}^3 \setminus \{0\}$. Thus

$$f(x) = \fint_{\mathbb{S}_r(x)} f(y)\mathrm{d}S(y)$$

Step 1 We consider the case μ is a uniform measure on a sphere. We want to prove that

$$\int\limits_{|y|=R} \frac{\mathrm{d}y}{|x-y|} = \int\limits_{|y|=R} \frac{\mathrm{d}y}{\max\{|x|, R\}}.$$

If $|x| > R$ the function $y \mapsto \frac{1}{|x-y|} =: f(y)$ is a harmonic function on $B(0, |x|)$, because $\Delta\left(\frac{1}{|x|}\right) = 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$. By the mean value theorem then

$$f(0) = \fint_{|y|=R} f(y)\mathrm{d}y \implies \frac{1}{|x|} = \fint_{|y|=R} \frac{\mathrm{d}y}{|x-y|}$$

and therefore

$$\int\limits_{|y|=R} \frac{\mathrm{d}y}{|x-y|} = |S_R(0)|\frac{1}{|x|} = \int\limits_{|y|=R} \mathrm{d}y\frac{1}{|x|} = \int\limits_{|y|=R} \frac{\mathrm{d}y}{\max\{|x|, R\}}.$$

If $|x| < R$

$$\int\limits_{|y|=R} \frac{\mathrm{d}y}{|x-y|} = R^2\int\limits_{\mathbb{S}^2} \frac{\mathrm{d}\omega}{|x-R\omega|} = R^2\int\limits_{\mathbb{S}^2} \frac{\mathrm{d}\omega}{\big||x|\omega - Ry_0\big|} \overset{\text{Case } |x|>R}{=\!=\!=}$$
$$= R^2\int\limits_{\mathbb{S}^2} \frac{\mathrm{d}\omega}{R} = \frac{|S_R(0)|}{R} = \int\limits_{|y|=R} \frac{\mathrm{d}y}{\max\{|x|, R\}}$$

If $|x| = R$ then by the Dominated Convergence Theorem

$$\int\limits_{|y|=R} \frac{\mathrm{d}y}{|x-y|} = \lim_{R_n\uparrow R} \int\limits_{|y|=R_n} \frac{\mathrm{d}y}{|x-y|} = \lim_{R_n\uparrow R} \frac{|S_{R_n}(0)|}{|x|} = \frac{|S_R(0)|}{|x|} = \int\limits_{|y|=R} \frac{\mathrm{d}y}{\max\{|x|, R\}}.$$

Thus we proved for all $R > 0$ and $x \in \mathbb{R}^3$

$$\int\limits_{|x-y|} \frac{\mathrm{d}y}{|x-y|} = \int\limits_{|y|=R} \frac{\mathrm{d}y}{\max\{|x|, |y|\}}$$

Step 2 For general μ, with μ radial

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}\mu(y)}{|x-y|} = \int_0^\infty r^2 \int_{\mathbb{S}^2} \frac{\mathrm{d}\mu(r\omega)}{|x-r\omega|} = \int_0^\infty r^2 \int_{\mathbb{S}^2} \frac{\mathrm{d}\mu(r\omega)}{\max\{|x|,r\}} = \int_{\mathbb{R}^3} \frac{\mathrm{d}\mu(y)}{\max\{|x|,|y|\}}$$

q.e.d.

Definition 7.7. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. We say that f is super-harmonic if $-\Delta f \geqslant 0$ in $\mathscr{D}'(\Omega)$. f is called sub-harmonic if $-\Delta f \leqslant 0$ in $\mathscr{D}'(\Omega)$. □

Remark 7.8. In one dimension super-harmonic is equivalent to $-f'' \geqslant 0$ i.e. f is a concave function.
If $T \in \mathscr{D}'(\Omega)$, then we say that $T \geqslant 0$ if $T(\varphi) \geqslant 0$ for all $\varphi \in \mathscr{D}(\Omega)$, for $\varphi \geqslant 0$. Actually by the Riesz-Markov representation theorem, $T \in \mathscr{D}'(\Omega)$, $T \geqslant 0$ iff there exists a positive Borel measure μ such that

$$\begin{cases} T(\varphi) = \int_\Omega \varphi(y)\mathrm{d}\mu(y), & \forall \varphi \in \mathscr{D}(\Omega) \\ \mu(K) < \infty, & \forall K \subset \Omega \text{ compact} \end{cases}$$

However, we shall not use this result in this course. One way to prove this is to define

$$\mu(K) = \inf\big\{T(\varphi)\,\big|\,\varphi \in \mathscr{D},\, \varphi \geqslant 0\, \varphi = 1 \text{ on } K\big\}$$
$$\mu(O) = \sup\big\{T(\varphi)\,\big|\,\varphi \in \mathscr{D},\, 0 \leqslant \varphi \leqslant 1, \operatorname{supp}\varphi \subset O\big\}$$

□

Theorem 7.9 (Mean-Value-Theorem)**.** *Let $f \in L^1_{loc}(\Omega)$. Then f is super-harmonic iff for a.e. $x \in \Omega$ and $R > 0$ such that $\overline{B_R(x)} \subset \Omega$*

$$f(x) \geqslant \fint_{B_R(x)} f(y)\mathrm{d}y$$

□

Proof. "Similar" to Theorem 7.2 for harmonic functions. First let $f \in \mathscr{C}^\infty$, if $-\Delta f \geqslant 0$, then

$$0 \geqslant \int\limits_{B_r(x)} \Delta f(y)\mathrm{d}y = r^{d-1}\frac{\mathrm{d}}{\mathrm{d}r}\int\limits_{S^{d-1}} f(x+r\omega)\mathrm{d}\omega$$

which means that $r \mapsto \int\limits_{\mathbb{S}^{d-1}} f(x+r\omega)\mathrm{d}\omega$ is non-increasing and therefore

$$f(x) \geqslant \fint_{B_R(x)} f(y)\mathrm{d}y.$$

Then for $f \in L^1_{\mathrm{loc}}$, replace f by $h_n * f \in \mathscr{C}^\infty$. *q.e.d.*

Theorem 7.10 (Strong Minimum Principle)**.** *Let $f \in L^1_{loc}(\Omega)$, $-\Delta f \geqslant 0$ in $\mathscr{D}'(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and path-connected. Let $E = \operatorname{ess\,inf}_\Omega f$. Then either*

1) *$f(x) > E$, for a.e. $x \in \Omega$*

2) *$f = const$ on Ω.*

□

Remark 7.11. The weak minimum principle tell us that $\operatorname{ess\,inf}_\Omega f = \operatorname{ess\,inf}_{\partial\Omega} f$. □

Proof. Assume that $f(x) \geqslant \fint_{B_R(x)} f(y)\mathrm{d}y$ holds for all $R > 0$, $B_R(x) \subset \Omega$ holds for all $x \in \Omega'$, i.e. $|\Omega \setminus \Omega'| = 0$. If $x \in \Omega'$ and $f(x) = E$, then

$$E = f(x) \geqslant \fint_{B_R(x)} \underbrace{f(y)}_{\geqslant E}\mathrm{d}y \geqslant E$$

i.e. equality has to occur and therefore $f(y) = E$ for a.e. $y \in B_R(x) \subset \Omega$. Now for every $z \in \Omega$ there exists a continuous curve connecting x and z. We can find $r > 0$ and finitely many points $x_1, \dots, x_N$ such that $x_1 = x$ and $x_N = z$ such that $B_r(x_m) \subset \Omega$ covering the curve. Then $f(X) = E$ implies that a.e. $y \in B_r(x)$ and by induction it follows that $f(x_m) = E$ and thus also $f(z) = E$. *q.e.d.*

Theorem 7.12 (Mean-Value Theorem for $(-\Delta + \mu^2)$)**.** *Let $f \in L^1_{loc}(\Omega)$, $-\Delta f + \mu^2 f \geqslant 0$ in $\mathscr{D}'(\Omega)$, $\mu \in \mathbb{R}$. Assume that Ω is open and path-connected.*

1) Then for a.e. $x \in \Omega$ we have

$$f(x) \geqslant C_R \int\limits_{B_R(x)} f(y)\mathrm{d}y$$

for all $R > 0$ such that $B_R(x) \subset \Omega$, where $C_R > 0$ depends only on $R > 0$.

2) If $f \geqslant 0$ and $f \not\equiv 0$, then $f(x) > 0$ for a.e. $x \in \Omega$. In fact for all $K \subset \Omega$ compact, we have

$$f(x) \geqslant C_K \int\limits_K f(y)\mathrm{d}y$$

for a.e. $x \in K$, where C_K depends only on K.

□

Proof.

Step 1. We can find a function $J : \mathbb{R}^d \to \mathbb{R}$ such that $J \geqslant 0$, $T \in L^\infty_{\mathrm{loc}}$, $J(0) = 1$ and J is radial and

$$(-\Delta + \mu^2)J(x) = 0, \qquad \text{pointwise.}$$

For example in 3-dimension this is

$$J(x) = \frac{\sinh(\mu|x|)}{\mu|x|}.$$

Step 2. Assume that $f \in \mathscr{C}^\infty$ and $-\Delta f + \mu^2 f \geqslant 0$ pointwise. Then

$$\int\limits_{B_r(0)} (-\Delta f + \mu^2 f)J \geqslant 0.$$

On the other hand

$$\int\limits_{B_r(0)} f(-\Delta J + \mu^2 J) = 0$$

i.e.

$$0 \leqslant \int\limits_{B_r(x)} ((-\Delta f)J - f(-\Delta J)) = -r^{n-1} \int\limits_{\mathbb{S}^{d-1}} (\nabla f J - f \nabla f) \cdot \omega \mathrm{d}\omega =$$

$$= -r^{d-1} \int\limits_{\mathbb{S}^{d-1}} \left(\frac{d}{dr} f(r\omega) j(r\omega) - f(r\omega) \frac{d}{dr} J(r\omega) \right) \mathrm{d}\omega =$$

$$= -r^{d-1} \left(\frac{d}{dr} \left(\int\limits_{\mathbb{S}^{d-1}} f(r\omega) \right) J(r) - \int\limits_{\mathbb{S}^{d-1}} f(r\omega) \frac{d}{dr} J(r) \right)$$

which implies that

$$\left(\frac{d}{dr} g \right) J - g \left(\frac{d}{dr} J \right) \leqslant 0 \implies \frac{d}{dr} \frac{g}{J} \leqslant 0$$

Thus $r \mapsto \frac{g}{J}$ is non-increasing and therefore

$$|\mathbb{S}^{d-1}| f(0) \geqslant \frac{g(R)}{J(R)} = \frac{1}{J(R)} \int\limits_{B_R(0)} f(R\omega) \mathrm{d}\omega$$

for all $R > 0$ such that $B_R(0) \subset \Omega$ and thus also that

$$f(0) \geqslant C_R \int\limits_{B_R(0)} f(y) \mathrm{d}y$$

and

$$f(x) \geqslant C_R \int\limits_{B_R(x)} f(y) \mathrm{d}y.$$

Step 3 Now let $f \in L^1_{\mathrm{loc}}$ and consider $h_n * f \in \mathscr{C}^\infty$, with $h_n * f \to f$ in $L^1_{\mathrm{loc}}(\Omega)$. From Step 2 we have

$$(h_n * f)(x) \geqslant C_R \int\limits_{B_R(x)} (h_n * f)(y) \mathrm{d}y = C_R \mathbf{1}_{B_R(0)} * (h_n * f) = C_R h_n * \left(\mathbf{1}_{B_R(0)} * f \right)$$

Taking $n \to 0$ we find that

$$f(x) \geqslant C_R\big(\mathbf{1}_{B_R(0)} * f\big)(x) = C_R \int\limits_{B_R(x)} f(y)\mathrm{d}y$$

for a.e. x.

Step 4 If $f \geqslant 0$ and $f \not\equiv 0$. Then the mean value inequality implies that

$$f(x) \geqslant C_R \int\limits_{B_R(x)} f(y)\mathrm{d}y$$

implies that $f(x) > 0$. The proof argument is the same as for the strong maximum principle.

Step 5 K is compact, we can find $x_1, \ldots, x_n$, $r > 0$ such that $K \subset \bigcup_{i=1}^N B_r(x) =: U$

$$\int\limits_K f(y)\mathrm{d}y \leqslant \int\limits_U f \leqslant \sum_{i=1}^N \int\limits_{B_r(x)} f.$$

And thus if we assume that $B_i \cap B_{i+1} \neq 0$ and $x \in B(x_1, r)$

$$f(x) \geqslant c \int\limits_{B_r(x)} f(y)\mathrm{d}y \geqslant \int\limits_{B(x_1,r)\cap B(x_2,r)} \geqslant c'|B_1 \cap B_2| \inf_{B_1\cap B_2} \geqslant c' \int\limits_{B_r(x_2)} f(y)\mathrm{d}y$$

if $|B_1 \cap B_2| \neq 0$ (or $B_i \cap B_{i+1} \neq \emptyset$ for all i). Thu

$$f(x) \geqslant c_1 \int\limits_{B_1} f(y)\mathrm{d}y \geqslant \cdots \geqslant c_n \int\limits_{B_N} f(y)\mathrm{d}y$$

hence

$$f(x) \geqslant \tilde{c} \int\limits_K f(y)\mathrm{d}y$$

$\tilde{c} = \inf c_i$.

q.e.d.

Theorem 7.13 (Uniqueness of Minimiser)**.** *Assume that* $V \in L^1_{loc}$ *and* E *has a minimiser. Assume that* $V_+ \in L^\infty_{loc}(\mathbb{R}^d)$, $V_+(x) = \max\{V(x), 0\}$. *Then there exists a unique*

$u_0 > 0$ minimiser for E. Moreover if u is another minimiser, then

$$u = cu_0$$

for a constant $c \in \mathbb{C}$, $|c| = 1$. □

Proof. By the diamagnetic inequality, $\mathcal{E}(u) \geqslant (|u|)$. We may thus assume that E has a minimiser $u_0 \geqslant 0$ and we have to prove that $u_0 > 0$.
Since u_0 is a minimiser, it satisfies

$$-\Delta u_0 + V u_0 = E u_0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^d)$$

Thus

$$-\Delta u_0 + V u_0 = E u_0$$

in $\mathscr{D}'(B)$ for all open balls in $\mathbb{R}^d$. Since $V_+ \in L^\infty(B)$ implies that $V \leqslant \mu^2$ in B for some constant $\mu \geqslant 0$. Thus

$$-\Delta u_0 + (\mu^2 - E) u_0 \geqslant 0 \qquad \text{in } \mathscr{D}'(B).$$

By the above theorem it follows that

$$u_0(x) \geqslant C_K \int\limits_K u_0(y) \mathrm{d}y$$

for all compact subsets of B and a.e. $x \in K$. This means that for every $y \in \mathbb{R}^d$, $r > 0$, that

$$u_0(x) \geqslant C_r \int\limits_{B_r(y)} u_0(z) \mathrm{d}z$$

Because $u_0 \geqslant 0$, $u_0 \not\equiv 0$ (as $\|u_0\|_2 = 1$), then

$$\int\limits_{B_R(0)} u(z) \mathrm{d}z > 0$$

for R big enough. Therefore $u_0(x) > 0$ for a.e. $x \in B_R(0)$ for all R large enough. Therefore $u_0(x) > 0$ for a.e. $x \in \mathbb{R}^d$.
Next assume that u is another minimiser. We can write $u = f + ig$, with $f, g : \mathbb{R}^d \to \mathbb{R}$.

$$\mathcal{E}(u) = \int |\nabla u|^2 + \int V|u|^2 = \int \left(|\nabla f|^2 + V|f|^2\right) + \int \left(|\nabla g|^2 + V|g|^2\right) = E = E\int |f|^2 + E\int |g|^2$$

But

$$\int |\nabla f|^2 + \int V|f|^2 \geqslant E \int |f|^2$$
$$\int |\nabla g|^2 + \int V|g|^2 \geqslant E \int |g|^2$$

By the definition of E. Thus $\frac{f}{\|f\|_2}$ and $\frac{g}{\|g\|_2}$ are also minimisers for E.
Then either u is real indeed, or we assume both $f, g \not\equiv 0$. Let us consider when both $f, g \neq 0$. Then $\frac{|f|}{\|f\|_2}, \frac{|g|}{\|g\|_2}$ are also minimisers by the diamagnetic inequality. We can therefore assume that $f > 0$ and $g > 0$.
Now we choose $|u|$, we know that

$$\int |\nabla u|^2 = \int |\nabla |u||^2$$

because u is minimiser. Because $f, g > 0$

$$\nabla |u| = \frac{f\nabla f + g\nabla g}{\sqrt{f^2 + g^2}}$$

which implies that

$$\int |\nabla f|^2 + |\nabla g|^2 = \int \frac{|f\nabla f + g\nabla g|^2}{f^2 + g^2}$$

On the other hand

$$\frac{|f\nabla f + g\nabla g|^2}{f^2 + g^2} \leqslant |\nabla f|^2 + |\nabla g|^2 \qquad \text{pointwise.}$$

Thus

$$\frac{|f\nabla f + g\nabla g|^2}{f^2 + g^2} = |\nabla f|^2 + |\nabla g|^2 \qquad \text{a.e.}$$

Hence $f = \text{const}g$. Consequently $u = f + ig = (1 + i\text{const})g = \text{const}g$, i.e. u is real valued and $u > 0$ up to a phase.
Finally, since both u and u_0 are minimisers (and positive)

$$\varphi = \frac{u + iu_0}{\|u + iu_0\|_2}$$

is also a minimiser and thus by the same argument we have $u = Cu_0$.

q.e.d.

Corollary 7.14. *If there exists a* $\lambda \in \mathbb{R}$ *and* $v \geqslant 0$ *such that*

$$-\Delta v + Vv = \lambda v \qquad \text{in } \mathscr{D}'\big(\mathbb{R}^d\big)$$

Then $\lambda = E$ *and* $v = u_0 > 0$ *(where* u_0 *is the unique minimiser of* $\mathcal{E}$*).* □

Proof. The PDE implies

$$\int \nabla v \cdot \nabla \varphi + \int V v \varphi = \lambda \int v \varphi$$

for all $\varphi \in \mathscr{D}\big(\mathbb{R}^d\big)$ and thus

$$\int \nabla v \cdot \nabla u_0 + \int V v u_0 = \lambda \int v u_0$$

(where we have omitted some conditions on V). Moreover,

$$-\Delta u_0 + V u_0 = E u_0$$

thus

$$\int \nabla u_0 \cdot \nabla v + \int V u_0 v = E \int v u_0$$

Thus $\lambda \int v u_0 = E \int v u_0$. Since $\int v u_0 > 0$ (as $v \geqslant 0$, $u_0 > 0$) which implies that $\lambda = E$, hence v is a minimiser and thus $v = u_0$. *q.e.d.*

Chapter 8

Smoothness of Weak Solutions

Consider the Poisson equation

$$-\Delta u = f \qquad \text{in } \mathscr{D}'(\mathbb{R}^d)$$

If $f \in \mathscr{C}(\mathbb{R}^d)$, can we conclude $u \in \mathscr{C}^2(\mathbb{R}^d)$. If $d = 1$ yes otherwise no. But $f \in \mathscr{C}(\mathbb{R}^d)$ implies that $u \in \mathscr{C}^1(\mathbb{R}^d)$.
However, there exists the Elliptical optimal estimate that if $f \in \mathscr{C}^\alpha$ then $u \in \mathscr{C}^{2+\alpha}$ for $0 < \alpha < 1$, where $\mathscr{C}^\alpha$ are the Hölder spaces.

