Final exam

(12.2.2021)

Surname: ________________ Given name: ________________

Birthday: ________________ Matriculation: ________________

- There are 3 problems with total $40 + 40 + 50 = 130$ points. You need 50 points to pass and 85 points to get the grade 1.0.
- You have 5 hours from 9:00 to 14:00.
- You can use the lecture notes and solutions of homework sheets.
- Discussion with other people is not allowed.
- Please send your solutions to "nam@math.lmu.de".

Problem 1	Problem 2	Problem 3	$\sum$	GRADE

Problem 1 (10+20+10 points). Here is an alternative proof of the Lieb–Thirring inequality in one dimension. Let $\{u_n\}_{n=1}^N \subset C_c^\infty(\mathbb{R})$ be an orthonormal family in $L^2(\mathbb{R})$ and denote

$$\gamma(x,y) = \sum_{n=1}^N u_n(x)\overline{u_n(y)}, \quad \rho(x) = \sum_{n=1}^N |u_n(x)|^2, \quad \forall x, y \in \mathbb{R}.$$

(a) Prove that for all $y, z \in \mathbb{R}$ we have

$$|\gamma(z,y)|^4 \le \left(\int_{\mathbb{R}} |\gamma(x,y)|^2 \mathrm{d}x\right)\left(\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}x\right).$$

Hint: You can use $g(x) = \int_{-\infty}^x g'(t)\mathrm{d}t = -\int_x^\infty g'(t)\mathrm{d}t$ with a suitable function g.

(b) Use (a) to prove that

$$\sum_{n=1}^N \int_{\mathbb{R}} |u_n'(x)|^2 \mathrm{d}x \ge \int_{\mathbb{R}} \rho(x)^3 \mathrm{d}x.$$

Hint: You can use $\rho(y) = \int_{\mathbb{R}} |\gamma(x,y)|^2 \mathrm{d}x$

(c) Use (b) to prove that for every function $0 \le V \in C_c^\infty(\mathbb{R})$ we have

$$\mathrm{Tr}(-\Delta - V)_- \ge -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} |V(x)|^{3/2} \mathrm{d}x.$$

Here $\mathrm{Tr}(-\Delta - V)_-$ is the sum of all negative eigenvalues of $-\Delta - V$.

Solutions: (a) For any function $g \in C^1(\mathbb{R})$ satisfying $\lim_{|x|\to\infty} g(x) = 0$ we have

$$g(x) = \int_{-\infty}^x g'(t)\mathrm{d}t = -\int_x^\infty g'(t)\mathrm{d}t.$$

Consequently, by the triangle inequality

$$|g(x)| \le \int_{-\infty}^x |g'(t)|\mathrm{d}t \quad \text{and} \quad |g(x)| \le \int_x^\infty |g'(t)|\mathrm{d}t.$$

Consequently

$$|g(x)| \le \frac{1}{2}\int_{-\infty}^x |g'(t)|\mathrm{d}t + \frac{1}{2}\int_x^\infty |g'(t)|\mathrm{d}t = \frac{1}{2}\int_{\mathbb{R}} |g'(t)|\mathrm{d}t.$$

Thus we get the basic Sobolev inequality

$$\|g\|_{L^\infty(\mathbb{R})} \le \frac{1}{2}\|g'\|_{L^1(\mathbb{R})}.$$

Replacing g by $|g|^2$ and using Hölder's inequality we obtain

$$\|g\|^2_{L^\infty(\mathbb{R})} \leq \frac{1}{2}\|(g^2)'\|_{L^1(\mathbb{R})} = \|gg'\|_{L^1(\mathbb{R})} \leq \|g\|_{L^2}\|g'\|_{L^2}.$$

Applying the latter bound to $g(x) = \gamma(x,y)$ we find that for all $y, z \in \mathbb{R}$

$$|\gamma(z,y)|^2 \leq \left(\int_{\mathbb{R}} |\gamma(x,y)|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}x\right)^{1/2}$$

which is equivalent to the desired inequality.

