Mathematical Quantum Mechanics II

Final Exam

Nachname: ____________________ Vorname: ____________________

Geburtstag: ____________________ Matrikelnr.: ____________________

Studiengang: ____________________ Fachsemester: ____________________

- This is individual work. Discussion with other people is not allowed.
- You can use the lecture notes, solutions of homework sheets, and other references.
- Justify your statements by referring to the material discussed in class or proving them.
- Submit your solutions via Uni2work. If you get technical problems with Uni2work, send your solutions to Dinh-Thi Nguyen “Thi.Nguyen@lmu.de”.
- The exam takes place on July 29, 2020, from 9:00 to 13:00 (4 hours). Good luck!

Problems	1	2	3	4	5
Maximum points	25	20	20	30	25
Scored points					

Final exam	Homework bonus	Total points	GRADE

Problem 1. (5+5+5+10 points) Let $\mathscr{H}$ be a separable Hilbert space. Let $u, v \in \mathscr{H}$, $\|u\| = \|v\| = 1$ and $|\langle u, v\rangle| < 1$. Consider

$$\Psi_N := c_N(u^{\otimes N} + v^{\otimes N})$$

where $c_N > 0$ is a normalization constant making Ψ_N a normalized vector in $\mathscr{H}^{\otimes_s N}$.

(a) Prove that

$$\lim_{N\to\infty} c_N = \frac{1}{\sqrt{2}}.$$

(b) Let $\gamma^{(1)}_{\Psi_N}$ be the one-body density matrix of Ψ_N. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\langle u, \gamma^{(1)}_{\Psi_N} u\rangle = \frac{1}{2}(1 + |\langle u, v\rangle|^2).$$

(c) Prove that

$$\lim_{N\to\infty} \frac{1}{N}\gamma^{(1)}_{\Psi_N} = \frac{1}{2}\Big(|u\rangle\langle u| + |v\rangle\langle v|\Big)$$

strongly in trace class.

(d) Let Ψ'_N be another normalized vector in $\mathscr{H}^{\otimes_s N}$ such that its one-body density matrix is $\gamma^{(1)}_{\Psi'_N} = N|u\rangle\langle u|$. Prove that $\Psi'_N = z u^{\otimes N}$ with some complex number $z \in \mathbb{C}$.

Hint: The transformation $U_N(u)$ in Chapter 7.2 (the lecture notes) may be useful.

Problem 2. (5+5+10 points) Let $\mathscr{H}$ be a separable Hilbert space. Let h be a self-adjoint operator on $\mathscr{H}$. Let $\mathbb{H} = \mathrm{d}\Gamma(h)$ be the second quantization of h on the bosonic Fock space $\mathcal{F}(\mathscr{H})$.

(a) Prove that $\mathbb{H}$ is bounded from below if and only if $h \geq 0$.

(b) Use Weyl's Criterion to prove that $\inf \sigma_{\mathrm{ess}}(\mathbb{H}) \leq \inf \sigma_{\mathrm{ess}}(h)$.

(c) Assume that $h > 0$. Prove that $\mu_2(\mathbb{H}) = \inf \sigma_{\mathrm{ess}}(\mathbb{H})$ if and only if $\mu_1(h) = \inf \sigma_{\mathrm{ess}}(h)$. Here we denote by $\mu_i(A)$ the i-th min-max value of A.

Problem 3. (5+10+5 points) Let $\mathscr{H}$ be a separable Hilbert space. Let $J : \mathscr{H} \to \mathscr{H}^*$ be the anti-linear map defined by $J(f)(g) = \langle f, g \rangle$ for all $f, g \in \mathscr{H}$.

(a) Let $\xi > 0$ be a self-adjoint operator on $\mathscr{H}$. Consider the block operator on $\mathscr{H} \oplus \mathscr{H}^*$

$$B = \begin{pmatrix} \xi & 0 \\ 0 & J\xi J^* \end{pmatrix}.$$

Prove that

$$\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(\xi)$$

(both sides can be empty).

(b) Let $\mathcal{V} \in \mathscr{G}$, namely $\mathcal{V}$ is a linear bounded operator on $\mathscr{H} \oplus \mathscr{H}^*$ such that

$$\mathcal{V} = \begin{pmatrix} U & V \\ JVJ & JUJ^* \end{pmatrix}, \quad \mathcal{V}^* \mathcal{S} \mathcal{V} = \mathcal{V} \mathcal{S} \mathcal{V}^* = \mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{Tr}(VV^*) < \infty.$$

Prove that

$$\sigma_{\text{ess}}(\mathcal{V}^* B \mathcal{V}) = \sigma_{\text{ess}}(B).$$

(c) Assume that

$$\mathcal{V}^* B \mathcal{V} = \begin{pmatrix} h & k \\ k^* & JhJ^* \end{pmatrix}$$

with a self-adjoint operator h on $\mathscr{H}$ and a compact operator $k : \mathscr{H}^* \to \mathscr{H}$. Prove that

$$\sigma_{\text{ess}}(h) = \sigma_{\text{ess}}(\xi).$$

Problem 4. (5+10+10+5 points) Let $\mathscr{H} = L^2(\mathbb{R}^d)$. Let Ψ be a normalized vector on the bosonic Fock space $\mathcal{F}(\mathscr{H})$ with $\langle \Psi, \mathcal{N}\Psi \rangle < \infty$. Here $\mathcal{N}$ is the number operator on $\mathcal{F}(\mathscr{H})$.