Theorem 8.1 (Basic Regularity)**.** *Assume that $u \in L^1_{loc}(\Omega)$, $f \in L^p_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open. If*

$$-\Delta u = f \qquad in\ \mathscr{D}'(\Omega)$$

Then

- *$u \in \mathscr{C}(\Omega)$ if $p > \frac{d}{2}$*
- *$u \in \mathscr{C}^1(\Omega)$ if $p > d$.*

□

Proof.

Step 1 $f \in L^p(\mathbb{R}^d)$ and f has compact support

$$-\Delta u = f \qquad \text{in } \mathscr{D}'(\Omega)$$

Then a solution is $u(x) = (G * f)(x) = \int\limits_{\mathbb{R}^d} G(x-y)f(y)\mathrm{d}y$ where

$$G(x) = \begin{cases} \frac{1}{(d-2)|\mathbb{S}^{d-1}|}\frac{1}{|x|^{d-2}}, & \text{if } d \neq 2 \\ -\frac{1}{2\pi}\ln|x|, & \text{if } d = 2. \end{cases}$$

Let us restrict ourselves to the case $d \geqslant 3$.

$$u(x) = c_d \int\limits_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}}\mathrm{d}y$$

is well-defined because

$$\int\limits_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}}\mathrm{d}y \leqslant \Bigg(\int\limits_{\mathbb{R}^d} |f|^p\Bigg)^{1/p} \Bigg(\int\limits_{\operatorname{supp} f} \frac{\mathrm{d}y}{|x-y|^{(d-2)q}}\Bigg)^{1/q} \leqslant C\|f\|_p$$

with $C < \infty$ if

$$(d-2)q < d \iff \frac{p}{p-1} < \frac{d}{d-2} \iff \frac{p-1}{p} > \frac{d-2}{d} \iff \frac{1}{p} < \frac{2}{d} \iff p > \frac{d}{2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Step 2 We prove prove that $u(x)$ as defined above is continuous if $p > \frac{d}{2}$.

$$u(x) - u(x') = c_d \int f(y)\left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|x'-y|^{d-2}}\right)\mathrm{d}y$$

thus

$$|u(x) - u(x')| \leqslant c_d \int |f(y)| \left|\frac{1}{|x-y|^{d-2}} - \frac{1}{|x'-y|^{d-2}}\right|\mathrm{d}y$$

Using the elementary inequality for $a, b \geqslant 0$ and $\alpha \geqslant 1$

$$\left|\frac{1}{a^\alpha} - \frac{1}{b^\alpha}\right| = \frac{|a^\alpha - b^\alpha|}{a^\alpha b^\alpha} \leqslant C|a-b|\frac{a^{\alpha-1} + b^{\alpha-1}}{a^\alpha b^\alpha} \leqslant C|a-b|^\varepsilon |a+b|^{1-\varepsilon}\frac{(a^{\alpha-1} + b^{\alpha-1})}{a^\alpha b^\alpha} \leqslant$$
$$\leqslant C|a-b|^\varepsilon \frac{1}{a^{\alpha+\varepsilon} + \frac{1}{b^{a+\varepsilon}}}$$

for $\varepsilon > 0$ small. Thus

$$\left|\frac{1}{|x-y|^{d-2}} - \frac{1}{|x'-y|^{d-2}}\right| \leqslant C|x-x'|\left|\frac{1}{|x-y|^{d-2+\varepsilon}} + \frac{1}{|x'-y|^{d-2+\varepsilon}}\right|$$

therefore

$$|u(x)-u(x')| \leqslant C|x-x'| \int |f(y)| \left(\frac{1}{|x-y|^{d-2+\varepsilon}} + \frac{1}{|x'-y|^{d-2+\varepsilon}} \right) dy \leqslant$$

$$\leqslant C|x-x'| \left(\int |f|^p \right)^{1/p} \left(\left(\int_{\operatorname{supp} f} \frac{1}{|x-y|^{(d-2+\varepsilon)q}} \right)^{1/q} + \left(\int_{\operatorname{supp} f} \frac{1}{|x'-y|^{(d-2+\varepsilon)q}} \right)^{1/q} \right)$$

Thus in total we have

$$|u(x)-u(x')| \leqslant C|x-x'|^{\varepsilon} \|f\|_p$$

if

$$(d-2+\varepsilon)q < d \iff \varepsilon \frac{d}{q} - (d-2) = \frac{d(p-1)}{p} - (d-2) = 2 - \frac{d}{p}.$$

Step 3 We prove that if $p > d$ then $u(x) = c_d \int \frac{f(y)}{|x-y|^{d-2}} \mathrm{d}y$ is $\mathscr{C}^1$.

$$\partial_{x_i} u(x) = c_d \int f(y) \frac{x_i - y_i}{|x-y|^d} \mathrm{d}y$$

and therefore

$$|\partial_{x_i} u(x) - \partial_{x_i} u(x')| \leqslant c_d \int |f(y)| \left| \frac{x_i - y_i}{|x-y|^d} - \frac{x'_i - y'_i}{|x'-y'|^d} \right| \mathrm{d}y$$

Let $a = |x-y|$, $a_i = x_i - y_i$, $b = |x'-y|$ and $b_i = x'_i - y_i$. We have

$$\left| \frac{a_i}{a^d} - \frac{b_i}{b^d} \right| \leqslant \frac{|a_i - b_i|}{a^d} + |b_i| \left| \frac{1}{a^d} - \frac{1}{b^d} \right| \leqslant |x-x'| \frac{1}{a^d} + |b| \left| \frac{1}{a^d} - \frac{1}{b^d} \right| \leqslant$$

$$\leqslant C|x-x'|^{\varepsilon} \left(\frac{1}{|x-y|^{d-1+\varepsilon}} + \frac{1}{|x'-y|^{d-1+\varepsilon}} \right)$$

hence

$$|\partial_{x_i} u(x) - \partial_{x_i} u(x')| \leqslant C|x-x'|^{\varepsilon} \int |f(y)| \left| \frac{1}{|x-y|^{d-1+\varepsilon}} + \frac{1}{|x'-y|^{d-1+\varepsilon}} \right| \mathrm{d}y \leqslant$$

$$\leqslant |x-x'|^{\varepsilon} \|f\|_p \left(\int_{\operatorname{supp} f} \left| \frac{\mathrm{d}y}{|x-y|^{(d-1+\varepsilon)q}} \right| \right)^{1/q} + \left(\int_{\operatorname{supp} f} \left| \frac{\mathrm{d}y}{|x-y|^{(d-1+\varepsilon)q}} \right)^{1/q} \right) \leqslant$$

$$\leqslant C|x-x'|^{\varepsilon} \|f\|_p$$

if

$$(d-1+\varepsilon)q < d \iff \varepsilon < \frac{d}{p} - (d-1) = \frac{d(p-1)}{p} - (d-1) = 1 - \frac{d}{p}.$$

Step 4 Now let $f \in L^p_{\text{loc}}(\Omega)$, $-\Delta u = f$ in $\mathscr{D}'(\Omega)$. Take an open ball B such that $\overline{B} \subset \Omega$. Take function u_1 such that $-\Delta u_1 = \mathbf{1}_B f$ in $\mathscr{D}'(\Omega)$, (i.e. $u_1 = G * (\mathbf{1}_B f)$) From Step 1,2,3 it follows that $u_1 \in \mathscr{C}(B)$ if $p > \frac{d}{2}$ and $\mathscr{C}^1(B)$ if $p > d$.

Further we also have

$$-\Delta(u - u_1) = f(1 - \mathbf{1}_B), \qquad \text{in } \mathscr{D}'(\Omega)$$

Thus

$$-\Delta(u - u_1) = 0, \qquad \text{in } \mathscr{D}'(B)$$

Thus $u - u_1$ is a harmonic function in B. Therefore $u - u_1 \in \mathscr{C}^\infty(B)$. If $u_1 \in \mathscr{C}(B)$ it follows that $u \in \mathscr{C}(B)$ and analogously for $\mathscr{C}^1$. Since the ball B was arbitrary with $\overline{B} \subset \Omega$, we have

$$u \in \mathscr{C}(\Omega), \qquad \text{if } p > \frac{d}{2}$$
$$u \in \mathscr{C}^1(\Omega), \qquad \text{if } p > d$$

q.e.d.

An application of this theorem would be

Theorem 8.2. *Assume that $u \in L^2(\mathbb{R}^3)$, $V \in \mathscr{C}^\infty(\mathbb{R}^3)$ and*

$$-\Delta u + Vu = 0, \qquad \text{in } \mathscr{D}'(\mathbb{R}^)$$

Then $u \in \mathscr{C}^\infty(\mathbb{R}^3)$. □

Proof. $-\Delta u + Vu = 0$ implies that $-\Delta u = -Vu$ in $\mathscr{D}'(\mathbb{R}^3)$, $u \in L^2$, $V \in \mathscr{C}^\infty$, hence $Vu \in L^2_{\text{loc}}(\mathbb{R}^3)$.
By the above theorem, we have as $p = 2$, $d = 3$, $p > \frac{d}{2}$ thus $u \in \mathscr{C}(\mathbb{R}^3)$. Then as $u \in \mathscr{C}$, $V \in \mathscr{C}^\infty$ implies that $Vu \in \mathscr{C}(\mathbb{R}^3) \subset L^\infty_{\text{loc}}(\mathbb{R}^3)$. By the same theorem as $p = \infty > d$, $u \in \mathscr{C}^1(\mathbb{R}^3)$. Since $V \in \mathscr{C}^\infty$, $u \in \mathscr{C}^1$ we have $Vu \in \mathscr{C}^1$ and therefore

$$-\Delta(\partial_{x_i} u) = \partial_{x_i}(-\Delta u) = \partial_{x_i}(-Vu) \in \mathscr{C}\big(\mathbb{R}^3\big)$$

Applying the same regularity theorem we find that $\partial_{x_i} u \in \mathscr{C}^1$ for all i and thus $u \in \mathscr{C}^2(\mathbb{R}^3)$. By induction

$$(-\Delta)(D^\alpha u) = D^\alpha(-\Delta u) = D^\alpha(-Vu) \in \mathscr{C}$$

for all $|\alpha| \leqslant 2$, thus $D^\alpha u \in \mathscr{C}^1$ and therefore $u \in \mathscr{C}^3$. *q.e.d.*

Chapter 9

Concentration Compactness Method

We call the functional

$$\mathcal{E}(u) = \int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \int\limits_{\mathbb{R}^3} \frac{Z}{|x|}|u|^2 \mathrm{d}x + \frac{1}{2} \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y$$

the ***Hartree Function*** for atoms, where $Z > 0$ is the nuclear charge, $|u(x)|^2$ is the density of electrons. Consider the variational problem

$$E(\lambda) := \inf\big\{\mathcal{E}(u) \,\big|\, u \in H^1\big(\mathbb{R}^3\big), \|u\|_2^2 = \lambda\big\}.$$

$E(\lambda)$ is called the ground state energy of the atom. If u_0 is a minimiser for $E(\lambda)$, then it satisfies the following PDE

$$-\Delta u_0 - \frac{Z}{|x|}u_0 + \big(|u_0|^2 * |\cdot|^{-1}\big)u_0 = \mu u_0, \qquad \text{in } \mathscr{D}'\big(\mathbb{R}^3\big)$$

with $\mu \leqslant 0$.

Lemma 9.1. *The map* $\lambda \mapsto E(\lambda)$ *is non-increasing on* $[0, \infty)$. □

Proof. Let $0 \leqslant \lambda_1 < \lambda_2$. We are going to prove that $E(\lambda_1) \geqslant E(\lambda_2)$. By a density argument we can find a $v_1 \in D$ such that $\int |v_1|^2 \mathrm{d}x = \lambda_1$ and $\mathcal{E}(v_1) \leqslant E(\lambda_1) + \varepsilon$, for $\varepsilon > 0$ small. Take another function $\varphi \in \mathscr{D}$ such that $\|\varphi\|_2^2 = \lambda_2 - \lambda_1 > 0$. Choose $v_2(x) = v_1(x) + \varphi(x - Rx_0)$, where $x_0 \in \mathbb{R}^3\backslash\{0\}$, $R > 0$. For R sufficiently large v_1 and $\varphi(\cdot - Rx_0)$ have disjoint supports, then $\|v_2\|_2^2 = \|v_1\|_2^2 + \|\varphi\|_2^2 = \lambda_2$. Moreover,

$$E(\lambda_2) \leqslant \mathcal{E}(v_2) = \mathcal{E}(v_1+\varphi(\cdot - Rx_0)) = \mathcal{E}(v_1)+\mathcal{E}(\varphi(\cdot - Rx_0))+ \int\limits_{\mathrm{supp}(v_1)\times(\mathrm{supp}\,\varphi+x_0R)} \frac{|v_1(x)|^2|\varphi(y-x_0R)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y$$

taking $R \to \infty$, we get the inequality

$$E(\lambda_2) \leqslant E(\lambda_1) + 2\varepsilon + \int |\nabla\varphi|^2 \mathrm{d}x$$

for all $\varphi \in \mathscr{D}$, $\|\varphi\|_2^2 = \lambda_2 - \lambda_1$. Rescaling φ, we can achieve $\|\nabla\varphi\|_2^2 < \varepsilon$ and taking $\varepsilon \to 0$, we get $E(\lambda_2) \leqslant E(\lambda_1)$

q.e.d.

Theorem 9.2. *a) If $0 \leqslant \lambda \leqslant Z$, then there exists a minimiser for $E(\lambda)$.*

b) If $\lambda > 2Z$, there does not exist a minimiser for $E(\lambda)$.

□

Proof. a) Let $(u_n)_{n\in\mathbb{N}}$ be a minimising sequence for $E(\lambda)$. By the diamagnetic inequality $|\nabla u| \geqslant |\nabla|u||$ we have $\mathcal{E}(u) \geqslant \mathcal{E}(|u|)$, thus w.l.o.g. we can assume $u_n \geqslant 0$ for all $n \in \mathbb{N}$. By the hydrogen atom theory

$$\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \int\limits_{\mathbb{R}^3} \frac{a}{|x|}|u|^2 \mathrm{d}x \geqslant -\frac{a^2}{4}\int\limits_{\mathbb{R}^3} |u|^2 \mathrm{d}x$$

for all $a \geqslant 0$. Thus for $a = 2Z$

$$\mathcal{E}(u) \geqslant \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\Big(\int |\nabla u|^2 \mathrm{d}x - \int \frac{2Z}{|x|}|u|^2\mathrm{d}x\Big) \geqslant \frac{1}{2}\int |\nabla u|^2 - \frac{Z^2\lambda}{2} \geqslant -\frac{Z^2\lambda}{2}.$$

for all $u \in H^1$ and $\|u\|_2^2 = \lambda$. Moreover, as $(u_n)_n$ is a minimising sequence,

$$\mathcal{E}(\lambda) = \lim_{n\to\infty} \mathcal{E}(u_n) \geqslant \frac{1}{2}\int |\nabla u_n|^2 - \frac{Z^2\lambda}{2}$$

thus $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$. By going to a subsequence and renaming it to the original, we may assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$. We have $\nabla u_n \overset{\nabla}{\rightharpoonup} u_0$

in $L^2(\mathbb{R}^3)$ which implies that

$$\liminf_{n\to\infty} \int\limits_{\mathbb{R}^3} |\nabla u_n|^2 \geqslant \int |\nabla u_0|^2$$

Moreover,

$$\frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} \xrightarrow{n\to\infty} \frac{|u_0(x)|^2|u_0(y)|^2}{|x-y|} \qquad \text{a.e. } x, y \in \mathbb{R}^3.$$

Thus by Fatou's lemma we have

$$\liminf_{n\to\infty} \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y \geqslant \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_0(x)|^2|u_0(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y$$

On the other hand from $u_n \overset{H^1}{\rightharpoonup} u_0$ we have that the Coloumb interaction term converges as we saw from the weak-continuity of this potential energy above. Thus we have

$$E(\lambda) = \lim_{n\to\infty} \mathcal{E}(u_n) \geqslant \mathcal{E}(u_0).$$

To conclude that u_0 is a minimiser, we need to prove that $\|u_0\|_2^2 = \lambda$. By $u_n \overset{L^2}{\rightharpoonup} u_0$, and $\|u_0\|^2 \leqslant \liminf_{n\to\infty} \|u_n\|_2^2 = \lambda$. The reverse inequality is non-trivial and we shall prove it by using $\lambda \leqslant Z$. Now assume that $\|u_0\|_2^2 < \lambda$. Then $\mathcal{E}(u_0) \leqslant \mathcal{E}(\lambda) \leqslant \mathcal{E}(v)$, for all $v \in H^1$ with $\|v\|_2^2 = \lambda$.