(b) Applying (a) with $z = y$ we have, for all $y \in \mathbb{R}$,

$$\rho(y)^4 \leq \left(\int_{\mathbb{R}} |\gamma(x,y)|^2 \mathrm{d}x\right) \left(\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}x\right).$$

Let us simplify the right side. We have

$$\begin{aligned} |\gamma(x,y)|^2 = \overline{\gamma(x,y)}\gamma(x,y) &= \Big(\sum_{m}^{N} \overline{u_m(x)} u_m(y)\Big)\Big(\sum_{n=1}^{N} u_n(x)\overline{u_n(y)}\Big) \\ &= \sum_{m,n=1}^{N} \overline{u_m(x)} u_n(x) u_m(y) \overline{u_n(y)}. \end{aligned}$$

Hence, using the fact that $\{u_n\}_{n=1}^N$ are orthonormal, we get

$$\begin{aligned} \int_{\mathbb{R}} |\gamma(x,y)|^2 \mathrm{d}x &= \sum_{m,n=1}^{N} \int_{\mathbb{R}} \overline{u_m(x)} u_n(x) \mathrm{d}x \left(u_m(y)\overline{u_n(y)}\right) \\ &= \sum_{m,n=1}^{N} \delta_{mn} \left(u_m(y)\overline{u_n(y)}\right) = \sum_{n=1}^{N} |u_n(y)|^2 = \rho(y). \end{aligned}$$

Thus from (a) we get

$$\rho(y)^4 \leq \rho(y) \left(\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}x\right).$$

which is equivalent to

$$\rho(y)^3 \leq \int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}x.$$

Integrating over y and using Fubini's theorem we obtain

$$\int_{\mathbb{R}} \rho(y)^3 \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}x\right) \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}y\right) \mathrm{d}x.$$

Similarly to the above computation, we have

$$\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}y = \sum_{m,n=1}^{N} \overline{u'_m(x)} u'_n(x) \left(\int_{\mathbb{R}} u_m(y) \overline{u_n(y)} \mathrm{d}y \right)$$

$$= \sum_{m,n=1}^{N} \overline{u'_m(x)} u'_n(x) \delta_{mn} = \sum_{n=1}^{N} |u'_n(x)|^2.$$

Thus we conclude that

$$\int_{\mathbb{R}} \rho(y)^3 \mathrm{d}y \leq \int_{\mathbb{R}} \sum_{n=1}^{N} |u'_n(x)|^2 \mathrm{d}x.$$

(c) By the min–max principle

$$\mathrm{Tr}(-\Delta - V) = \inf \left\{ \sum_{n=1}^{N} \langle u_n, (-\Delta - V) u_n \rangle : \{u_n\}_{n=1}^{N} \subset C_c^\infty \text{ and orthonormal in } L^2(\mathbb{R}) \right\}.$$

For every family $\{u_n\}_{n=1}^{N} \subset C_c^\infty$ and orthonormal in $L^2(\mathbb{R})$ from (c) we have

$$\sum_{n=1}^{N} \langle u_n, (-\Delta - V) u_n \rangle = \sum_{n=1}^{N} \int_{\mathbb{R}} |u'_n(x)|^2 \mathrm{d}x - \int_{\mathbb{R}} V(x)\rho(x) \mathrm{d}x \geq \int_{\mathbb{R}} \rho(x)^3 \mathrm{d}x - \int_{\mathbb{R}} V(x)\rho(x)\mathrm{d}x$$

with $\rho(x) = \sum_{n=1}^{N} |u_n(x)|^2$. Using the AM-GM inequality

$$a + b + c \geq 3(abc)^{1/3}, \quad a, b, c \geq 0$$

with $a = \rho(x)^3$, $b = c = (V(x)/3)^{3/2}$ we get

$$\rho(x)^3 + \frac{2}{3\sqrt{3}} V(x)^{3/2} \geq \rho(x) V(x), \quad \forall x \in \mathbb{R}.$$

Thus

$$\int_{\mathbb{R}} \rho(x)^3 \mathrm{d}x - \int_{\mathbb{R}} V(x)\rho(x)\mathrm{d}x \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} V(x)^{3/2} \mathrm{d}x.$$

Hence, we conclude that

$$\mathrm{Tr}(-\Delta - V) \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} V(x)^{3/2} \mathrm{d}x.$$

Problem 2 (10+10+20 points). Consider the operator

$$A = -\Delta - |x|^{-1/2}$$

on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$. We know that A is self–adjoint and has infinitely many eigenvalues.