(a) Assume that $\Psi = (\lambda_n \Psi_n)_{n=0}^{\infty}$ with $\Psi_n \in \mathscr{H}^{\otimes_s n}$, $\|\Psi_n\| = 1$ and $\sum_{n\geq 0} |\lambda_n|^2 = 1$. Denote

$$\rho_\Psi(x) := \sum_{n\geq 1} |\lambda_n|^2 \rho_{\Psi_n}(x)$$

where $\rho_{\Psi_n}(x) := \gamma^{(1)}_{\Psi_n}(x,x)$ is the one-body density functional of Ψ_n. Prove that

$$\int_{\mathbb{R}^d} \rho_\Psi(x) \mathrm{d}x = \langle \Psi, \mathcal{N}\Psi \rangle.$$

(b) Consider the kinetic operator $\mathrm{d}\Gamma(-\Delta)$ on $\mathcal{F}(\mathscr{H})$. Prove that

$$\langle \Psi, \mathrm{d}\Gamma(-\Delta)\Psi \rangle \geq \int_{\mathbb{R}^d} |\nabla \sqrt{\rho_\Psi}|^2.$$

(c) Let $w : \mathbb{R}^d \to \mathbb{R}$ be an even function such that $0 \leq \widehat{w} \in L^1(\mathbb{R}^d)$. Consider the interaction operator on $\mathcal{F}(\mathscr{H})$

$$\mathbb{W} = 0 \oplus 0 \oplus \bigoplus_{n=2}^{\infty} \Big(\sum_{1\leq i<j\leq n} w(x_i - x_j) \Big).$$

Prove that

$$\langle \Psi, \mathbb{W}\Psi \rangle \geq \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \rho_\Psi(x)\rho_\Psi(y) w(x-y) \mathrm{d}x\mathrm{d}y - C_w \int_{\mathbb{R}^d} \rho_\Psi.$$

(d) For any $N \in \mathbb{N}$, consider the energy

$$E_N := \inf \Big\{ \langle \Psi, \mathrm{d}\Gamma(-\Delta)\Psi \rangle + \frac{1}{N} \langle \Psi, \mathbb{W}\Psi \rangle \,|\, \Psi \in \mathcal{F}(\mathscr{H}), \|\Psi\| = 1, \langle \Psi, \mathcal{N}\Psi \rangle = N \Big\}.$$

Prove that when $N \to \infty$ we have

$$E_N = N e_{\mathrm{H}} + \mathcal{O}(1)$$

where

$$e_{\mathrm{H}} := \inf \Big\{ \int_{\mathbb{R}^d} |\nabla u(x)|^2 \mathrm{d}x + \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(x)|^2 |u(y)|^2 w(x-y) \mathrm{d}x\mathrm{d}y \,|\, u \in L^2(\mathbb{R}^d), \|u\|_{L^2} = 1 \Big\}.$$

Problem 5. (5+5+10+5 points) Let $Z > 1$. Consider the Hartree functional

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} |\nabla u(x)|^2 \mathrm{d}x - Z \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} \mathrm{d}x + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} \mathrm{d}x \mathrm{d}y.$$

Given that there exists a unique Hartree minimizer $0 \le u_0 \in H^1(\mathbb{R}^3)$, $\|u_0\|_{L^2} = 1$, and it is the unique solution (up to a constant factor) to the equation $hu = 0$ where

$$h := -\Delta - \frac{Z}{|x|} + \int_{\mathbb{R}^3} \frac{|u_0(y)|^2}{|x-y|} \mathrm{d}y - \mu, \qquad \text{for some } \mu \in (-\infty, 0).$$

(a) Prove that $h + \mu$ has infinitely many negative eigenvalues.

(b) Let K be an operator on $L^2(\mathbb{R}^3)$ with kernel

$$K(x,y) = u_0(x)|x-y|^{-1} u_0(y).$$

Prove that K is a Hilbert-Schmidt operator and $K \ge 0$.

(c) Consider the quadratic Hamiltonian on the bosonic Fock space $\mathcal{F}(\mathscr{H}_+)$, $\mathscr{H}_+ = \{u_0\}^\perp \subset L^2(\mathbb{R}^3)$,

$$\mathbb{H}_{\mathrm{Bog}} := \sum_{m,n \ge 1} \langle u_m, (h+K) u_n \rangle a_m^* a_n + \frac{1}{2} \sum_{m,n \ge 1} \Big(\langle u_m, K \overline{u_n} \rangle a_m^* a_n^* + h.c. \Big)$$

where $a_n = a(u_n)$ with $\{u_n\}_{n \ge 1}$ an orthonormal basis for $\mathscr{H}_+$. Prove that $\mathbb{H}_{\mathrm{Bog}}$ can be diagonalized by a Bogoliubov transformation $\mathbb{U}$, namely

$$\mathbb{U}^* \mathbb{H}_{\mathrm{Bog}} \mathbb{U} = \mathrm{d}\Gamma(\xi) + e_{\mathrm{Bog}}$$

with $e_{\mathrm{Bog}} \in \mathbb{R}$ and with a self-adjoint operator $\xi > 0$ on $\mathscr{H}_+$.

(d) What is $\sigma_{\mathrm{ess}}(\xi)$?