Let $\varphi \in \mathscr{D}(\mathbb{R}^3)$, $\varphi \geqslant 0$ by the above Lemma. For $\varepsilon \in \mathbb{R}$, with $|\varepsilon|$ small we have

$$\int |u_0 + \varepsilon\varphi|^2 \leqslant \lambda \implies \mathcal{E}(u_0) \leqslant \mathcal{E}(u_0 + \varepsilon\varphi) \implies \frac{1}{2}\frac{d^2}{d\varepsilon^2}\mathcal{E}(u_0+\varepsilon\varphi)\bigg|_{\varepsilon=0} \geqslant 0$$

and thus

$$\begin{aligned}
0 &\leqslant \frac{1}{2}\frac{d^2}{d\varepsilon^2}\cdots\bigg|_{\varepsilon=0} = \\
&= \frac{1}{2}\frac{d^2}{d\varepsilon^2}\left(\int\limits_{\mathbb{R}^3} |\nabla(u_0+\varepsilon\varphi)|^2\mathrm{d}x - \int \frac{Z}{|x|}|u_0+\varepsilon\varphi|^2\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_0(x)+\varepsilon\varphi(x)|^2|u_0(y)+\varepsilon\varphi(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y\right) \\
&= \int |\nabla\varphi|^2\mathrm{d}x - \int \frac{Z}{|x|}|\varphi|^2\mathrm{d}x + \int\int \frac{|u_0(y)||\varphi(x)|^2}{|x-y|} + 2\int\int \frac{u_0(x)u_0(y)\varphi(x)\varphi(y)}{|x-y|}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

Choosing φ to be radial and letting $\varphi = 0$ if $|x| < R$, where $\varphi \in \mathscr{D}$, $\varphi \geqslant 0$, we find by

Newton's theorem that

$$\int\int \frac{|u_0(y)||\varphi(x)|^2}{|x-y|} = \int\int \frac{|u_0(y)||\varphi(x)|^2}{\max\{|x|,|y|\}}\mathrm{d}x\mathrm{d}y \leqslant \int u_0 \mathrm{d}y \int \frac{\varphi^2(x)}{|x|}\mathrm{d}x$$

$$\int\int \frac{u_0(x)u_0(y)\varphi(x)\varphi(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \leqslant \left(\int\limits_{|x|,|y|\geqslant R} \frac{|u_0(x)||\varphi(y)|^2}{|x-y|}\right)^{1/2}\left(\int\limits_{|x|,|y|>R} \frac{|u_0(y)|^2\varphi(x)^2}{|x-y|}\right)^{1/2} \leqslant$$

$$\leqslant \left(\int_{|y|\geqslant R} u_0(y)^2\mathrm{d}y\right)\int \frac{\varphi(x)^2}{|x|}\mathrm{d}x$$

Altogether

$$0 \leqslant \int |\nabla\varphi|^2 + \left(-Z + \|u_0\|_2^2 + 2\int\limits_{|y|>R} |u_0(y)|^2 dy\right)\int \frac{\varphi(x)^2}{|x|}\mathrm{d}x$$

Choose $\varphi(x) := \varphi_0\left(\frac{x}{R}\right)$, $\varphi_0 \in \mathscr{D}$, $\varphi_0 \geqslant 0$ and $\varphi_0 = 0$ in $\overline{B_1(0)}$, $\varphi_0 \not\equiv 0$ and φ_0 radial. Then

$$0 \leqslant \left(\frac{1}{R}\int |\nabla\varphi_0|^2 + \left(-Z + \|u_0\|_2^2 + 2\int\limits_{|y|>R} u_0^2\right)R^2\int \frac{\varphi_0^2(x)}{|x|}\mathrm{d}x\right)R^2$$

by passing $R \to \infty$ it follows that

$$0 \leqslant -Z + \int u_0^2 \implies \lambda > \|u_0\|^2 \geqslant Z$$

which is a contradiction.

b) If $\lambda > 2Z$, then $E(\lambda)$ has no minimiser. Assume that u_0 is a minimiser. By the diamagnetic inequality we can assume that $u_0 \geqslant 0$. Then for all $f \in H^1(\mathbb{R}^3)$

$$0 = \frac{1}{2}\frac{d}{d\varepsilon}\mathcal{E}\left(\sqrt{\lambda}\frac{u_0+\varepsilon f}{\|u_0+\varepsilon f\|_2}\right)\bigg|_{\varepsilon=0} =$$

$$= \int \nabla u_0\cdot\nabla f - \int \frac{Z}{|x|}u_0 f + \int\int \frac{u_0(x)^2 u_0(y)}{f}(y)|x-y|\mathrm{d}x\mathrm{d}y - \mu\int u_0 f$$

with $\mu \leqslant 0$. Now choose $f := \varphi^2 u_0$, with $\varphi \in \mathscr{D}$, $\varphi \geqslant 0$, $\varphi(x) = |x|$ if $|x| \leqslant R$ and $|\nabla\varphi| \leqslant 1$ if $|x| \geqslant R \geqslant 1$.

We have

$$0 = \int \nabla u_0 \cdot \nabla\big(\varphi^2 u_0\big) - \int \frac{Z}{|x|}\varphi^2 u_0^2 + \int\int \frac{\varphi(x)^2 u_0(x)^2 u_0(y)^2}{|x-y|} - \underbrace{\mu \int \varphi^2 u_0^2 \mathrm{d}x}_{\geqslant 0} \geqslant$$

$$\geqslant \int |\nabla(\varphi u_0)|^2 - \int |\nabla\varphi|^2 |u_0|^2 - \int Z u_0^2 + \int \int_{|x|\leqslant R} \frac{\varphi(x)^2 u_0(x)^2 u_0(y)^2}{|x-y|} =$$

$$= \int \frac{\varphi^2 u_0^2}{4|x|^2} - \int |\nabla\varphi|^2 |u_0|^2 - Z\lambda + \frac{1}{2}\int \int_{|x|\leqslant R} \underbrace{\frac{|x|+|y|}{|x-y|}}_{\geqslant 1} u_0(x)^2 u_0(y)^2 \geqslant$$

$$\geqslant \int_{|x|\leqslant R} \underbrace{\left(\frac{\varphi^2}{4|x|^2} - |\nabla\varphi|^2\right)}_{=0} u_0^2 - \int_{x>R} \underbrace{|\nabla\varphi|^2}_{\leqslant 1} |u_0|^2 - Z\lambda + \frac{1}{2}\int \int_{|x|\leqslant R} u_0(x)^2 u_0(y)^2 \geqslant$$

$$\geqslant - \int_{x>R} u_0^2 - Z\lambda + \frac{1}{2}\left(\int_{|x|\leqslant R} u_0(x)^2\right)^2$$

Thus

$$0 \geqslant - \int_{|x|\geqslant R} u_0^2 - Z\lambda + \frac{1}{2}\left(\int_{|x|\leqslant R} u_0^2\right)^2$$

for all $R \geqslant 1$ and thus taking $R \to \infty$ we have

$$0 \geqslant -Z\lambda + \frac{\lambda^2}{2} \implies \lambda \leqslant 2Z$$

which is a contradiction.

q.e.d.

For all $u, v \geqslant 0$ we have

$$\frac{\mathcal{E}(u)+\mathcal{E}(v)}{2} \geqslant \mathcal{E}\left(\sqrt{\frac{u^2+v^2}{2}}\right)$$

with strict inequality if $u \neq v$. Consequently $\lambda \mapsto \mathcal{E}(\lambda)$ is convex. Thus there must exists a λ^* such that it minimises $E(\lambda)$. Numerically one finds that $\lambda* \approx 1.21Z$.

Now we shall consider a general functional

$$\mathcal{E}(u) = \int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} V|u|^2 + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} |u(x)|^2 w(x-y)|u(y)|^2 \mathrm{d}x\mathrm{d}y$$

where V is an external potential and w is an interaction potential.

Remark 9.3 (Assumptions)**.** We shall assume that $|v|, |w| \in L^p + L^q$, for $p, q > \max\{\frac{d}{2}, 1\}$ and $w(x) = w(-x)$. □

Example 9.4. 1) Hartree $V = -\frac{Z}{|x|}$, $w = \frac{1}{|x|}$ (Coulomb potential).

2) Chequard-Pekar $w = \frac{1}{|x|}$ (Newton potential).

Definition 9.5.

$$E(\lambda) = \inf\{\mathcal{E}(u) \,|\, u \in H^1(\mathbb{R}^d), \|u\|_2^2 = \lambda\}$$
$$E^0(\lambda) = \inf\left\{\mathcal{E}^0(u) = \int |\nabla u|^2 + \frac{1}{2}\int |u(x)|^2 w(x-y)|u(y)|^2 \,\middle|\, u \in H^1(\mathbb{R}^d), \|u\|_2^2 = \lambda\right\}$$

where the second minimiser is for problems at infinity. □

Theorem 9.6 (Concentration-Compactness Prinicple)**.** *We always have*

$$E(\lambda) \leqslant E(\lambda - \lambda') + E^0(\lambda')$$

for all $0 \leqslant \lambda' \leqslant \lambda$. *Moreover, if we have the strict binding inequality*

$$E(\lambda) < E(\lambda - \lambda') + E^0(\lambda')$$

for all $0 < \lambda' \leqslant \lambda$ *then* $E(\lambda)$ *has a minimiser.* □

For the Hartree functional $E^0(\lambda') = 0$ (by scaling).

Lemma 9.7. *If* $|v|, |w| \in L^p + L^q$, *with* $p, q > \max\{\frac{d}{2} + 1\}$, *then*

$$\int |V||u|^2 \mathrm{d}x \leqslant C(\|V\|_{L^p+L^q})\|u\|_{H^1}^2$$
$$\big\| |w| * |u|^2 \big\|_\infty \leqslant C(\|w\|_p + \|w\|_q)\|u\|_{H^1}^2$$

where we $\|V\|_{L^p+L^q} = \inf\{\|V_1\|_p + \|V_2\|_q \,|\, V_1 \in L^p, V_2 \in L^q, V_1 + V_2 = V\}$ *Moreover, for* *all* $\varepsilon > 0$

$$\int |V||u|^2 \mathrm{d}x \leqslant \varepsilon \int |\nabla u|^2 + C_\varepsilon \int |u|^2$$
$$\left\| |w| * |u|^2 \right\|_\infty \leqslant \varepsilon \int |\nabla u|^2 + C_\varepsilon \int |u|^2$$

□

Proof. If $V = V_1 + V_2$, with $V_1 \in L^p$, $V_2 \in L^q$, we have

$$\int |V_1||u|^2 \leqslant \left(\int |V_1|^p\right)^{1/p} \left(\int |u|^{2p'}\right)^{1/p'} \leqslant C\|V_1\|_p \|u\|_{H^1}^2$$

where we used the Sobolev inequality in the second inequality, which we were allowed to as $2p' <$ Sobolev power. We have the same inequality for V_2. Thus

$$\int |V||u|^2 \leqslant C\|V\|_{L^p+L^q} \|u\|_{H^1}^2$$

For the second inequality we have the same method

$$|w| * |u|^2 = \int |w(x-y)||u(y)|^2 \mathrm{d}y$$

We can write $w = w_1 + w_2$, $w_1 \in L^p$, $w_2 \in L^q$ and thus

$$|w_1| * |u|^2 = \int |w_1(x-y)||u(y)|^2 \mathrm{d}y \leqslant \left(\int |w_1(x-y)|^p \mathrm{d}y\right)^{1/p} \left(\int |u(y)|^{2p'} \mathrm{d}y\right)^{1/p'} \leqslant C\|w_1\|_p \|u\|_{H^1}^2$$

By the same bound for w_2, we get the bound for W.

Now take $\varepsilon > 0$. Since $V \in L^p + L^q$, we can decompose it into

$$V = V_\varepsilon + V_\infty$$

where $\|V_\varepsilon\|_{L^p+L^q} \leqslant \varepsilon$ and $V_\infty \in L^\infty$. Then

$$\int |V||u|^2 \leqslant \int |V_\varepsilon||u|^2 + \int |V_\infty||u|^2 \leqslant C \underbrace{\|V_\varepsilon\|_{L^p+L^q}}_{\leqslant \varepsilon} \|u\|_{H^1} + \underbrace{\|V_\infty\|_\infty}_{\leqslant C_\varepsilon} \|u\|_2^2.$$

q.e.d.

For our general interaction energy have by this Lemma

$$\mathcal{E}(u) \geqslant (1-\varepsilon)\int |\nabla u|^2 - C_\varepsilon \int |u|^2$$

for all $\varepsilon > 0$ and thus

$$\mathcal{E}(u) \geqslant \frac{1}{2}\int |\nabla u|^2 - C$$

Thus $E(\lambda) = \inf\big\{\mathcal{E}(u)\,\big|\, u \in H^1, \|u\|_2^2 = \lambda\big\} \geqslant -C\lambda > \infty$.

Now take a minimising sequence $u_n \in H^1$, $\int |u_n|^2 = \lambda$ and $\mathcal{E}(u_n) \to E(\lambda)$. By the diamagnetic inequality we have $|\nabla u_n| \geqslant |\nabla |u_n||$ (pointwise), $\mathcal{E}(u_n) \geqslant \mathcal{E}(|u_n|)$, so we can assume that $u_n \geqslant 0$.

Because $\frac{1}{2}\int |\nabla u_n|^2 - C \leqslant \mathcal{E}(u_n) \to E(\lambda)$. We have u_n is bounded in H^1. By choosing a subsequence we can assume that $u_n \rightharpoonup u_0$ weakly in H^1.

Lemma 9.8. *If $u_n \rightharpoonup u_0$ weakly in H^1, then*

$$\lim_{n\to\infty}\big(\mathcal{E}(u_n) - \mathcal{E}(u_0) - \mathcal{E}^0(u_n - u_0)\big) = 0.$$

□

Proof. Let us denote $v_n = u_n - u_0$, the $v_n \rightharpoonup 0$ weakly in H^1.

$$\int |\nabla u_n|^2 - \int |\nabla u_0|^2 - \int |\nabla v_n|^2 = 2\int \nabla u_0 \cdot \nabla v_n \longrightarrow 0$$

by weak convergence.

Second we have for the external potential

$$\int V|u_n|^2 - \int V|u_0|^2 \longrightarrow 0$$

because $u_n \rightharpoonup u_0$ and $V \in L^p + L^q$, as we have already proven above.

For the interaction term we have

$$\int\int |u_n(x)|^2 w(x-y)|u_n(y)|^2 \mathrm{d}x\mathrm{d}y - \int\int |u_0(x)|^2 w(x-y)|u_0(y)|^2 \mathrm{d}x\mathrm{d}y - \\ - \int\int |v_n(x)|^2 w(x-y)|v_n(y)|^2 \mathrm{d}x\mathrm{d}y$$

$$\begin{aligned}\int |u_n(x)|^2 w(x-y)|u_n(y)|^2 \mathrm{d}x\mathrm{d}y &= \int \big(|u_n(x)|^2 - |u_0(x)|^2 - |v_n(x)|^2\big) w(x-y)|v_n(y)|^2 \mathrm{d}x\mathrm{d}y + \\ &+ \int \big(|u_0(x)|^2 + |v_n(x)|^2\big) w(x-y)\big(|u_n(y)|^2 - |u_0(y)|^2 - |v_n(y)|^2\big)\mathrm{d}x\mathrm{d}y + \\ &+ \int \big(|u_0(x)|^2 + |v_n(x)|^2\big) w(x-y)\big(|u_0(y)|^2 + |v_n(y)|^2\big)\mathrm{d}x\mathrm{d}y \\ &= \int \big(|u_n(x)|^2 - |u_0(x)|^2 - |v_n(x)|^2\big) w(x-y)|v_n(y)|^2 \mathrm{d}x\mathrm{d}y + \\ &+ \int \big(|u_0(x)|^2 + |v_n(x)|^2\big) w(x-y)\big(|u_n(y)|^2 - |u_0(y)|^2 - |v_n(y)|^2\big)\mathrm{d}x\mathrm{d}y + \\ &+ \int |u_0(x)|^2 w(x-y)|u_0(y)|^2 \mathrm{d}x\mathrm{d}y + \\ &+ \int |v_n(x)|^2 w(x-y)|v_n(y)|^2 \mathrm{d}x\mathrm{d}y + \\ &+ 2\int |u_0(x)|^2 w(x-y)|v_n(y)|^2 \mathrm{d}x\mathrm{d}y\end{aligned}$$

We shall now estimate the first (I), second (II) and last term (III) and other terms cancel. For (III) we shall prove that

$$\int |u_0(x)|^2 w(x-y)|v_n(y)|^2 \mathrm{d}x\mathrm{d}y \longrightarrow 0$$

for this we split the integral into

$$\int\limits_{\mathbb{R}^d\times\mathbb{R}^d} |u_0(x)|^2 w(x-y)|v_n(y)|\mathrm{d}x\mathrm{d}y = \int\limits_{|y|\leqslant R} + \int\limits_{\substack{|y|\geqslant R\\ |x-y|\leqslant \frac{R}{2}}} + \int\limits_{\substack{|y|\geqslant R\\ |x-y|\geqslant \frac{R}{2}}} = \mathrm{III}_a + \mathrm{III}_b + \mathrm{III}_c$$

We have

$$\mathrm{III}_a = \int_{|y|\leqslant R} |u_0(x)|w(x-y)|v_n(y)|^2 = \int\limits_{|y|\leqslant R} \big(|w| * |u_0|^2\big)|v_n(y)|\mathrm{d}y$$

Since $\|w * |u_0|^2\|_\infty \leqslant C\|w\|_{L^p+L^q}\|u_0\|_{H^1}^2$ and thus

$$\mathrm{III}_a \leqslant C \int\limits_{|y|\leqslant R} |v_n(y)|_2 \mathrm{d}y \xrightarrow{n\to\infty} 0$$

for all $R > 0$, as $v_n \rightharpoonup 0$ weakly in H^1 and Sobolev.

$$\begin{aligned}
\mathrm{III}_b = \int\limits_{|y|\geqslant R |x-y|\leqslant \frac{R}{2}} &\leqslant \int\limits_{|x|\geqslant \frac{R}{2}} |u_0(x)|^2 |w(x-y)||v_n(y)|^2 = \int\limits_{|x|\geqslant \frac{R}{2}} |u_0(x)|\big(|w| * |v_n|^2\big)\mathrm{d}x \leqslant \\
&\leqslant C \underbrace{\|v_n\|_{H^1}^2}_{\substack{\text{bounded as} \\ n\to\infty}} \int\limits_{|x|\geqslant \frac{R}{2}} |u_0(x)|^2 \leqslant C \int\limits_{|x|\geqslant \frac{R}{2}} |u_0(x)|^2 \xrightarrow{R\to\infty} 0
\end{aligned}$$

and or the third term

$$\begin{aligned}
\mathrm{III}_c = \int\limits_{\substack{|y|\geqslant R \\ |x-u|>\frac{R}{2}}} &\leqslant \int |u_0(x)|^2 \Big(\mathbf{1}_{|x-y|>\frac{R}{2}} w(x-y)\Big)|v_n(y)|^2 \mathrm{d}x\mathrm{d}y \leqslant C\|\mathbf{1}_{B_{\frac{R}{2}}(0)^C} w\|_{L^p+L^q} \|u_0\|_{L^2}^2 \|v_n\|_{H^1}^2 \leqslant \\
&\leqslant C\|\mathbf{1}_{B_{\frac{R}{2}}(0)^C} w\|_{L^p+L^q} \xrightarrow{R\to\infty} 0.
\end{aligned}$$

For I we have

$$\begin{aligned}
\mathrm{I} &= 2\int |u_0(x)||v_n(x)||w(x-y)||u_0(y)|^2 \mathrm{d}y\mathrm{d}x \leqslant \\
&\leqslant \underbrace{\left(\int |u_0(x)|^2 |w(x-y)||v_n(y)|^2\right)^{1/2}}_{\leqslant C\|w\|_{L^p+L^q}\|u_0\|_2^2\|u_n\|_{H^1}^2 \leqslant C} \underbrace{\left(\int |v_n(x)|^2 |w(x-y)||u_n(y)|^2\right)^{1/2}}_{\text{Simiilar to III} \xrightarrow{n\to\infty} 0}
\end{aligned}$$

and the proof II goes similarly. *q.e.d.*

Proof of Theorem 9.6. Recall that u_n is a minimising sequence, $u_n \rightharpoonup u_0$, $v_n = u_n - u_0 \rightharpoonup 0$ weakly in H^1, then

$$\mathcal{E}(u_n) - \mathcal{E}(u_0) - \mathcal{E}^0(v_n) \xrightarrow{n\to\infty} 0$$

On the other hand we have

$$\begin{aligned}
\mathcal{E}(u_n) &\longrightarrow E(\lambda) \\
\mathcal{E}(u_0) &\geqslant E(\lambda - \lambda'), \qquad \text{for } \lambda - \lambda' = \int |u_0|^2 \leqslant \lambda \\
\mathcal{E}^0(v_n) &\geqslant E^0\Big(\int |v_n|^2\Big) \longrightarrow E^0(\lambda')
\end{aligned}$$

since

$$\int |v_n|^2 = \int |u_n - u_0|^2 = \underbrace{\int |u_n|^2}_{=\lambda} + \underbrace{\int |u_0|^2}_{=\lambda-\lambda'} - 2 \underbrace{\int u_n u_0}_{\to \int |u_0|^2 = \lambda - \lambda'} \longrightarrow \lambda'$$

Thus $E(\lambda) \geqslant E(\lambda - \lambda') + E^0(\lambda')$. However, by the strict binding inequality we have

$$E(\lambda) < E(\lambda - \lambda') + E^0(\lambda')$$

for all $0 < \lambda' \leqslant \lambda$. Thus we have to conclude that $\lambda' = 0$, which means that $\int |u_0|^2 = \lambda - \lambda' = \lambda$ and $\mathcal{E}(u_n) - \mathcal{E}(u_0) \to 0$ since $\mathcal{E}^0(v_n) \to 0$ as $\int |v_n|^2 \to \lambda' = 0$. Thus $\mathcal{E}(u_0) = E(\lambda)$ and $\int |u_0|^2 = \lambda$. So u_0 is a minimiser. *q.e.d.*

Theorem 9.9. □

Remark 9.10. □

Translation Invariant Cases

$$\mathcal{E}^0(u) = \int |\nabla u|^2 + \frac{1}{2} \int \int |u(x)|^2 w(x-y) |u(y)|^2 \mathrm{d}x\mathrm{d}y$$
$$E^0(\lambda) = \inf\{\mathcal{E}^0(u) \mid u \in H^1,\ \|u\|_2^2 = \lambda\}$$

Remark 9.11. $\mathcal{E}^0(u) = \mathcal{E}^0(u(\cdot + z))$ for all $z \in \mathbb{R}^d$.