(a) Let E_N be the sum of the first N eigenvalues of A. Prove that

$$\lim_{N\to\infty} \frac{E_N}{N^{7/9}} = E^{\mathrm{TF}} := \inf_{\substack{0\le f\in L^1\cap L^{5/3}\\ \int_{\mathbb{R}^3} f\le 1}} \int_{\mathbb{R}^3} \Big(\frac{3}{5}(6\pi^2)^{2/3} f(x)^{5/3} - \frac{f(x)}{|x|^{1/2}}\Big)\mathrm{d}x.$$

Hint: A has the same spectrum with $A_\ell = \ell^2(-\Delta) - \ell^{1/2}|x|^{-1/2}$ for every $\ell > 0$.

(b) Prove that E^{TF} has a unique minimizer f_0 and compute f_0.

(c) Denote $\rho_N(x) = \sum_{i=1}^N |u_n(x)|^2$ with $\{u_n\}_{n=1}^N$ being the first N eigenfunctions of A. Define the 'half–radius' $R_N > 0$ by

$$\int_{|x|<R_N} \rho_N(x)\mathrm{d}x = \int_{|x|>R_N} \rho_N(x)\mathrm{d}x = \frac{N}{2}.$$

Prove that the following limit exists

$$\lim_{N\to\infty} \frac{R_N}{N^{4/9}}.$$

Solutions: (a) By changing the variables, namely by using the unitary operator $f \mapsto \ell^{3/2} f(\ell\cdot)$, we see that A has the same spectrum with

$$A_\ell = \ell^2(-\Delta) - \ell^{1/2}|x|^{-1/2} = \ell^{1/2}(\ell^{3/2}(-\Delta) - |x|^{-1/2})$$

for every $\ell > 0$. In particular, taking

$$\ell^{3/2} = N^{-2/3} \iff \ell = N^{-4/9}$$

we find that

$$\sigma(A) = N^{-2/9}\sigma(N^{-2/3}(-\Delta) - |x|^{-1/2}).$$

Thus $N^{2/9}E_N$ is equal to the sum of the first N eigenvalues of $N^{-2/3}(-\Delta) - |x|^{-1/2}$. By Pauli's exclusion principle, $N^{2/9}E_N$ is equal to the ground state energy of the Hamiltonian

$$H_N = \sum_{i=1}^N (N^{-2/3}(-\Delta_{x_i}) - |x_i|^{-1/2}), \quad x_n \in \mathbb{R}^3$$

on the anti–symmetric space $L^2_a(\mathbb{R}^{3N})$. Note that we can write the potential $-|x|^{-1/2}$ as

$$-|x|^{-1/2} = -|x|^{-1/2}\mathbb{1}(|x| \le 1) + -|x|^{-1/2}\mathbb{1}(|x| \ge 1) \in L^4(\mathbb{R}^3) + L^8(\mathbb{R}^3)$$

where $4, 8 \in [1+3/2, \infty)$. Thus we can apply the convergence to the Thomas–Fermi theory in the lecture notes (Section 8.4) and obtain

$$\lim_{N\to\infty} \frac{N^{2/9} E_N}{N} = E^{\rm TF}.$$

(b) The existence of a minimizer ρ_0 for $E^{\rm TF}$ follows from Homework 8.3 (our situation here is even simpler since there is no interaction potential). The minimizer is unique since the Thomas–Fermi functional

$$\mathcal{E}^{\rm TF}(f) = \int_{\mathbb{R}^3} \Big(\frac{3}{5} (6\pi^2)^{2/3} f(x)^{5/3} - \frac{f(x)}{|x|^{1/2}} \Big) {\rm d}x$$

is strictly convex (the potential term is linear in f and the mapping $f \mapsto f^{5/3}$ is strictly convex).