- If u_n is a minimising sequence for $E^0(\lambda)$ then $\tilde{u}_n := u_n(\cdot + x_n)$, $x_n \in \mathbb{R}^d$ then $\tilde{u}_n$ is also a minimising sequence. But if $u_n \to u$ strongly in H^1 and $x_n \to \infty$, then $u_n \rightharpoonup 0$ weakly in H^1, i.e. we lack compactness or in other words one has compactness up to translation.

□

Definition 9.12 (Vanishing Sequence)**.** Let $(u_n)_n$ be bounded in $H^1(\mathbb{R}^d)$. We call $(u_n)_n$

a vanishing sequence if for all $(x_n)_n \subset \mathbb{R}^d$ and all subsequences of $(u_n)_n$, $u_n(\cdot + x_n) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^d)$. □

Theorem 9.13 (Characterisation of Vanishing Sequences). *If $(u_n)_n$ is bounded in $H^1(\mathbb{R}^d)$ and $(u_n)_n$ is vanishing, then*

- *For all $R > 0$*

$$\sup_{x\in\mathbb{R}^n} \int\limits_{B_R(x)} |u_n(y)|^2 \mathrm{d}y \xrightarrow{n\to\infty} 0$$

- *$u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$ for all $2 < p < p^*$ with*

$$p^* = \begin{cases} \frac{2d}{d-2}, & \text{if } d > 2 \\ \infty, & \text{if } d = 1, 2 \end{cases}$$

□

Proof. Let us assume that there exists a $R > 0$, $\varepsilon > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int\limits_{B_R(x)} |u_n|^2 \geqslant \varepsilon > 0.$$

Then there exists a sequence $(x_n)_n \subset \mathbb{R}^d$ such that

$$\int\limits_{B_R(x)} |u_n(x)|^2 \geqslant \frac{\varepsilon}{2} > 0$$

for all $n \in \mathbb{N}$. Define $v_n(x) = u_n(x + x_n)$. Then for all $n \in \mathbb{N}$

$$\int\limits_{B_R(0)} |v_n|^2 \geqslant \frac{\varepsilon}{2} > 0,$$

hence $v_n \not\rightharpoonup 0$ weakly in $H^1(\mathbb{R}^d)$ by Sobolev embedding, which is a contradiction. Thus for all $R > 0$,

$$\sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2 \xrightarrow{n\to\infty} 0$$

We shall consider the case $d \geqslant 3$. Let $p = 2 + \frac{4}{d}$, then

$$\int\limits_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \leqslant \left(\int\limits_{\mathbb{R}^d} |u_n|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{d}} \left(\int\limits_{\mathbb{R}^d} |u_n|^2\right)^{\frac{2}{d}} \leqslant c\left(\int\limits_{\mathbb{R}^d} |\nabla u_n|^2\right)\left(\int\limits_{\mathbb{R}^d} |u_n|^2\right)^{\frac{2}{d}} \leqslant C$$

Now we shall use a localisation argument. Take $Q := \left[-\frac{1}{2}, \frac{1}{2}\right]^d \subset \mathbb{R}^d$. Take $\varphi \in \mathscr{C}_c^\infty$, $0 \leqslant \varphi \leqslant 1$ with $\varphi\big|_Q \equiv 1$ and $\varphi|_{(2Q)^C} \equiv 0$. Take $z \in \mathbb{Z}^d$ and define $Q_z := Q + z$, and $\varphi_z = \varphi(\cdot + z)$. We have

$$1 \leqslant \sum_{z\in\mathbb{Z}} \varphi_z(x) \leqslant C, \qquad \sum_{z\in\mathbb{Z}^d} |\nabla\varphi_z(x)|^2 \leqslant C$$

and thus

$$\int\limits_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} = \sum_{z\in\mathbb{Z}^d} \int\limits_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \leqslant \sum_{z\in\mathbb{Z}^d} \left(\int\limits_{Q_z} |u_n|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{d}} \left(\int\limits_{Q_z} |u_n|^2\right)^{\frac{2}{d}}$$

Now note that

$$\|\mathbf{1}_{Q_z} u_n\|^2_{\frac{2d}{d-2}} \leqslant \|\varphi_z u_n\|^2_{\frac{2d}{d-2}} \leqslant C\|\nabla(\varphi_z u_n)\|_2^2 \leqslant 2C \int \big(|\nabla\varphi_z(x)|^2|u_n(x)|^2 + |\varphi_z(x)|^2|\nabla u_n(x)|^2\big)\mathrm{d}x$$

and thus

$$\begin{aligned}\int\limits_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} &\leqslant C\sum_{z\in\mathbb{Z}^d}\left(\int\limits_{\mathbb{R}^d} \big(|\nabla\varphi_z(x)|^2|u_n(x)|^2 + |\varphi_z(x)|^2|\nabla u_n(x)|^2\big)\mathrm{d}x\right)\left(\int\limits_{Q_z} |u_n|^2\right)^{\frac{2}{d}} \leqslant \\ &\leqslant C\sup_{z'\in\mathbb{Z}^d}\left(\int\limits_{Q'_z} |u_n|^2\right)^{\frac{2}{d}} \sum_{z\in\mathbb{Z}^d}\left(\int\limits_{\mathbb{R}^d} \big(|\nabla\varphi_z(x)|^2|u_n(x)|^2 + |\varphi_z(x)|^2|\nabla u_n(x)|^2\big)\mathrm{d}x\right) \leqslant \\ &\leqslant C\sup_{z'\in\mathbb{Z}^d}\left(\int\limits_{Q'_z} |u_n|^2\right)^{\frac{2}{d}}\left(\int\limits_{\mathbb{R}^d} \big(|u_n(x)|^2 + |\nabla u_n(x)|^2\big)\mathrm{d}x\right) \leqslant C\sup_{z\in\mathbb{Z}^d}\left(\int\limits_{Q_z} |u_n|^2\right)^{\frac{2}{d}} \longrightarrow 0\end{aligned}$$

by the convergence proven above as $\int\limits_{Q_z} |u_n|^2 \leqslant \int\limits_{B_2(z')} |u_n|^2$. Thus

$$\int\limits_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \xrightarrow{n\to\infty} 0$$

Now we prove $\int\limits_{\mathbb{R}^d} |u_n|^p \to 0$ for all $2 < p < p^* = \frac{2d}{d-2}$. By interpolation, if $2 < p < 2 + \frac{4}{d} = p_1$,

$$\|u_n\|_p \leqslant \underbrace{\|u_n\|_2^a}_{\leqslant C} \underbrace{\|u_n\|_{p_1}^{1-a}}_{\to 0}$$

for $a \in (0,1)$. Similarly $p_1 < p < p^*$ as $\|u_n\|_{p^*} \leqslant \|\nabla u\|_2 \leqslant C$. *q.e.d.*

We shall apply this to

$$\mathcal{E}^0(u) = \int |\nabla u|^2 + \frac{1}{2}\int\int |u(x)|^2 w(x-y)|u(y)|\mathrm{d}x\mathrm{d}y$$

for $w \in L^p + L^q$, with $\max\{1, \frac{d}{2}\} < p, q < \infty$ and

$$E^0(\lambda) = \inf\{\mathcal{E}^0(u) \,|\, u \in H^1(\mathbb{R}^d),\, \|u\|_2^2 = \lambda\}$$

Theorem 9.14 (Concentration Compactness for the Translation Invariant Case)**.** *Assume that $w \in L^p + L^q$ and*

$$E^0(\lambda) < E^0(\lambda - \lambda') + E^0(\lambda')$$

for all $0 < \lambda' < \lambda$ and $E^0(\lambda') < 0$ for all $0 < \lambda' \leqslant \lambda$, then $E^0(\lambda)$ has a minimiser. □

Proof. Take u_n to be a minimising sequence for $E^0(\lambda)$. Recall that for all $\varepsilon > 0$

$$E^0(\lambda) \longleftarrow \mathcal{E}^0(u_n) \geqslant (1-\varepsilon)\int |\nabla u_n|^2 - C_\varepsilon$$

thus u_n is bounded in $H^1(\mathbb{R}^d)$. We want to prove that u_n is non-vanishing. Assume by contradiction that u_n is vanishing,

$$0 > E^0(\lambda) \longleftarrow \mathcal{E}^0(u_n) = \int |\nabla u_n|^2 + \frac{1}{2}\int |u_n(x)|^2 (w * |u_n|^2)(x)\mathrm{d}x$$

which implies that

$$\int |u_n(x)|^2 \big(w * |u_n|^2\big)(x)\mathrm{d}x < -\varepsilon < 0$$

or all n large enough for some $\varepsilon > 0$.

However,

$$-\varepsilon > \int |u_n(x)|^2 \big(w * |u_n|^2\big)(x)\mathrm{d}x \geqslant \int\limits_{\mathbb{R}^d} |u_n(x)|^2 \mathrm{d}x \inf_{z\in\mathbb{R}^d} \big(w * |u_n|^2\big)(z)$$

which implies that

$$\inf_{z\in\mathbb{R}^d} \big(w * |u_n|^2\big)(z) < -\frac{\varepsilon}{\lambda}$$

for n large and therefore there exists a sequence $(z_n)_n \subset \mathbb{R}^d$ such that

$$\big(w * |u_n|^2\big)(z_n) < -\frac{\varepsilon}{2\lambda}$$

for n large. Thus

$$\int |u_n(x+z_n)|^2 w(x)\mathrm{d}x < -\frac{\varepsilon}{2\lambda}$$

and therefore

$$\int |u_n(x+z_n)|^2 w(-x)\mathrm{d}x < -\frac{\varepsilon}{2\lambda}$$

It follows that $u_n(\cdot + z_n) \rightharpoonup 0$ weakly in $H^1\big(\mathbb{R}^d\big)$, then

$$\int |u_n(x+z_n)|^2 w(-x)\mathrm{d}x \xrightarrow{n\to\infty} 0$$

because $w \in L^p + L^q$. Thus $u_n(\cdot + z_n) \not\rightharpoonup 0$ weakly. We know that $u_n(\cdot + z_n) \not\rightharpoonup 0$ weakly in $H^1(\mathbb{R}^d)$. Because $u_n(\cdot + z_n)$ is also a minimising sequence we can assume that $z_n = 0$, $u_n \not\rightharpoonup 0$ weakly in $H^1\big(\mathbb{R}^d\big)$ (otherwise we consider $\tilde{u}_n(x) = u_n(x + x_n)$). Since u_n is bounded in H^1, we can go to a subsequence such that $u_n \rightharpoonup u_0 \not\equiv 0$ weakly in H^1. Assume that $\int |u_n|^2 = \lambda$ and that for $\lambda' > 0$, $\int |u_n - u_0|^2 \to \lambda'$. We have already proven that

$$\underbrace{\mathcal{E}^0(u_n)}_{\to E^0(\lambda)} - \underbrace{\mathcal{E}^0(u_0) - \mathcal{E}^0(u_n - u_0)}_{\geqslant E^0(\|u_n - u_0\|_2^2) \to E(\lambda')} \xrightarrow{n\to\infty} 0$$

from which follows that

$$\mathcal{E}^0(u_0) \leqslant E^0(\lambda) - E^0(\lambda') \leqslant E^0(\lambda - \lambda')$$

Thus u_0 is minimiser for $E^0(\lambda-\lambda')$ and $E^0(\lambda)+E^0(\lambda-\lambda')$. By the strict inequality $\lambda-\lambda'=\lambda$ and thus $\|u_n\|_2^2=\lambda$ and u_0 is a minimiser for $E(\lambda)$. *q.e.d.*

Applications of the Concentration-Compactness Principle

Definition 9.15 (Choquard-Pekar Problem)**.**

$$\mathcal{E}(u) := \int\limits_{\mathbb{R}^3} |\nabla u|^2 + \int\limits_{\mathbb{R}^3} V(x)|u(x)|^2 \mathrm{d}x - \frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y$$

$$E(\lambda) := \inf\left\{\mathcal{E}(u), \middle|\, u \in H^1\left(\mathbb{R}^3\right),\ \|u\|_2^2=\lambda\right\}$$

□

Theorem 9.16. *If $V \in L^p + L^q(\mathbb{R}^3)$, $p,q \in \left(\frac{3}{2},\infty\right)$ and $V \leqslant 0$ then for all $\lambda>0$, $E(\lambda)$ has a minimiser. Moreover, the minimiser solves*

$$-\Delta u_0 + V u_0 - \left(|u_0|^2 * \frac{1}{|x|}\right)u_0 = \mu u_0 \qquad \text{in } \mathscr{D}'\left(\mathbb{R}^3\right).$$

□

Proof.

$V \equiv 0$

$$\mathcal{E}^0(u) := \int\limits_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y$$

$$E^0(\lambda) := \inf\left\{\mathcal{E}^0(u), \middle|\, u \in H^1\left(\mathbb{R}^3\right),\ \|u\|_2^2=\lambda\right\}$$

From the concentration compactness principle, we need to check

a) $E^0(\lambda) < 0$

b) $E^0(\lambda) < E^0(\lambda-\lambda') + E^0(\lambda')$ for all $0<\lambda'<\lambda$.

Proof. a) Take $\varphi \in H^1(\mathbb{R}^3)$, $\varphi \not\equiv 0$, $\|\varphi\|_2^2 = \lambda$. For $\ell > 0$, let $\varphi_\ell(x) = \ell^{3/2}\varphi(\ell x)$, $\|\varphi_\ell\|_2^2 = \|\varphi\|_2^2 = \lambda$ and

$$\mathcal{E}^0(\varphi_\ell) = \ell^2 \int |\nabla\varphi|^2 - \ell\frac{1}{2}\int\int \frac{\varphi(x)\varphi(y)}{|x-y|}\mathrm{d}x\mathrm{d}y = A\ell^2 - B\ell < 0$$

if $\ell > 0$ small enough, as $A > 0, B > 0$. Thus $E^0(\lambda) \leqslant \mathcal{E}^0(\varphi_\ell) < 0$ if $\ell > 0$ small enough.

b) It follows from the following lemma that for all $0 < \lambda' < \lambda$

$$E^0(\lambda) = \frac{\lambda - \lambda'}{\lambda} E^0(\lambda) + \frac{\lambda'}{\lambda} < E(\lambda - \lambda') + E^0(\lambda')$$

q.e.d.

We can thus conclude that $E^0(\lambda)$ has a minimiser. Then by using variational formulae

$$\mathcal{E}^0\left(\frac{(u_0 + \varepsilon\varphi)\sqrt{\lambda}}{\|u_0 + \varepsilon\varphi\|_2}\right) \geqslant \mathcal{E}^0(u_0)$$

for all $\varepsilon \in \mathbb{R}$ small and thus

$$0 = \frac{d}{d\varepsilon}(\cdots)\Big|_{\varepsilon=0} \implies -\Delta u_0 - \left(|u_0|^2 * \frac{1}{|x|}\right) u_0 = \mu u_0$$

with $\mu \leqslant 0$, and $\lambda \mapsto E^0(\lambda)$ is decreasing.

$V \leqslant 0. V \not\equiv 0$ We need to prove the binding inequality

$$E(\lambda) < E(\lambda - \lambda') + E^0(\lambda')$$

for all $0 < \lambda' \leqslant \lambda$. Using the second of the following lemmata we can conclude that

$$E(\lambda) = \frac{\lambda - \lambda'}{\lambda} E(\lambda) + \frac{\lambda'}{\lambda} E(\lambda) \leqslant E(\lambda - \lambda') + E(\lambda')$$

To conclude, we need to show that $E(\lambda') < E^0(\lambda')$ for all $0 < \lambda' \leqslant \lambda$. By the case $V \equiv 0$, we have $E^0(\lambda')$ has a minimiser, $u_{\lambda'}$ and

$$E(\lambda') - E^0(\lambda') \leqslant \mathcal{E}(u_{\lambda'}) - \mathcal{E}^0(u_{\lambda'}) = \int V(x)|u_{\lambda'}(x)|^2 \mathrm{d}x$$

Assume for the sake of contradiction that $E(\lambda') = E^0(\lambda')$ for which we would need

$$\int V(x)|u_{\lambda'}(x)|^2 \mathrm{d}x \geqslant 0$$

and thus $V(x)|u_{\lambda'}(x)|^2 = 0$ a.e. (since $V \leqslant 0$). Thus $V(x) = 0$ for a.e. x such that

$u_{\lambda'}(x) \neq 0$. Since $\mathcal{E}^0(u)$ is translation invariant, $u_{\lambda'}$ is a minimiser for $E^0(\lambda')$. Thus $u_{\lambda'}(\cdot + y)$ is also a minimiser for $E^0(\lambda')$ for all $y \in \mathbb{R}^3$.

By the above argument it follows that $V(x) = 0$ for a.e. x such that $u_{\lambda'}(x+y) \neq 0$ for all $y \in \mathbb{R}^3$.

Here $\int |u_{\lambda'}|^2 = \lambda' > 0$ and thus $u_{\lambda'} \not\equiv 0$, hence there must exists a ball $B_r(z)$ such that $u_{\lambda'} \neq 0$ for a.e. $x \in B_r(z)$. Hence $V(x) = 0$ for a.e. $x \in \mathbb{R}^3$ which is a contradiction to the assumption $V \not\equiv 0$.

Thus $E(\lambda') < E^0(\lambda')$ for all $0 < \lambda'$ and $E(\lambda) < E(\lambda - \lambda') + E^0(\lambda')$ for all $0 < \lambda' < \lambda$.

Therefore, $E(\lambda)$ has a minimiser and the equation follows similarly to $E^0(\lambda)$.

q.e.d.

Lemma 9.17. *For all $\lambda > 0$, for all $0 < \vartheta < 1$*

$$\vartheta E^0(\lambda) < E^0(\vartheta\lambda).$$

□

Proof. Take f_n a minimising sequence for $E^0(\vartheta\lambda)$, i.e. $\|f_n\|_2^2 = \vartheta\lambda$, $\mathcal{E}^0(f_n) \to E^0(\vartheta\lambda)$. Define $g_n = \frac{f_n}{\sqrt{\vartheta}}$, $\|g_n\|_2^2 = \lambda$.
Thus

$$\begin{aligned} E^0(\lambda) \leqslant \mathcal{E}^0(g_n) = \mathcal{E}^0\left(\frac{f_n}{\sqrt{\vartheta}}\right)\frac{1}{\vartheta}\int |\nabla f_n|^2 - \frac{1}{\vartheta^2}\int\int \frac{|f_n(x)|^2|f_n(y)|^2}{|x-y|} = \\ = \frac{1}{\vartheta}\mathcal{E}^0(f_n) + \left(\frac{1}{\vartheta} - \frac{1}{\vartheta^2}\right)\frac{1}{2}\int\int \frac{|f_n(x)|^2|f_n(y)|^2}{|x-y|} \end{aligned}$$

Using $\mathcal{E}^0(f_n) \to E^0(\vartheta\lambda)$ and

$$\frac{1}{2}\int\int \frac{|f_n(x)|^2||f_n(y)|^2}{|x-y|} = \int |\nabla f_n|^2 - \mathcal{E}^0(f_n) \geqslant -\mathcal{E}^0(f_n) \longrightarrow -E(\vartheta\lambda)$$

and thus

$$E^0(\lambda) \leqslant \frac{1}{\vartheta}E^0(\vartheta\lambda) + \left(\frac{1}{\vartheta} - \frac{1}{\vartheta^2}\right)\left(-E^0(\vartheta\lambda)\right) = \frac{E^0(\vartheta\lambda)}{\vartheta^2} < \frac{E^0(\vartheta\lambda)}{\vartheta}$$

since $0 < \vartheta < 1, E^0(\vartheta\lambda) < 0$.

q.e.d.

Lemma 9.18. *Suppose that* $V \leqslant 0, V \not\equiv 0$. *For all* $\lambda > 0$, *for all* $0 < \vartheta < 1$

$$\vartheta E(\lambda) \leqslant E(\vartheta\lambda).$$

□

Proof. Similar to the previous lemma. *q.e.d.*

Gagliardo-Nirenberg Interpolation Inequality

$$\|\nabla u\|_2^\alpha \|u\|_2^{d-\alpha} \geqslant c\|u\|_p$$

for all $2 < p < p^*$ with $p^* = \frac{2d}{d-2}$ if $d \geqslant 3$ and $p^* = \infty$ if $d = 1, 2$. By a scaling argument $\frac{1}{p} = \frac{d-2}{2d}\alpha + \frac{1-\alpha}{2}$, $\alpha \in (0,1)$.

Remark 9.19. $u_\ell(x) = \ell^{\frac{d}{2}} u(\ell x)$, $\|u_\ell\|_2 = \|u\|_2$

$$\frac{\|\nabla u_\ell\|_2^\alpha \|u_\ell\|_2^{1-\alpha}}{\|u_\ell\|_p}$$

is independent of ℓ. □

Theorem 9.20. *For these* p, α, *then the variational problem*

$$E = \inf\left\{ \frac{\|\nabla u\|_2^\alpha \|u_2\|_2^{1-\alpha}}{\|u\|_p} \,\middle|\, u \in H^1\big(\mathbb{R}^d\big), u \not\equiv 0 \right\}$$

has a minimiser. The minimiser can be chosen such that $Q \geqslant 0$ *and*

$$-\Delta Q + Q - Q^{p-1} = 0, \qquad \text{in } \mathscr{D}'(\mathbb{R}^n).$$

□

Thomas Fermi Problem

$$\mathcal{E}(\rho) = \int\limits_{\mathbb{R}^3} \rho^{5/3} - \int \frac{Z}{|x|}\rho(x)\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^3} \frac{1}{2}\int\limits_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y$$

$$E(\lambda) = \left\{ \inf \mathcal{E}(\rho) \,\middle|\, \rho \geqslant 0, \rho \in L^1 \cap L^{5/3}, \int \rho = \lambda \right\}$$

Theorem 9.21. *Let $Z > 0$ constant. Then for all $\lambda \in (0, Z]$, $E(\lambda)$ has a unique minimiser. Moreover the minimiser ρ_0 satisfies*

$$\frac{5}{3}\rho_0^{2/3}(x) = \left[\frac{Z}{|x|} - \rho_0 * \frac{1}{|x|} + \mu\right]_+$$

for some constant $\mu \leqslant 0$. Moreover, $E(\lambda)$ has no minimiser if $\lambda > Z$. □

Proof. Take a minimising sequence ρ_n for $E(\lambda)$. We want to prove ρ_n is bounded in $L^{5/3}$.

$$\int \frac{Z}{|x|}\rho_n(x) = \int\limits_{|x|\leqslant 1} + \int\limits_{|x|>1} \leqslant Z\left(\int\limits_{|x|\leqslant 1} \frac{1}{|x|^{5/2}}\right)^{2/5}\left(\int\limits_{|x|\leqslant 1} \rho_n^{5/3}\right)^{3/5} + Z\int\limits_{|x|>1} \rho_n(x)\mathrm{d}x \leqslant CZ\left(\int \rho_n^{5/3}\right)^{3/5} + Z\lambda$$

This implies that

$$E(\lambda) \longleftarrow \mathcal{E}(\rho_n) - \int \rho_n^{5/3} - CZ\left(\int \rho_n^{5/3}\right)^{3/5} - Z\lambda$$

Thus $E(\lambda) > -\infty$ and ρ_n is bounded in $L^{5/3}$. By going to a subsequence we may assume that $\rho_n \rightharpoonup \rho_0$ weakly in $L^{5/3}$. We have to prove that

$$\liminf \mathcal{E}(\rho_n) \geqslant \mathcal{E}(\rho_0)$$

By weak convergence we have

$$\liminf_{n\to\infty}\int \rho_n^{5/3} \geqslant \int \rho_0^{5/3}$$
$$\lim_{n\to\infty}\int \frac{Z\rho_n(x)}{|x-y|} = \int \frac{Z\rho_0(x)}{|x-y|}$$
$$\liminf_{n\to\infty}\int\int \frac{\rho_n(x)\rho_n(y)}{|x-y|} \geqslant \int\int \frac{\rho_0(x)\rho_0(y)}{|x-y|}$$

where the last one is an exercise. Thus

$$E(\lambda) = \lim \mathcal{E}(\rho_n) \geqslant \mathcal{E}(\rho_0) \geqslant E(\lambda_0)$$

with $\lambda_0 = \int \rho_0$.

To prove that ρ_0 is a minimiser for $E(\lambda)$, need to prove that $\lambda_0 = \lambda$. Assuming that $\lambda_0 < \lambda$. Then $E(\lambda) \geqslant \mathcal{E}(\rho_0) \geqslant E(\lambda_0) \geqslant E(\lambda)$, hence

$$\mathcal{E}(\rho_0) = E(\lambda_0) = E(\lambda) = E(\lambda')$$

for all $\lambda' \in [\lambda_0, \lambda]$.

Concerning the variational equation for ρ_0 we have $\mathcal{E}(\rho_0 + \varepsilon\varphi) \geqslant \mathcal{E}(\rho_0)$ for all $\varphi \in L^1 \cap L^{5/3}$, $\varphi \geqslant 0$ and $\varepsilon \geqslant 0$ small enough. Thus

$$\frac{d}{d\varepsilon}\mathcal{E}(\rho_0 + \varepsilon\varphi)\Big|_{\varepsilon=0} \geqslant 0$$

Thus

$$\int \frac{5}{3}\rho^{2/3}\varphi - \int \frac{Z}{|x|}\rho_0\varphi + \int \left(\rho_0 * \frac{1}{|x|}\right)\varphi \geqslant$$

and therefore

$$\int \left(\frac{5}{3}\rho_0^{2/3} - \frac{Z}{|x|} + \rho_* * \frac{1}{|x|}\right)\varphi \geqslant 0$$

for all $\varphi \in L^1 \cap L^{5/3}$, and $\varphi \geqslant 0$. Using he following lemma it follows that

$$\frac{5}{3}\rho_0^{2/3}(x) - \frac{Z}{|x|} + \rho_0 * \frac{1}{|x|} \geqslant 0$$

Contradiction to $\int \rho_0 = \lambda_0 < \lambda \leqslant Z$. Using the convexity we find that ρ_0 is a minimiser for $E(\lambda_0)$ implies that ρ_0 is unique.

??????????????????????????????????

Assume that $E(\lambda)$ has a minimiser ρ_0 (λ not necessarily $\leqslant Z$), then for all $\rho \in L^1 \cap L^{5/3}$, $\int \rho = \lambda$

$$\mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho)$$

Choose $\rho_\varepsilon = \rho_0 + \varepsilon\varphi$, for $\varphi \in L^1 \cap L^{5/3}$, $\int \varphi = 0$ and $\varphi(x) \geqslant -C\rho_0(x)$ for all x. Then

$$\mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho_\varepsilon)$$

for all $\varepsilon \geqslant 0$ small enough. This implies that

$$\frac{d}{d\varepsilon}\mathcal{E}(\rho_\varepsilon)\bigg|_{\varepsilon=0} \geqslant 0$$

And therefore

$$\int \underbrace{\left(\frac{5}{3}\rho^{2/3} - \frac{Z}{|x|} + \rho_0 * \frac{1}{|x|}\right)}_{=:W} \varphi \geqslant 0$$

Choose $\varphi = g - \frac{\int g}{\lambda}\rho_0$, $\int \varphi = \int g - \frac{\int g}{\lambda}\int \rho_0 = 0$ with $g \in L^1 \cap L^{5/3}$, $g(x) \geqslant -C\rho_0(x)$. This implies that

$$0 \leqslant \int W\varphi = \int W\left(g - \frac{\int \rho}{\lambda}\rho_0\right) = \int Wg - \frac{\int W\rho_0}{\lambda}\int \rho = \int (W-\mu)\rho$$

with $\mu := \frac{\int W\rho_0}{\lambda} \in \mathbb{R}$. We deduce that

$$\begin{cases} W(x) - \mu = 0, & \text{if } \rho_0(x) > 0 \\ W(x) - \mu \geqslant 0, & \text{for all } x \in \mathbb{R}^3 \end{cases}$$

and therefore

$$\frac{5}{3}\rho_0^{2/3} - \frac{Z}{|x|} + \rho_0 * \frac{1}{|x|} - \mu \begin{cases} = 0, & \text{if } \rho_0(x) > 0 \\ \geqslant 0, & \text{for all } x \in \mathbb{R}^3 \end{cases}$$

which in turn implies

$$\frac{5}{3}\rho_0^{2/3} \begin{cases} = \frac{Z}{|x|} - \rho_0 * \frac{1}{|x|} + \mu, & \text{if } \rho_0(x) > 0 \\ \geqslant \frac{Z}{|x|} - \rho_0 * \frac{1}{|x|} + \mu, & \text{for all } x \in \mathbb{R}^3 \end{cases}$$

and thus

$$\frac{5}{3}\rho_0^{2/3} = \left[\frac{Z}{|x|} - \rho_0 * * \frac{1}{|x|} + \mu\right]_+$$

Now we shall show that $\mu \leqslant 0$. Assume that $\mu > 0$. Then the Thomas Fermi equation reads

$$\frac{5}{3}\rho^{2/3} \geqslant \mu - \left|\frac{Z}{|x|} - \rho_0 * \frac{1}{|x|}\right|$$

and

$$\rho_0 * \frac{1}{|x|} = \int \frac{\rho_0(y)}{|x-y|}\mathrm{d}y = \int \frac{\rho_0(y)}{\max\{|x|,|y|\}} \leqslant \frac{\int \rho_0}{|x|} = \frac{\lambda}{|x|}$$

which implies that

$$\mu \leqslant \frac{5}{3}\underbrace{\rho_0^{2/3}}_{\in L^{3/2}} + \frac{Z+\lambda}{|x|}$$

and therefore $\mu \leqslant 0$.

$$\frac{5}{3}\rho^{2/3} \geqslant \mu - \frac{Z+\lambda}{|x|} \overset{\mu>0}{\geqslant} \frac{\mu}{2}$$

for $|x|$ large this implies

$$\underbrace{\left(\frac{5}{3}\rho_0^{2/3}\right)^{\frac{3}{2}}}_{\in L^1} \geqslant \left(\frac{\mu}{2}\right)^{3/2}$$

for $|x|$ large.

We remark here that $\mu < 0$ if $\lambda = \int \rho < Z$ and that $\mu = 0$ if $\lambda = Z$, the proof of which is left as an exercise.

We shall now prove the non-existence of a minimiser for $\lambda > Z$. The proof presented was first given by Simon-Lieb. We have the Thomas Fermi equation

$$\frac{5}{3}\rho^{2/3} = \left[\frac{Z}{|x|} - \rho_* * \frac{1}{|x|} + \mu\right]_+, \qquad \mu \leqslant 0$$

Assume that $\int \rho_0 > Z$, and define

$$f(x) := \frac{Z}{|x|} - \rho_* * \frac{1}{|x|} + \mu$$

- $f(x) < 0$ if $|x|$ large. Since

$$f(x) \leqslant \frac{Z}{|x|} - \rho_* * \frac{1}{|x|} = \frac{Z}{|x|} - \int \frac{\rho(y)}{\max\{|x|,|y|\}} \mathrm{d}y \leqslant \frac{Z}{|x|} - \int\limits_{|y|\leqslant R} \frac{\rho_0(y)}{\max\{|x|,|y|\}} \mathrm{d}y =$$
$$= \left(Z - \int\limits_{|y|\leqslant R} \rho_0 \right) \frac{1}{|x|}$$

if $|x| \geqslant R$. Since

$$\int\limits_{|y|\leqslant R} \rho_0 \xrightarrow{R\to\infty} \int\limits_{\mathbb{R}^3} \rho_0 = \lambda > Z \implies Z - \int\limits_{|y|\leqslant R} \rho_0 < 0$$

if R large.

- $f(x) > 0$ if $|x|$ is small enough

$$f(x) = \frac{Z}{|x|} - \rho_0 * \frac{1}{|x|} + \mu = \frac{Z}{|x|} - \int \frac{\rho_0(y)}{\max\{|x|,|y|\}} \mathrm{d}y + \mu \geqslant \frac{Z}{|x|} - \int \frac{\rho_0(y)}{|y|} + \mu > 0$$

if $|x|$ small.

- The Thomas Fermi equation reeds

$$\frac{5}{3}\rho_0^{2/3} = [f(x)]_+ \implies \rho_0 = 0$$

if $|x|$ large enough since $f(x) < 0$. Define $\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, f(x) < 0 \right\}$. Ω is open, $\Omega \neq \emptyset$ and $0 \notin \Omega$.

On Ω, we have

$$\Delta f(x) = \Delta \left(\frac{Z}{|x|} - \rho_* * \frac{1}{|x|} + \mu \right) = 4\pi\rho_0 \overset{\text{TF}}{=} 0$$

as $-\Delta \frac{1}{|x|} = 4\pi \mathrm{d}elta_0$. Thus f is harmonic on Ω.

By the maximum principle $\inf_\Omega f \geqslant \inf_{\partial\Omega} f = 0$, which is a contradiction.

We shall now present a second proof. Using the Thomas-Fermi equation

$$\frac{5}{3}\rho_0^{2/3} = \left[\frac{Z}{|x|} - \rho_0 * \frac{1}{|x|} + \mu \right]_+$$

which implies that

$$\underbrace{\frac{5}{3}\rho_0^{5/3}}_{\geqslant 0} = \frac{Z}{|x|}\rho_0 - \left(\rho_* * \frac{1}{|x|}\right)\rho_* + \underbrace{\mu\rho_0}_{\leqslant 0}$$

and thus

$$\frac{Z}{|x|}\rho_0(x) \geqslant \left(\rho_0 * \frac{1}{|x|}\right)\rho_0(x)$$

for all x. Integrating against $|x|^k \mathbf{1}_{|x|\leqslant R}$ we find that

$$\int\limits_{|x|\leqslant R} \frac{Z}{|x|}|x|^k\rho(x)\mathrm{d}x \geqslant \int\limits_{|x|\leqslant R} \left(\rho_0 * \frac{1}{|x|}\right)|x|^k\rho_0(x)\mathrm{d}x \leqslant \int\limits_{|x|\leqslant R}\int\limits_{|y|\leqslant R} \frac{\rho_0(y)|x|^k\rho(y)}{\max\{|x|,|y|\}}\mathrm{d}x\mathrm{d}y$$

Using the elementary inequality

$$\forall x, y \in \mathbb{R}^3 \setminus \{0\} : \frac{|x|^k + |y|^k}{2\max\{|x|,|y|\}} \geqslant \frac{|x|^{k-1} + |y|^{k-1}}{2}\left(1 - \frac{1}{k}\right).$$

Now

$$\int\limits_{|x|\leqslant R} Z|x|^{k-1}\rho_0(x)\mathrm{d}x \geqslant \int\limits_{|x|\leqslant R}\int\limits_{|y|\leqslant R} \rho_0(x)\rho_0(y)\left(1 - \frac{1}{k}\right)\left(\frac{|x|^{k-1} + |y|^{k-1}}{2}\right)\mathrm{d}x\mathrm{d}y =$$

$$= \left(\int\limits_{|x|\leqslant R} \rho_0(x)|x|^{k-1}\mathrm{d}x\right)\left(\int\limits_{|y|\leqslant R} \rho_0(y)\mathrm{d}y\right)\left(1 - \frac{1}{k}\right)$$

which implies that

$$Z \geqslant \left(\int\limits_{|y|\leqslant R} \rho_0(y)\mathrm{d}y\right)\left(1 - \frac{1}{k}\right)$$

for all $R > 0$, for all $k \in \mathbb{N}$. Passing $R \to \infty$ and $k \to \infty$ we find that

$$Z \geqslant \int \rho_0 = \lambda$$

To prove of the elementary inequality we need to prove that for $M \geqslant m > 0$, then

$$\frac{M^k + m^k}{M} \geqslant \left(1 - \frac{1}{k}\right)\left(M^{k-1} + m^{k-1}\right) \iff \left(M^{k-1} + \frac{m^k}{M}\right)k \geqslant (k-1)\left(M^{k-1} + m^{k-1}\right) \iff$$

$$\iff M^{k-1} + k\frac{m^k}{M} \geqslant (k-1)m^{k-1}$$

Using the Arithmetic Mean- Geometric Mean, in equality, i.e. that for all $a_1, \ldots, a_k \geqslant 0$

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geqslant \sqrt[k]{a_1 a_2 \cdots a_k}$$

consequently

$$M^{k-1} + \underbrace{\frac{m^k}{M} + \frac{m^k}{M} + \cdots + \frac{m^k}{M}}_{\text{k -1}} \geqslant k \left(M^{k-1} \left(\frac{m^k}{M} \right)^{k-1} \right)^{\frac{1}{k-1}}$$

from which the inequality follows.

q.e.d.

Lemma 9.22. $\lambda \mapsto E(\lambda)$ *is decreasing.* $\square$

Lemma 9.23. *If* $\int f\varphi \geqslant 0$*, for all* $\varphi \in \mathscr{D}$*,* $\varphi \geqslant 0$*, then* $f \geqslant 0$ *a.e.* $\square$

Lemma 9.24. $\rho \mapsto \mathcal{E}(\rho)$ *is a convex functional.*

$$\mathcal{E}(\rho_1) + \mathcal{E}(\rho_2) > 2\mathcal{E}\left(\frac{\rho_1 + \rho_2}{2} \right)$$

$\square$

Chapter 10

Boundary Value Problem

Example 10.1. Let Ω be open, bounded in $\mathbb{R}^d$.