It remains to compute f_0. It is similar to Homework 7.3 but let us recall the argument here. First, note that $f_0 \not\equiv 0$ since $E^{\rm TF} < 0$. Indeed, by choose $g_\ell = \ell^3 g(\ell x)$ with a fixed function $0 < g \in L^1 \cap L^{5/3}$ we obtain

$$\mathcal{E}^{\rm TF}(g_\ell) = \ell^2 \int_{\mathbb{R}^3} \frac{3}{5} (6\pi^2)^{2/3} g(x)^{5/3} {\rm d}x - \ell^{1/2} \int_{\mathbb{R}^3} \frac{g(x)}{|x|^{1/2}} {\rm d}x < 0$$

if $\ell > 0$ is small enough.

Next, arguing exactly as in Homework 7.3 (the details of the potential is not important for this part), we obtain the Thomas–Fermi equation

$$(6\pi^2)^{2/3} f_0(x)^{2/3} = \Big[\frac{1}{|x|^{1/2}} - \mu \Big]_+ \iff f_0(x) = \frac{1}{6\pi^2} \Big[\frac{1}{|x|^{1/2}} - \mu \Big]_+^{3/2}.$$

for a constant

$$0 \le \mu = - \frac{\int_{\mathbb{R}^3} f_0(x) \left[(6\pi^2)^{2/3} f_0(x)^{2/3} - |x|^{-1/2} \right] {\rm d}x}{\int_{\mathbb{R}^3} f_0}.$$

If $\int_{\mathbb{R}^3} f_0 < 1$, then we have

$$\mathcal{E}^{\rm TF}((1+t) f_0) \ge \mathcal{E}^{\rm TF}(f_0), \quad \forall t \in (-\varepsilon, \varepsilon)$$

for some $\varepsilon > 0$ small enough. Hence

$$0 = \frac{d}{dt}_{|t=0} \mathcal{E}^{\rm TF}((1+t) f_0) = \int_{\mathbb{R}^3} f_0(x) \left[(6\pi^2)^{2/3} f_0(x)^{2/3} - |x|^{-1/2} \right] {\rm d}x$$

which implies that $\mu = 0$. However, in this case the Thomas–Fermi equation implies that

$$f_0(x) = \frac{1}{6\pi^2} \frac{1}{|x|^{3/4}}$$

which is not integrable.

Thus $\int_{\mathbb{R}^3} f_0 = 1$. Using the Thomas–Fermi equation we can compute μ:

$$1 = \int_{\mathbb{R}^3} f_0(x)\mathrm{d}x = \frac{1}{6\pi^2}\int_{\mathbb{R}^3}\left[\frac{1}{|x|^{1/2}} - \mu\right]_+^{3/2}\mathrm{d}x = \frac{1}{6\pi^2}4\pi\int_0^{\mu^{-2}}\left[\frac{1}{r^{1/2}} - \mu\right]^{3/2} r^2\mathrm{d}r = \frac{7}{768\mu^{9/2}}.$$

Thus

$$\mu = \left(\frac{7}{768}\right)^{2/9}.$$

(c) **Step 1.** By the rescaling argument in (a), the functions $\{u_n^{(\ell)}\}_{n=1}^N$ with

$$u_n^{(\ell)}(x) = \ell^{3/2}u_n(\ell x), \quad \ell = N^{-4/9}$$

are the first N eigenfunctions of the operator $N^{-2/3}(-\Delta) - |x|^{-1}$. Hence, the Slater determinant