1) The Dirichlet problem

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

2) The von Neumann problem

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= \eta \qquad \text{on } \partial\Omega \end{aligned}$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot n$, where n is the unit normal vector field to the boundary surface, if it exists.

We need

- Sobolev spaces in Ω
- Value of H^1 function on $\partial\Omega \rightsquigarrow$ trace theorem, as for $d \geqslant 2$ $H^1(\mathbb{R}^d) \not\subset \mathscr{C}(\mathbb{R}^d)$.

Definition 10.2.

$$H^m(\Omega) := \left\{ f \in L^2(\Omega) \,\middle|\, D^\alpha f \in L^2(\Omega),\ |\alpha| \leqslant m \right\}$$

where $D^\alpha f = g$ in $\mathscr{D}'(\Omega)$ iff

$$(-1)^{|\alpha|}\int f(D^\alpha\varphi) = \int g\varphi$$

for all $\varphi \in \mathscr{C}_c^\infty(\Omega)$. □

Theorem 10.3. *$H^m(\Omega)$ is a Hilbert space for every $m \in \mathbb{N}$, with norm*

$$\|u\|_{H^m}^2 := \sum_{|\alpha|\leqslant m} \|D^\alpha u\|_2^2$$

□

We want given $u \in H^1(\Omega)$, find a $\tilde{u} \in H^1(\mathbb{R}^d)$ such that $\tilde{u}\big|_\Omega = u$. For this we need some smoothness of $\partial\Omega$.

Example 10.4. Extension by reflection. Let $x \in \mathbb{R}^d$, with $x = (x', x_d)$, $x' = (x_1, \dots, x_{d-1})$. Let

$$Q = \{x \in \mathbb{R}^d \,\big|\, |x'| < 1, |x_d| < 1\}$$

which for example is a cylinder in $d = 3$. Further let

$$Q_+ := \{x \in Q \,\big|\, x_d > 0\}, \quad Q_0 := \{x \in Q \,\big|\, x_d = 0\}, \quad Q_- := \{x \in Q \,\big|\, x_d < 0\}$$

Theorem 10.5. *Given $u \in H^1(Q_+)$, define*

$$u^*(x', x_d) := \begin{cases} u(x', x_d), & \textit{if } (x', x_d) \in Q_+ \\ -u(x', -x_d), & \textit{if } (x', x_d) \in Q_- \end{cases}$$

Then $u^* \in H^1(Q)$ and $\|u^*\|_{H^1(Q)} \leqslant 2\|u\|_{H^1(Q_+)}$ and $\|u^*\|_{L^2(Q)} \leqslant 2\|u\|_{L^2(Q_+)}$ □

Proof. We have

$$\partial_{x_i} u^* = (\partial_{x_i} u)^*$$

if $i = 1, \dots, d-1$ and

$$\partial_{x_d} u^* = \begin{cases} \partial u_d(x', x_d), & \text{if } x_d > 0 \\ -\partial u_d(x', -x_d), & \text{if } x_d < 0 \end{cases}$$

in the distributional sense. If $u \in \mathscr{C}^\infty$ then this is trivial. In the general case $u \in H^1(\Omega)$ and let $\varphi \in \mathscr{C}_c^\infty(Q)$. We want to prove

$$\int_Q u^*(x', x_d)\partial_{x_d}\varphi \mathrm{d}x = -\Big(\int_{Q_+} \partial_{x_d}u^*(x', x_d)\varphi \mathrm{d}x + \int_{Q_-} (\partial_{x_d}u^*(x', -x_d))\varphi \mathrm{d}x\Big)$$

Defining $\tilde{\varphi}(x', x_d) = \varphi(x', x_d) - \varphi(x', -x_d)$ with $(x', x_d) \in Q_+$ then this is equivalent to

$$\int\limits_{Q_+} u\partial_{x_d}\tilde{\varphi} = -\int\limits_{Q_+} \partial_{x_d}u\tilde{\varphi}$$

This is trivial if $\tilde{\varphi} \in \mathscr{C}_c^\infty(Q_+)$. More generally consider $\eta_\varepsilon\tilde{\varphi} \in \mathscr{C}_c^\infty(Q_+)$ with $\eta_\varepsilon(x_d) = \eta\big(\frac{x_d}{\varepsilon}\big)$ with $\eta(t) =$ if $t \leqslant \frac{1}{2}$, $\eta(t) = 1$ if $t \geqslant 1$ and $\eta \in \mathscr{C}^\infty$. Per definitionem of $\partial_{x_d}u$ in Q_+, we have

$$\int\limits_{Q_+} u(\partial_{x_d}(\eta_\varepsilon\tilde{\varphi})) = \int\limits_{Q_+} \partial_d u(\eta_\varepsilon\tilde{\varphi}).$$

Taking $\varepsilon \to 0$ we find that

$$\int\limits_{Q_+} \partial_{x_d}u(\eta_\varepsilon\tilde{\varphi}) \longrightarrow \int \partial_{x_d}u\tilde{\varphi}$$

by dominated convergence as $\eta_\varepsilon(x_d) \to 1$ and

$$|\partial_{x_d}u(\eta_\varepsilon\tilde{\varphi})| \leqslant C|\partial_{x_d}u\tilde{\varphi}| \in \mathrm{Ł}^1(Q_+)$$

Moreover,

$$\int\limits_{Q_+} u(\partial_{x_d}(\eta_\varepsilon\tilde{\varphi})) = \int\limits_{Q_+} u(\partial_{x_d}\eta_\varepsilon)\tilde{\varphi} + \int\limits_{Q_+} u\eta_\varepsilon\partial_{x_d}\tilde{\varphi}$$

Here

$$\int\limits_{Q_+} u\eta_\varepsilon\partial_{x_d}\tilde{\varphi} \longrightarrow \int u\partial_{x_d}\tilde{\varphi}$$

by dominated convergence. It remains to prove that $\int_{Q_+} u(\partial_{x_d}\eta_\varepsilon)\tilde{\varphi} \to 0$. Because $\eta_\varepsilon = \eta\big(\frac{x_d}{\varepsilon}\big)$ we have

$$|\partial_{x_d}\eta_\varepsilon| \leqslant \frac{C}{\varepsilon}\mathbf{1}_{\{0<|x_d|<\varepsilon\}}$$

And $\mathscr{C}^1(Q) \ni \tilde{\varphi}(x', x_d) = \varphi(x', x_d) - \varphi(x', -x_d)$ and $\varphi(x', 0) = 0$. Thus we have

$$|\tilde{\varphi}(x', x_d)| \leqslant C|x_d| \leqslant C\varepsilon$$

if $0 < |x_d| < \varepsilon$. Thus

$$\left|\int_{Q_+} u(\partial_{x_d}\eta_\varepsilon)\tilde{\varphi}\right| \leqslant \int_{Q_+} u\frac{C}{\varepsilon}\mathbf{1}_{\{0<|x_d|<\varepsilon\}}c\varepsilon = Cc \int_{Q_+\cap\{0<|x_d|<\varepsilon\}} u \longrightarrow 0$$

by dominated convergence. We conclude that needed equality is correct.

q.e.d.

Definition 10.6 (Extension Problem)**.** If $u \in H^1(\Omega)$, when does there exist a $Pu \in H^1(\mathbb{R}^d)$ such that, $Pu\big|_\Omega = u$, $\|Pu\|_{H^1} \leqslant C\|u\|_{H^1}$. □

Example 10.7. Let $\Omega = [0,1]^d \subset \mathbb{R}^d$. Then extension is easy by reflection we can extend $u \in H^1(\Omega)$ by $\tilde{u} \in H^1(\Omega')$ with $\overline{\Omega} \subset \Omega'$ such that $\eta = 1$ on Ω. Define $\eta\tilde{u} \in H^1(\Omega')$ and as compact support. Extend $\eta\tilde{u}$ to $H^1(\mathbb{R}^d)$ setting it to 0 outside Ω'. Thus the of $u \in H^1(\Omega)$

Theorem 10.8 (Urysohn's Lemma)**.** *If Ω, Ω' are open with $\overline{\Omega} \subset \Omega'$ then there exists $\eta \in \mathscr{C}_c^\infty(\Omega')$ such that $\eta = 1$ on Ω.* □

Definition 10.9 ($\mathscr{C}^1$- boundary condition on Ω)**.** Let Ω be open, bounded set in $\mathbb{R}^d$. We say that $\partial\Omega$ is $\mathscr{C}^1$ if for all $x \in \partial\Omega$, there exists an open neighbourhood such that there exists $h : U \to Q$ satisfying

- $h \in \mathscr{C}^1$ and $h^{-1} \in \mathscr{C}^1$,
- $h(U \cap \Omega) = Q_+$,
- $h(U \cap \partial\Omega) = Q_0$.

□

Theorem 10.10. *Assume that Ω is open and bounded and has a $\mathscr{C}^1$ boundary. Then for all $u \in H^1(\Omega)$ there exists a $Pu \in H^1(\mathbb{R}^d)$ such that $Pu\big|_\Omega = u$, $\|Pu\|_{H^1(\mathbb{R}^d)} \leqslant$*

$C\|u\|_{H^1(\Omega)}$, $\|Pu\|_{L^2(\mathbb{R}^d)} \leqslant C\|u\|_{L^2(\Omega)}$ and Pu has compact support. Here the constant C depends only on Ω, but is independent of u. □

Proof.

Step 1 **(Local Map)** By the definition of $\mathscr{C}^1$ condition, for all $x \in \partial\Omega$ there exist open neighbourhood U_x satisfying the Q conditions. Thus $\partial\Omega \subset \bigcup_{x\in\partial\Omega} U_x$. Since $\partial\Omega$ is compact, there exists a finite subcover $\{U_{x_1}, \dots U_{x_n}\}$ also covering $\partial\Omega$.

Step 2 **(Partition of Unity)** Let $U_i := U_{x_i}$ if $i = 1, \dots, n$ and $U_0 = \Omega$. Then there exist $\eta_i \in \mathscr{C}_c^\infty(U_i)$ for all $i = 0, \dots, n$ such that $\eta_i \geqslant 0$ and $\sum_{i=0}^N \eta_i\big|_\Omega = 1$, as follows from the existence of partitions of unity subordinate to the cover $\left\{\Omega, U_1, \dots, U_n, \overline{\Omega}^C\right\}$.

Step 3 We write $u = \sum_{i=0}^n \eta_i u = \sum_{i=0}^n u_i$ where $u_i := \eta_i u$, $i = 0, \dots, n$. We want to extend every u_i to a function $H^1(\mathbb{R}^d)$. For $i = 0$ we can do this by defining

$$\tilde{u}_0(x) := \begin{cases} u_0(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega \end{cases}$$

Then $\tilde{u}_0 \in H^1(\mathbb{R}^d)$ and $\tilde{u}_0\big|_\Omega = u_0$.

For $1 \leqslant i \leqslant n$. Per definitionem there exist $h_i : U_i \to Q$ satisfying all conditions in $\mathscr{C}^1$-boundary condition. As $u_i = \eta_i u \in H^1(U_i \cap \Omega)$ it follows that $v_i := u_i \circ h_i^{-1} \in H^1(Q_+)$, because $h^{-1} \in \mathscr{C}^1$.

We can extend v_i to $v_i^* \in H^1(Q)$ by reflection. Define $\tilde{u}_i := v_i^* \circ h_i \in H^1(U_i)$ as $h_i \in \mathscr{C}^1$. Since $u_i = \eta_i u$ with $\eta_i \in \mathscr{C}_c^\infty(U_i)$ it follows that $\tilde{u}_i$ has compact support in U_i and thus can be extended trivially to all $\mathbb{R}^d$

Conclusion Defining $\tilde{u} := \sum_{i=0}^n \tilde{u}_i$ we have

- $\tilde{u}|_\Omega = \sum_{i=0}^n \tilde{u}_i\big|_\Omega = \sum_{i=0}^n u_i = u$
- $\tilde{u}$ has compact support.
- $\|\tilde{u}\|_{H^1(\mathbb{R}^d)} \leqslant C\|u\|_{H^1(\Omega)}$ and $\|\tilde{u}\|_{L^2(\mathbb{R}^d)} \leqslant c\|u\|_{L^2(\Omega)}$ follows from the construction.

q.e.d.

Theorem 10.11 (Sobolev Inequality in Ω)**.** *Assume that Ω is open, bounded and has*

$\mathscr{C}^1$ boundary. Then $\|u\|_{H^1(\Omega)} \geqslant C\|u\|_{L^p(\Omega)}$ for all p with

$$\begin{cases} p \leqslant \frac{2d}{d-2}, & \text{if } d \geqslant 3 \\ p < \infty, & \text{if } d = 2 \\ p \leqslant \infty, & \text{if } d = 1 \end{cases}$$

Moreover, if $\{u_n\}$ is bounded in $H^1(\Omega)$, then there exists a subsequence such that $u_n \to u$ strongly in $L^p(\Omega)$ for all p with

$$\begin{cases} p < \frac{2d}{d-2}, & \text{if } d \geqslant 3 \\ p < \infty, & \text{if } d = 2 \\ p \leqslant \infty, & \text{if } d = 1 \end{cases}$$

In particular $H^1(\Omega) \subset \mathscr{C}\big(\overline{\Omega}\big)$, if $\Omega \subset \mathbb{R}$. □

Proof. If $u \in H^1(\Omega)$, then there exists $\tilde{u} \in H^1\big(\mathbb{R}^d\big)$ such that $\tilde{u}\big|_\Omega = u$ and $\|\tilde{u}\|_{H^1(\mathbb{R}^d)} \leqslant C\|\tilde{u}\|_{H^1(\Omega)}$. By the Sobolev inequality in $H^1\big(\mathbb{R}^d\big)$ we have

$$\|u\|_{L^p(\Omega)} \leqslant \|\tilde{u}\|_{L^p(\mathbb{R}^d)} \leqslant C\|\tilde{u}\|_{H^1} \leqslant c\|u\|_{H^1}.$$

The remaining assertions are similarly to Sobolev compact embedding. *q.e.d.*

Remark 10.12. The constant C is independent of u. □

Theorem 10.13 (Density)**.** *$\mathscr{C}^\infty\big(\overline{\Omega}\big)$ is dense in $H^1(\Omega)$ but $\mathscr{C}_c^\infty(\Omega)$ is not dense in $H^1(\Omega)$.* □

Definition 10.14.

$$H_0^1(\Omega) := \overline{\mathscr{C}_c^\infty(\Omega)}^{H^1(\Omega)} \subsetneq H^1(\Omega) = \overline{\mathscr{C}_c^\infty(\Omega)}^{H^1(\Omega)}.$$

□

Example 10.15. In one dimension $H^1(\Omega) \subset \mathscr{C}(\overline{\Omega})$ for all $\Omega \subset \mathbb{R}$. If $u \in H^1(\Omega)$, then $u(x_0)$ is well-defined, i.e. there exists exactly one continuous representative of the equivalence class u which we may use define $u(x_0)$.
If $\Omega = (0,1)$, and $u \in H^1_0((0,1))$, then $u(0) = u(1) = 0$.

Proof. $u \in H^1_0((0,1))$ implies that there exists a sequence $(u_n)_n \in \mathscr{C}^\infty_c((0,1))$ such that $u_n \xrightarrow{n\to\infty} u$ in H^1. Thus $u_n(x) \to u(x)$ for all $x \in (0,1)$ because $H^1((0,1)) \subset \mathscr{C}((0,1))$, and therefore $u(0) = u(1) = 0$. *q.e.d.*

Indeed we shall prove that

$$H^1_0((0,1)) = \left\{u \in H^1((0,1)) \,\middle|\, u(0) = u(1) = 0\right\} \subsetneq H^1(0,1).$$

10.1 Trace on $\mathbb{R}^d$ ($d \geqslant 1$)

Consider the set

$$\mathbb{R}^d_+ = \left\{x = (x', x_d) \in \mathbb{R}^d \,\middle|\, x_d > 0\right\}$$

If $u \in H^1\left(\mathbb{R}^d_+\right)$, the is $u\big|_{\mathbb{R}^d_0}$ well-defined?

Theorem 10.16 (Trace Theorem in $\mathbb{R}^d_+$). *If $u \in \mathscr{C}^\infty_c\left(\mathbb{R}^d\right)$, then for $\Gamma = \mathbb{R}^{d-1} \times \{0\}$*

$$\|u\|_{L^2(\Gamma)} \leqslant C\|u\|_{H^1\left(\mathbb{R}^d_+\right)}$$

where C is independent of u. □

Proof.

$$|u(x',0)|^2 = \left|-\int_0^\infty \frac{d}{dx_d}|u(x',x_d)|^2 \mathrm{d}x_d\right| \leqslant \int_0^\infty 2|u(x',x_d)|\left|\frac{d}{dx_d}u(x',x_d)\right|\mathrm{d}x_d \leqslant$$

$$\leqslant \int_0^\infty \left(|u(x',x_d)|^2 + \left|\frac{d}{dx_d}u(x',x_d)\right|^2\right)\mathrm{d}x_d$$

Integrating over $x' \in \mathbb{R}^{d-1}$ one finds that

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^2 \mathrm{d}x' \leqslant \int_{\mathbb{R}^{n-1}} \int_0^\infty \left(|u(x',x_d)|^2 + \left| \frac{d}{dx_d} u(x',x_d) \right|^2 \right) \mathrm{d}x_d \leqslant \|u\|^2_{H^1(\mathbb{R}^d_+)}$$

q.e.d.

Thus we can define the trace operator

$$\mathrm{tr} : \begin{aligned} \mathscr{C}^\infty_c\left(\mathbb{R}^d\right) &\longrightarrow L^2(\Gamma) \\ u &\longmapsto u\big|_\Gamma \end{aligned}$$

This is a bounded linear function (i.e. continuous) on a dense subset of $H^1\left(\mathbb{R}^d_+\right)$ and therefore may be uniquely extended to the whole space.

Theorem 10.17 (Trace Theorem in Ω)**.** *Let $\Omega \subset \mathbb{R}6d$ be bounded, open, $\partial\Omega \in \mathscr{C}^1$. Then the there exists a trace operator*

$$\mathrm{tr} : \begin{aligned} H^1(\Omega) &\longrightarrow L^2(\partial\Omega) \\ u &\longmapsto u\big|_{\partial\Omega} \end{aligned}$$

satisfying

- *if $u \in H^1(\Omega) \cap \mathscr{C}\left(\overline{\Omega}\right)$, then $u\big|_{\partial\Omega} = u$ restricted to $\partial\Omega$.*
- *$\|u\|_{L^2(\Omega)} \leqslant C\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$, with C independent of u.*

□

Proof. As in the proof of Theorem 10.10 we have $\partial\Omega \subset \bigcup_{i=1}^n U_i$ with U_i open and for all i there exists a $h_i : U_i \to Q$, with $h_i, h_i^{-1} \in \mathscr{C}^1$, $h_i(U_i) = Q$, $h_i(U_i \cap \Omega) = Q_+$ and $h_i(U_i \cap \partial\Omega) = Q_0$. Also there exists a smooth partition of unity $(\vartheta_i)_i$ subordinate to the cover $\left\{\Omega, U_1, \dots, U_n, \overline{\Omega}^C\right\}$. Define $u_i = \vartheta_i u$.
For every $i = 1, \dots, n$, we have $w_i = u_i \circ h_i^{-1}$ and $w_i \in H^1(Q_i)$. Indeed, we can extend w_i to $H^1\left(\mathbb{R}^d_+\right)$ by setting $w_i(x) = 0$, if $x \notin Q$. By the Trace theorem in $\mathbb{R}^d_+$ we can define $w_i\big|_{Q_0} \in L^2(Q_0)$, with $\|w_i\big|_{Q_0}\|_{L^2(Q_0)} \leqslant \|w_i\|_{H^1(Q_+)}$. Define

$$u_i\big|_{\partial\Omega \cap U_i} := w_i\big|_Q \circ h_i \in L^2(\partial\Omega \cap \Omega)$$

and define

$$u\big|_{\partial\Omega} := \sum_{i=1}^{n} u_i\big|_{\partial\Omega\cap\Omega} \in L^2(\partial\Omega)$$

Moreover

$$\|u\|_{L^2(\partial\Omega)} \leqslant C\sum_{i=1}^{n}\|u_i\|_{L^2(\partial\Omega\cap U_i)} \leqslant C\sum_{i=1}^{n}\|w_i\|_{L^2(Q_+)} \leqslant C\sum_{i=1}^{n}\|w_i\|_{H^1(Q_+)} \leqslant C\sum_{i=1}^{n}\|u_i\|_{H^1(\Omega)} \leqslant C\|u\|_{H^1(\Omega)}$$

q.e.d.

Remark 10.18. The trace operator $u \mapsto u\big|_{\partial\Omega}$ is bounded. □

Theorem 10.19. *The trace operator $u \mapsto u\big|_{\partial\Omega}$ is bounded as an operator $H^1(\Omega) \to H^{1/2}(\partial\Omega)$. Consequently, $u \mapsto u\big|_{\partial\Omega}$ is a compact mapping $H^1(\Omega) \to L^2(\partial\Omega)$.*

$$H^1(\Omega) \overset{cont.}{\subset} H^{1/2}(\partial\Omega) \overset{comp.}{\subset\subset} L^2(\partial\Omega)$$

□

Definition 10.20 (Fractional Sobolev Spaces)**.**

$$H^{1/2}(\mathbb{R}^d) := \left\{ u \in L^2\left(\mathbb{R}^d\right) \,\middle|\, \int_{\mathbb{R}^d} (1 + 2\pi|k|)|\hat{u}(k)|^2 \mathrm{d}k < \infty \right\}$$

with the norm

$$\|u\|^2_{H^{1/2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + 2\pi|k|)|\hat{u}(k)|^2 \mathrm{d}k.$$

□

Remark 10.21. This definition extend the notion of n^{th} using the equivalent definition of the standard Sobolev

$$H^1(\mathbb{R}^d) := \left\{ u \in L^2\left(\mathbb{R}^d\right) \,\middle|\, \int_{\mathbb{R}^d} (1 + 2\pi|k|)^2|\hat{u}(k)|^2 \mathrm{d}k < \infty \right\}$$

Further we may define use this definition to define $\sqrt{-\Delta}$, via

$$\left\langle u, \sqrt{-\Delta} u\right\rangle = \langle \hat{u}, |\, k\,|\, \hat{u}\rangle = \int_{\mathbb{R}^d} 2\pi |k| |\hat{u}(k)|^2 \mathrm{d}k.$$

□

Theorem 10.22 (Sobolev Inequality for $H^{1/2}(\mathbb{R}^d)$)**.**

$$\|u\|_{H^{1/2}(\mathbb{R}^d)} \geqslant C\|u\|_{L^q(\mathbb{R}^d)}$$

for all $q \leqslant q^*$ *with*

$$q^* = \begin{cases} \frac{2d}{d-1}, & \text{if } d \geqslant 2 \\ \infty, & \text{if } d = 1 \end{cases}$$

And if $\{u_n\}$ *is bounded in* $H^{1/2}(\mathbb{R}^d)$*, then* $u_n \rightharpoonup u$ *in* $H^{1/2}(\mathbb{R}^d)$ *and* $u_n \mathbf{1}_B \to u \mathbf{1}_B$ *strongly in* $L^2(B)$ *for all* B *bounded.*

□

Corollary 10.23. *If* Ω *is bounded and* $\partial\Omega \in \mathscr{C}^1$*, then*

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega), \,|\, u\big|_{\partial\Omega} = 0\right\}$$

Moreover

$$\|u\|_{H_0^1}^2 := \int_\Omega |\nabla u|^2 \geqslant C\|u\|_{H^1(\Omega)}.$$

□

Proof. Since $H_0^1(\Omega) = \overline{\mathscr{C}_c^\infty(\Omega)}^{H^1(\Omega)}$, if $u \in H_0^1(\Omega)$ there exists $(u_n)_n \subset \mathscr{C}_c^\infty(\Omega)$ such that $u_n \xrightarrow{n\to\infty} u$ strongly in $H^1(\Omega)$. Then by continuity of the trace operator

$$0 = u_n\big|_{\partial\Omega} \longrightarrow u\big|_{\partial\Omega} \implies u\big|_{\partial\Omega} = 0$$

For the converse, let $u \in H^1(\Omega)$ and suppose that $u\big|_{\partial\Omega}$, then $u \in H_0^1$ (which is left as an exercise).

To prove

$$\int_\Omega |\nabla u|^2 \geqslant C\|u\|^2_{H^1(\Omega)} = C\int_\Omega \left(|\nabla u|^2 + |u|^2\right) \iff \int_\Omega |\nabla u|^2 \geqslant C\|u\|^2_{H^1(\Omega)} = C\int_\Omega |u|^2$$

Assume by contradiction that the latter inequality fails. Then there exits a sequence $(u_n)_n \subset H_0^1(\Omega)$ such that $\int_\Omega |u_n|^2 = 1$, but $\int_\Omega |\nabla u_n|^2 \to 0$. Since u_n is bounded in $H^1(\Omega)$, we can descend to a subsequence and assume that $u_n \to u$ weakly in $H^1(\Omega)$ and thus strongly in $L^2(\Omega)$. We have

$$\int_\Omega |u|^2 = \lim_{n\to\infty} \int_\Omega |u_n|^2 = 1$$

$$\int_\Omega |\nabla u|^2 \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 = 0$$

i.e. $u = \text{const}$ on Ω, which means that $u = \text{const} \neq 0$. But

$$0 = u_n\big|_{\partial\Omega} \longrightarrow u\big|_{\partial\Omega}$$

strongly in $L^2(\partial\Omega)$ and thus $u\big|_{\partial\Omega} = 0$ which is a contradiction. ↯ *q.e.d.*

Consider the Dirichlet problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

Theorem 10.24. *If $f \in L^2(\Omega)$, then there exits a unique $u \in H_0^1(\Omega)$ such that u is a solution of the Dirichlet problem in the distributional sense. Further*

$$\int_\Omega \nabla u \cdot \nabla\varphi + \int_\Omega u\varphi = \int_\Omega f\varphi$$

for all $\varphi \in H_0^1(\Omega)$, and u minimises

$$E = \inf\left\{ \frac{1}{2}\|v\|_{H^1} - \int_\Omega fv \,\middle|\, v \in H_0^1 \right\}$$

□

Proof. Using that $T : \varphi \mapsto \int f\varphi$ is a continuous functional on $L^2(\Omega)$ it follows that T is continuous on $H_0^1(\Omega)$, then by the Riesz representation theorem if follows that there exists a unique $u \in H_0^1$ such that $\langle u, \cdot\rangle_{H^1} = \langle f, \cdot\rangle_{L^2}$ (where we used that $H_0^1(\Omega)$ is a Hilbert space with norm $\|\cdot\|_{H^1(\Omega)}$). Thus for all $\varphi \in H_0^1(\Omega)$

$$\int f\varphi = \int \nabla u \cdot \nabla\varphi + \int u\varphi$$

and for $\varphi \in \mathscr{C}_c^\infty(\Omega)$

$$\int f\varphi = -\int u\Delta\varphi + \int u\varphi$$

which implies that

$$f = -\Delta u + u \qquad \text{in } \mathscr{D}'(\Omega)$$

q.e.d.

Consider the von Neumann problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

Theorem 10.25. *For all $f \in L^2(\Omega)$ there exists a unique $u \in H^1(\Omega)$ such that it solves the von Neumann problem in the distributional sense and*

$$\int_\Omega \nabla u \cdot \nabla\varphi + \int_\Omega u\varphi = \int_\Omega f\varphi$$

for all $\varphi \in H^1(\Omega)$. Moreover, u minimises

$$E = \inf\left\{ \|v\|^2_{H^1(\Omega)} - \int_\Omega fv \,\middle|\, v \in H^1(\Omega) \right\}$$

□

Remark 10.26. $\frac{\partial u}{\partial n}\big|_{\partial\Omega}$ is well-defined if $u \in H^2(\Omega)$, since then

$$H^1 \ni \nabla u \longmapsto \nabla u\big|_{\partial\Omega}$$

makes sense by the trace theorem, therefore need some regularity.
To motivate this consider the case $u \in \mathscr{C}^2(\Omega)$, $-\Delta u + u = f$ pointwise. Using

$$\int\limits_{\Omega} \nabla u \cdot \nabla \varphi = \int\limits_{\Omega} (-\Delta u)\varphi + \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}\varphi = \int f\varphi - \int u\varphi$$

by the PDE. But

$$\int (-\Delta)\varphi = \int f\varphi - \int u\varphi$$

by equation $-\Delta u = fu$ and there

$$\int\limits_{\partial\Omega} \frac{\partial u}{\partial n}\varphi = 0$$

for all $\varphi \in \mathscr{C}^2(\mathbb{R}^d)$, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. □

When is a weak solution in $H^2(\Omega)$? Does $f \in L^2(\Omega)$ imply that $\Delta u \in L^2(\Omega)$. If $\Omega = \mathbb{R}^d$, it is true that $u, \Delta u \in L^2$, then $u \in H^2$ (via the Fourier transform). If Ω is a bounded set one has to be more careful.

Definition 10.27. We say that $\partial\Omega \in \mathscr{C}^2$ if for all $x \in \partial\Omega$, there exists an open neighbourhood U of x, such that

- there exists $h : U \to Q$ such that $h \in \mathscr{C}^2\big(\overline{U}\big)$, $h \in \mathscr{C}^2\big(h\big(\overline{U}\big)\big)$.
- $h(U \cap \Omega) = Q_+$
- $h(U \cap \partial\Omega) = Q_0$.

□

Theorem 10.28 (Regularity)**.** *Assume that Ω has $\partial\Omega \in \mathscr{C}^2$ and $f \in L^2$.*

1) If $u \in H_0^1(\Omega)$*, for all* $\varphi \in H_0^1(\Omega)$

$$\int \nabla u \cdot \nabla \varphi + \int u\varphi = \int f\varphi$$

then $u \in H^2(\Omega)$*.*

2) If $u \in H^1(\Omega)$*, for all* $\varphi \in H^1(\Omega)$

$$\int \nabla u \cdot \nabla \varphi + \int u\varphi = \int f\varphi$$

then $u \in H^2(\Omega)$ *and*

$$\frac{\partial u}{\partial n} = \boldsymbol{n} \cdot \nabla u = 0, \qquad \text{on } \partial\Omega.$$

□

We shall prove this via the translation method by Nirenberg. But first we shall need a lemma.

Definition 10.29. For $h \in \mathbb{R}^d$ we define

$$(D_h u)(x) = \frac{u(x+h) - u(x)}{|h|}.$$

□

Lemma 10.30. *Let* $u \in L^2(\Omega)$*, then the following are equivalent*

(i) $u \in H^1(\Omega)$

(ii)

$$\sup_{\substack{\varphi \in \mathscr{D}(\Omega) \\ \|\varphi\|_2 \leqslant 1}} \left| \int_\Omega u \partial_{x_i} \varphi \right| < \infty$$

(iii) For all h small, and all $\Omega' \subset\subset \Omega$

$$\|D_h u\|_{L^2(\Omega')} \leqslant C$$

$\square$

Proof.

(i)⇒(ii) Obvious as for all $\varphi \in \mathscr{C}_c^\infty(\Omega)$

$$\left|\int_\Omega u\partial_{x_i}\varphi\right| = \left|-\int_\Omega \partial_{x_i}u\varphi\right| \leqslant \|\nabla u\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}$$

(ii)⇒(i) Define for all $\varphi \in \mathscr{D}(\Omega)$

$$T(\varphi) = \int_\Omega u\partial_{x_i}\varphi`$$

Then T is linear and bounded, as $|T(\varphi)| \leqslant C\|\varphi\|_{L^2(\Omega)}$.

Thus T can be extended to a linear, bounded mapping in $L^2(\Omega)$ by the Riesz theorem there exists $v \in L^2(\Omega)$ such that for all $\varphi \in L^2(\Omega)$

$$T(\varphi) = \int_\Omega v\varphi$$

In particular if $\varphi \in \mathscr{D}$. Thus

$$\int_\Omega v\varphi = T(\varphi) = \int_\Omega u\partial_{x_i}\varphi.$$

which implies that $\partial_{x_i} = -v \in L^2(\Omega)$.

(iii)⇒(ii) For all $\varphi \in \mathscr{D}(\Omega)$, and defining $y = x + h$

$$\int_\Omega (D_h u)\varphi = \int_\Omega \frac{u(x+h)-u(x)}{|h|}\varphi(x)\mathrm{d}x = \int_\Omega \frac{u(y)\varphi(y-h)-u(y)\varphi(y)}{h}\mathrm{d}x = \int_\Omega u(D_{-h}\varphi)$$

Thus

$$\left|\int_\Omega u(D_{-h}\varphi)\right| = \left|\int_\Omega (D_h u)\varphi\right| \leqslant \|D_h u\|_{L^2(\Omega')}\|\varphi\|_{L^2}$$

Choosing $h = (0, \dots, h_i, \dots, 0)$ and $h_i \to 0$ then

$$\left|\int_\Omega u\partial_{x_i}h\right| \leqslant C\|\varphi\|_{L^2(\Omega)}$$

for all $\varphi \in \mathscr{D}(\Omega)$.

(i)⇒(iii) Let $u_n \in \mathscr{C}^\infty(\overline{\Omega})$ and $u_n \to u$ strongly in $H^1(\Omega)$. Then

$$(D_h u_n)(x) = \frac{u_n(x+h) - u_n(x)}{|h|} = \frac{1}{|h|} \int_0^1 h \cdot \nabla u_n(x+th)\mathrm{d}t$$

$$|(D_h u_n)(x)|^2 = \left| \int_0^1 \frac{h}{|h|} \cdot \nabla u_n(x+th)\mathrm{d}t \right|^2 \leqslant \int_0^1 |\nabla u_n(x+th)|^2 \mathrm{d}t$$

$$\int_{\Omega'} |D_h u_n|^2 \leqslant \int_{\Omega'} \int_0^1 |\nabla u_n(x+th)|^2 \mathrm{d}t\mathrm{d}x = \int_0^1 \underbrace{\int_{\Omega'} |\nabla u_n(x+th)|^2 \mathrm{d}x}_{\leqslant \int_\Omega |\nabla u_n|^2} \mathrm{d}t = \|\nabla u_n\|^2_{L^2(\Omega)}$$

where h has to be chosen small enough so that $\Omega' + h \subset \Omega$.

Taking n to infinity $\|D_h u\|^2_{L^2(\Omega)}$ we find that

$$\|D_h u\|^2_{L^2(\Omega')} \leqslant \|\nabla u_n\|^2_{L^2(\Omega)}$$

Thus (iii) holds with $C = \|\nabla u_n\|^2_{L^2(\Omega)}$ for all $u \in H^1(\Omega)$.

q.e.d.

Proof of Theorem 10.28. In the case $\Omega = \mathbb{R}^d$, it follows from the variational formula

$$\int \nabla u \cdot \nabla \varphi + \int u\varphi = \int f\varphi$$

for all $\varphi \in H^1(\mathbb{R}^d)$. We can choose $\varphi = D_{-h}(D_h u) \in H^1(\mathbb{R}^d)$ for all $h \neq 0$. Thus

$$\int u\varphi = \int u D_{-h}(D_h u) = \int D_h u \cdot D_h u = \int |D_h u|^2$$
$$\int \nabla u \cdot \nabla \varphi = \int \nabla u \cdot \nabla D_{-h}(D_h(u)) = \int \nabla u D_{-h}(D_h(\nabla u)) \int |D_h(\nabla u)|^2$$

Thus

$$\int |D_h(\nabla u)|^2 + \int |D_h u|^2 = \int f D_{-h}(D_h u) \leqslant \|f\|_2 \|D_{-h}(D_h u)\|_2 \leqslant \|f\|_2 \|\nabla(D_h u)\|_2 = \|f\|_2 \|D_h(\nabla u)\|_2$$

and thus

$$\|D_h(\nabla u)\|_2 \leqslant \|f\|_2, \qquad \|D_h(\nabla u)\|_2 \leqslant \|f\|_2$$

from which follows that $\nabla u \in H^1(\mathbb{R}^d)$ by the lemma and therefore $u \in H^2(\mathbb{R}^d)$, i.e. $\partial_{x_i}\partial_{x_j} u \in L^2$.

Now we shall consider the case $\Omega = \mathbb{R}^d_+$.

Assume that $u \in H^1\big(\mathbb{R}^d_+\big)$ and

$$\int \nabla u \cdot \nabla \varphi + \int u\varphi = \int f\varphi$$

for all $\varphi \in H^1(\mathbb{R}^d_+)$ (for the von Neumann problem! For the Dirichlet problem we only need to change H^1 to H^1_0). By the same argument we have

$$\|D_h(\nabla u)\|_2 \leqslant \|f\|_2$$

for all h parallel to Γ. Choosing $h = (0, \dots, h_i, \dots, 0)$ for $i = 1, \dots, d-1$ and $h_i \to 0$, it follows from the lemma that $\partial_{x_i} \nabla u \in L^2$ for all $i = 1, \dots, d-1$ and thus $\partial_{x_i}\partial_{x_j} u \in L^2$ for $j = 1, \dots, d$ and $i = 1, \dots, d-1$.

Is $\partial^2_{x_d} u \in L^2$? Yes, because $-\sum_{i=1}^d \partial^2_{x_i} u = -\Delta u = f - u \in L^2(\Omega)$ and therefore

$$\partial^2_{x_d} u = -\Delta + \sum_{i=1}^{d-1} \partial^2_{x_i} u \in L^2(\Omega)$$

For the general case of Ω open, bounded and $\partial\Omega \in \mathscr{C}^2$.

We know that there exist a finite cover of $\Omega =: U_0$ via charts and a smooth partition of unity $\{\vartheta_i\}$ subordinate to that cover.

Defining $u_i = \vartheta_i u$ we only need to prove that $u_i \in H^2$.

For $i = 0$, $-\Delta u + u = f$ in $\mathscr{D}'(\Omega)$ because for all $\varphi \in \mathscr{C}^\infty_c$

$$-\Delta(\vartheta_0 u) = -\Delta\vartheta_0 u - 2\Delta\nabla\vartheta_0 \cdot \nabla u - \vartheta_0 \Delta u = -\Delta\vartheta_0 u - 2\Delta\nabla\vartheta_0 \cdot \nabla u - \vartheta_0(f-u) + \vartheta_0 u \equiv g \in L^2(\Omega)$$

Since $\vartheta_0 u \in H^1(\Omega)$ and $\vartheta_0 u$ has compact support we return to the case $\Omega = \mathbb{R}^d$ and thus $\vartheta_0 u \in H^2$.

For $i = 1, \dots, N$, $u_i = theta_i u$ satisfies

$$-\Delta(\vartheta_i u) + \vartheta_i u = g_i \in L^2(U_i \cap \Omega)$$

Define $v_i = u_i \circ h_i^{-1}$. The function v_i satisfies a second order elliptical equation

$$\sum_{k,l=1}^{d} \int_{Q_+} a_{kl}\partial_{x_k} v_i \partial_{x_l}\varphi + +\vartheta_0 u b v_i \varphi = +\vartheta_0 u \tilde{g}_i \varphi$$

for all $\varphi \in H^1(Q_+)$. By a similar argument in $\mathbb{R}^d_+$, we can show that $v_i \in H^2(Q_+)$. Since the matrix a is symmetric we can change variables to return to the standard $-\Delta$ case.

Because $v_i \in H^2(Q_+)$ and $h, h^{-1} \in \mathscr{C}^2$, it follows that $u_i \in H^2$. Thus $u = \sum_i u_i \in H^2$.

To prove the von Neumann problem $\partial_{\boldsymbol{n}} u = 0$ we shall need the Green Formulae, which proven below.

By regularity we have $u \in H^2(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega = \int_\Omega f\varphi$$

for all $\varphi \in H^1(\Omega)$. If we choose $\varphi \in \mathscr{D}$ then

$$-\Delta u + u = f$$

in $\mathscr{D}'(\Omega)$, and $u \in H^2(\Omega)$implies that the equality holds in the L^2 sense. Integrating against $\varphi \in H^1(\Omega)$ and using the second Green formula we find that

$$\int (-\Delta u)\varphi + \int u\varphi = \int f\varphi$$

implies

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_{\partial\Omega} \frac{\partial u}{\partial n}\varphi + \int_\Omega u\varphi = \int_\Omega f\varphi$$

for all $\varphi \in H^1(\Omega)$. It follows for all $\varphi \in H^1(\Omega)$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\varphi = 0$$

and therefore

$$\frac{\partial u}{\partial n} \qquad \text{on } \partial\Omega$$

q.e.d.

Theorem 10.31 (Green Formulae)**.** *For Ω open and bounded with $\partial\Omega \in \mathscr{C}^1$. If $u, \varphi \in H^1(\Omega)$, then*

$$\int_\Omega \partial_{x_i} u\varphi \mathrm{d}x = -\int_\Omega u\partial_{x_i}\varphi \mathrm{d}x + \int_{\partial\Omega} u\big|_{\partial\Omega}\varphi\big|_{\partial\Omega} n_i \mathrm{d}S(x)$$

where $\boldsymbol{n}$ is the outward pointing unit normal vector to $\partial\Omega$.
Moreover, if $u \in H^2(\Omega)$

$$\int_\Omega (\Delta u)\varphi = -\int_\Omega \nabla u \cdot \nabla\varphi + \int_{\partial\Omega} \frac{\partial u}{\partial n}\varphi \mathrm{d}S(x).$$

□

Proof. These formulae follow from the continuous case as the trace operator is continuous. *q.e.d.*

Example 10.32 (Von Neumann Problem)**.** Let $\Omega = (0, 1)$ and consider the von Neumann problem for $f \in L^2((0, 1))$

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

We can prove that there exists a unique $u \in H^1((0, 1))$ such that

$$\int u'\varphi' + \int u\varphi = \int f\varphi$$

for all $\varphi \in H^1((0, 1))$. If we choose $\varphi \in \mathscr{D}$ it follows that

$$-u'' + u = f \qquad \text{in } \mathscr{D}'((0, 1))$$

But $u, f \in L^2$ and therefore $u'' = u - f \in L^2$ which implies that $u \in H^2((0, 1))$.
And therefore

$$0 = \int_0^1 (-u'' + u - f)\varphi = \int u'\varphi' + \int u \; phi \int f\varphi + u'\varphi\big|_0^1$$

for all $\varphi \in H^2((0,1))$. And thus we have

$$u'(1)\varphi(1) - u'(0)\varphi(0) = 0$$

for all $\varphi \in H^1(0,1)$. Choosing $\varphi(x) = x$ implies that $u'(1) = 0$ and $\varphi(x) = 1-x$ implies $u'(0) = 0$.

Example 10.33 (Periodic Problem)**.** Consider the periodic problem, for $f \in L^2$

$$\begin{cases} -u'' + u = f \\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases}$$

To solve this consider the set

$$H = \left\{ u \in H^1((0,1)) \,\middle|\, u(0) = u(1) \right\}$$

H is a Hilbert space, with H^1 inner product. Thus there exists a unique u such that

$$\int u'\varphi' + \int u\varphi = \int f\varphi$$

for all $\varphi \in H$. From this we can deduce that $u \in H^2$, and $u'(0) = u'(1)$ which is left as an exercise.

Example 10.34 (Inhomogeneous Von Neumann Problem)**.** Consider the Robin problem, for $f \in L^2$ real valued

$$\begin{cases} -u'' + u = f \\ u'(0) = \alpha \\ u'(1) = \beta \end{cases}$$

Theorem 10.35. *For all $f \in L^2((0,1))$ there exists a unique solution $u \in H^2((0,1))$ to the inhomogeneous von Neumann problem.* □

Proof. What is the variational formula? Assume that $u \in H^2((0,1))$ is a solution then

$$\int_0^1 (-u'' + u - f)\varphi = 0$$

for all $\varphi \in H^1((0,1))$. Integrating by parts

$$\int_0^1 -u''\varphi = \int_0^1 u'\varphi' - u'(1)\varphi(1) + u'(0)\varphi(0)$$

which yields

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi - \int_0^1 f\varphi - u'(1)\varphi(1) + u'(0)\varphi(0) = 0$$

for all $\varphi \in H^1((0,1))$. If $u'(0) = \alpha, u'(1) = \beta$ this reduces to

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi + \beta\varphi(1) - \alpha\varphi(0)$$

for all $\varphi \in H^1((0,1))$.

Thus define the linear functional

$$\mathscr{L} : \begin{array}{l} H^1((0,1)) \longrightarrow \mathbb{R} \\ \varphi \longmapsto \displaystyle\int_0^1 f\varphi + \beta\varphi(1) - \alpha\varphi(0) \end{array}$$

which is bounded as

$$|\mathscr{L}(\varphi)| \leqslant \left| \int_0^1 f\varphi + \beta\varphi(1) - \alpha\varphi(0) \right| \leqslant \|f\|_2\|\varphi\|_2 + (|\beta| + |\alpha|)\|\varphi\|_\infty \leqslant C\|\varphi\|_{H^1}$$

where the last inequality follows from the one dimensional Sobolev inequality.

Thus $\mathscr{L}$ is a linear, bounded functional on H^1 and therefore there exists a unique $u \in H^1((0,1))$ such that

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi = \langle u, \varphi \rangle_{H^1} = \mathscr{L}(\varphi)$$

for all $\varphi \in H^1((0,1))$. Hence we have found a unique H^1 solution the problem integrated by parts. To finish the proof we need to show that $u \in H^2((0,1))$.
For this purpose we note that for all $\varphi \in \mathscr{D}((0,1))$

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi \implies -u'' + u = f \qquad \text{in } \mathscr{D}'((0,1)) \implies u'' = u - f \in L^2$$

and thus $u \in H^2((0,1))$. Therefore if for all $\varphi \in H^1((0,1))$

$$\int_0^1 (-u'' + u - f)\varphi = 0$$

then

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi - \int_0^1 f\varphi - u'(1)\varphi(1) + u'(0)\varphi(0) = 0$$

but we already know that

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi - \int_0^1 f\varphi - \beta\varphi(1) + \alpha\varphi(0) = 0$$

and therefore

$$-u'(1)\varphi(1) + u'(0)\varphi(0) = -\beta\varphi(1) + \alpha\varphi(0)$$

for all $\varphi \in H^1((0,1))$. Choosing $\varphi(x)$ and $\varphi(x) = 1 - x$ imply respectively $-u'(1) = -\beta$ and $u'(0) = \alpha$. *q.e.d.*

Example 10.36 (Robin Problem)**.** Consider the Robin problem, for $f \in L^2$

$$\begin{cases} -u'' + u = f \\ u'(0) = u(0) \\ u(1) = 0 \end{cases}$$

There exists a unique $H^2(0,1)$ for this problem.

Theorem 10.37. *For all $f \in L^2((0,1))$ there exists a unique $u \in H^2((0,1))$ solving the Robin problem.* □

Proof. Assume that u is a solution. Then for all $\varphi \in H^1$

$$0 = \int_0^1 (-u'' + u - f)\varphi = \int_0^1 (u'\varphi' + u\varphi - f\varphi) - u'(1)\varphi(0) + \underbrace{u'(0)\varphi(0)}_{=u(0)\varphi(0)}$$

which is equivalent to

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi + u(0)\varphi(0) = \int_0^1 f\varphi$$

for all $\varphi \in H^1$.

Now define the linear functional

$$\mathscr{L} : \begin{array}{c} H^1((0,1)) \longrightarrow \mathbb{R} \\ \varphi \longmapsto \int_0^1 f\varphi \end{array}$$

and define the new Hilbert space $\mathscr{H} = H^1$ with inner product

$$\langle u, \varphi \rangle_{\mathscr{H}} = \int_0^1 u'\varphi' + \int_0^1 u\varphi + u(0)\varphi(0)$$

We claim that $\mathscr{H}$ is a Hilbert space and that

$$\|u\|_{H^1} \leqslant \|u\|_{\mathscr{H}} \leqslant C\|u\|_{H^1}$$

which follows from $|u(0)|^2 \leqslant C\|u\|_{H^1}^2$.

Applying the Riesz theorem for $\mathscr{H}$ we find that there exists a unique $u \in \mathscr{H} = H^1((0,1))$ such that

$$\langle u, \varphi \rangle_{\mathscr{H}} = \mathscr{L}(\varphi) = \int_0^1 u\varphi$$

for all $\varphi \in \mathscr{H} = H^1((0,1))$. Thus there exists a unique H^1 solution to

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi + u(0)\varphi(0) = \int_0^1 f\varphi$$

for all $\varphi \in H^1$.
To prove that $u \in H^2((0,1))$ note that for $\varphi \in \mathscr{D}$ we have

$$-u'' + u = f \in \mathscr{D}' \implies u'' \in L^2 \implies u \in H^2 \implies -u'' + u = f \in L^2$$

and thus same as above we find that $u'(1) = 0$ and $u'(0) = u(0)$.

q.e.d.

Chapter 11

Schrödinger Dynamics

$$\boxed{i\partial_t\psi = H\psi}$$

with some initial condition $\psi(t=0)=\psi_0$. Here ψ represents the wave function and $|\psi(x)|^2$ represents the probability density of a particle in configuration space and $|\hat{\psi}(p)|^2$ represents the probability density of a particle in momentum space.
H here is an (unbounded) operator on $L^2(\mathbb{R}^d)$ the Hamiltonian and

$$\langle\psi, H\psi\rangle = \text{energy of } \psi$$

Example 11.1. Consider fore example for some measurable function $V:\mathbb{R}^d\to\mathbb{R}$ the operator

$$H = -\Delta + V(x) \qquad \text{in } L^2(\mathbb{R}^d).$$

For this problem to have a solution we need some conditions on H. Let $\mathscr{H}$ be a Hilbert space. For an inner product $\langle\cdot,\cdot\rangle$ we require

$$\forall\psi\in D(H): \langle\psi, H\psi\rangle\in\mathbb{R}$$

where

$$D(H) = \left\{\psi \,\middle|\, H\psi\in\mathscr{H}\right\}.$$

Lemma 11.2. *Let H be a linear operator on $\mathscr{H}$ with domain $D(H)$ (dense in $\mathscr{H}$).*

Then

$$\forall \psi \in D(H): \langle \psi, H\psi \rangle \in \mathbb{R} \iff \forall u, v \in D(H): \langle u, Hv \rangle = \langle Hu, v \rangle$$

We call H a symmetric operator in this case. □

Definition 11.3 (Adjoint)**.** Let H be an operator on a Hilbert space $\mathscr{H}$ with dense domain $D(H)$. Then we define

$$H^* : D(H^*) \longrightarrow \mathscr{H}$$

which satisfies

$$\forall u \in D(H^*) \, \forall v \in D(H): \langle u, Hv \rangle = \langle H^* u, v \rangle$$

where

$$D(H^*) = \left\{ u \in \mathscr{H} \;\middle|\; \langle u, H\cdot \rangle \text{ is a linear functional on } v \right\}$$

The map is well-defined as $D(H)$ is dense in $\mathscr{H}$. □

Proposition 11.4. *If $u \in D(H^*)$, then there exists $f \in H$ such that for all $v \in D(H)$*

$$\langle u, Hv \rangle = \langle f, v \rangle$$

and thus we can define uniquely $H^* u := f$ □

Proposition 11.5. *If H is symmetric, then $H \subset H^*$, i.e. $D(H) \subset D(H^*)$ and $H^*\big|_{D(H)} = H$.* □

Definition 11.6. H is called a self-adjoint operator iff $H^* = H$ (in particular $D(H^*) = D(H)$). □

Proposition 11.7. *In finite dimensions, if $H = (H_{ij})_{ij}$ is a matrix, then it self-adjoint w.r.t. to the standard inner product iff $H_{ji} = \overline{H_{ij}}$.* □

Example 11.8. $-\Delta$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$ with domain $D(-\Delta) = H^2\big(\mathbb{R}^d\big)$.

Example 11.9. $\mathscr{H} = L^2(\Omega, \mu)$ is a measure space, $f : \Omega \to \mathbb{R}$ measurable, then the multiplication operator

$$T_f : \qquad (T_f u)(x) = f(x)u(x)$$

is a self-adjoint operator with domain

$$D(T_f) = \big\{u \in L^2(\Omega, \mu) \,\big|\, fu \in L^2(\Omega, \mu)\big\}.$$

Theorem 11.10 (Spectral Theorem)**.** *Assume that A is a self-adjoint operator on a Hilbert space $\mathscr{H}$ with domain $D(A)$. Then there exists a unitary operator $U : \mathscr{H} \to L^2(\Omega, \mu)$ and a measurable function $f : \Omega \to \mathbb{R}$ such that*

$$UAU^{-1} = T_f.$$

□

Definition 11.11. We call $A \geqslant 0$ iff for all $u \in D(A)$ $\langle u, Au\rangle \geqslant 0$. Further $A \geqslant B$ iff $A - B \geqslant 0$. □

Theorem 11.12 (Friedrichs Extension)**.** *If $A \geqslant -C$, where A is a symmetric operator and $C \in \mathbb{R}$, then there exists unique self-adjoint extension $\tilde{A}$ of A and*

$$\inf_{\substack{u \in D(\tilde{A}) \\ \|u\|=1}} \Big\langle u, \tilde{A}u \Big\rangle = \inf_{\substack{u \in D(A) \\ \|u\|=1}} \langle u, Au\rangle\,.$$

□

Theorem 11.13 (Kato-Rellich)**.** *If A is a self-adjoint operator and B symmetric with $D(B) \supset D(A)$, and*

$$\|Bu\| \leqslant a\|Au\| + C\|u\|$$

for all $u \in D(A)$ with $a < 1$, then $A + B$ is self-adjoint with $D(A+B) = D(A)$. □

Example 11.14. If $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, then $-\Delta + V$ is self-adjoint on $H^2(\mathbb{R}^3)$.

Proof. Consider $A = -\Delta$, $B = V$. For every $\varepsilon > 0$ we can write $V = V_1 + V_2$, with $\|V_1\|_2 \leqslant \varepsilon$, $V_2 \in L^\infty$. Therefore

$$\begin{aligned}\|Vu\|_2 &\leqslant \|V_1 u\|_2 + \|V_2 u\|_2 \leqslant \|V_1\|_2\|u\|_\infty + \|V_2\|_\infty\|u\|_2 \leqslant C\varepsilon\|u\|_{H^1} + C_\varepsilon\|u\|_2 \leqslant \\ &\leqslant C\varepsilon\|\Delta u\|_2 + C_\varepsilon\|u\|_2\end{aligned}$$

by the Sobolev embedding as $L^\infty \subset H^2$. Choosing $a = C\varepsilon < 1$ we find the desired result. *q.e.d.*

Theorem 11.15. *If A is self-adjoint, then the equation*

$$\begin{cases} i\partial_t u = Au \\ u(t=0) = u_0 \end{cases}$$

has a unique solution, if $u_0 \in D(A)$ and

$$u(t,\cdot) \in \mathscr{C}^1((0,\infty),\mathscr{H}) \cap \mathscr{C}([0,\infty), D(A))$$

with $\|u\|_{D(A)} = \|u\| + \|Au\|$ for all $D(A)$.
"Symbolically" we can write

$$u(t) = e^{-itA}u_0$$

□

*Proof*Step 1 Assume that A is bounded. Then e^{-itA} well-defined by

$$e^{-itA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(-iA)^n$$

which is in the operator (norm) topology as

$$\left\|e^{-itA}\right\| \leqslant \sum_{n=0}^{\infty} \frac{t^n}{n!}\|A\|^n = e^{t\|A\|} < \infty$$

Thus we can define $u(t) = e^{-itA}u_0$ and check that it satisfies

$$i\partial_t\left(e^{-itA}\right) = Ae^{-itA}$$

Step 2 Assume that $A \geqslant 0$. Then we can define $A_n = \frac{nA}{A+n^2}$ is a bounded operator.

By step 1 there exists a solution u_n to the corresponding problem with A_n.

If we can prove that $u_n(t) \xrightarrow{n\to\infty} u(t)$ (in L^2) then we have found a solution.

Noting that e^{-itA} is unitary it follows that $\frac{d}{dt}\|u_n(t)\|_2 = 0$ which implies that $\|u_n(t)\| = u_0$ and therefore we find that

$$\begin{aligned}\frac{d}{dt}\|u_n - u_m\|^2 &= \frac{d}{dt}\left(\|u_n\|^2 + \|u_m\|^2 + 2\,\Re\left\langle u_m, U - n\right\rangle\right) = 2\,\Re(\langle -iA_m u_n, u_m\rangle + \langle u_n, iA_n u_m\rangle) = \\ &= 4\Im\left\langle u_n, (A_m - A_n)u_m\right\rangle \xrightarrow{n,m\to\infty} 0\end{aligned}$$

as

$$A_m - A_n = \frac{nA}{A+n} - \frac{mA}{A+m} = \frac{(m-n)}{(A+m)(A+n)} \sim \frac{m-n}{mn} \xrightarrow{m,n\to\infty} 0.$$

This implies that $u_n(t)$ converges to some $u(t)$ in $\mathscr{H}$ which solves the equation.

q.e.d.