$$u_1^{(\ell)} \wedge u_2^{(\ell)} \wedge ... \wedge u_N^{(\ell)}$$

is a ground state for the Hamiltonian H_N. In particular,

$$\rho_N^{(\ell)}(x) = \sum_{n=1}^N |u_n^{(\ell)}(x)|^2 = \sum_{n=1}^N \ell^3|u_n(\ell x)|^2 = \ell^3\rho_N(\ell x)$$

is the one–body density of this ground state. By the convergence to the Thomas–Fermi theory in the lecture notes (Section 8.4) we find that

$$N^{-1}\rho_N^{(\ell)} \rightharpoonup f_0$$

weakly in $L^{5/3}(\mathbb{R}^3)$. Consequently, for every constant $R > 0$ we have

$$\int_{|x|<R} f_0(x)\mathrm{d}x = \lim_{N\to\infty} N^{-1}\int_{|x|<R}\rho_N^{(\ell)}(x)\mathrm{d}x = \lim_{N\to\infty} N^{-1}\int_{|x|<R}\ell^3\rho_N(\ell x)\mathrm{d}x.$$

On the other hand, from the definition of R_N we have

$$\frac{1}{2} = N^{-1}\int_{|x|<R_N}\rho_N(x)\mathrm{d}x = N^{-1}\int_{|x|<\ell R_N}\rho_N(\ell x)\mathrm{d}x.$$

Step 2. Let us show that ℓR_N is bounded. Indeed, if ℓR_N is unbounded, then up to a subsequence as $N \to \infty$ we have $\ell R_N \to \infty$. This implies that for every $R > 0$, we have $\ell R_N \geq R$ for N large and hence

$$\frac{1}{2} = N^{-1}\int_{|x|<\ell R_N}\rho_N(\ell x)\mathrm{d}x \geq N^{-1}\int_{|x|<R}\rho_N(\ell x)\mathrm{d}x \to_{N\to\infty} \int_{|x|<R} f_0(x)\mathrm{d}x.$$

Since it holds for all $R > 0$, we can take $R \to \infty$ and find that

$$\frac{1}{2} \geq \lim_{R\to\infty} \int_{|x|<R} f_0(x)\mathrm{d}x = \int_{\mathbb{R}^3} f_0(x)\mathrm{d}x = 1$$

which is a contradiction. Thus ℓR_N is bounded.

Step 3. Since $\ell R_N \to R_0$ is bounded, up to a subsequence we can assume that $\ell R_N \to R_0$. Let us show that

$$\frac{1}{2} = \int_{|x|<R_0} f_0(x)\mathrm{d}x.$$

Indeed, for every $\varepsilon > 0$ we have $\ell R_N \geq (1-\varepsilon)R_0$ for N large, and hence

$$\frac{1}{2} = N^{-1} \int_{|x|<\ell R_N} \rho_N(\ell x)\mathrm{d}x \geq N^{-1} \int_{|x|<(1-\varepsilon)R_0} \rho_N(\ell x)\mathrm{d}x \to_{N\to\infty} \int_{|x|<(1-\varepsilon)R_0} f_0(x)\mathrm{d}x.$$

Since it holds for every $\varepsilon > 0$ we obtain

$$\frac{1}{2} \geq \lim_{\varepsilon\to 0^+} \int_{|x|<(1-\varepsilon)R_0} f_0(x)\mathrm{d}x = \int_{|x|<R_0} f_0(x)\mathrm{d}x.$$

Similarly,

$$\frac{1}{2} = N^{-1} \int_{|x|<\ell R_N} \rho_N(\ell x)\mathrm{d}x \leq N^{-1} \int_{|x|<(1+\varepsilon)R_0} \rho_N(\ell x)\mathrm{d}x \to_{N\to\infty} \int_{|x|<(1+\varepsilon)R_0} f_0(x)\mathrm{d}x$$

and hencee

$$\frac{1}{2} \leq \lim_{\varepsilon\to 0^+} \int_{|x|<(1+\varepsilon)R_0} f_0(x)\mathrm{d}x = \int_{|x|<R_0} f_0(x)\mathrm{d}x.$$

Thus

$$\frac{1}{2} = \int_{|x|<R_0} f_0(x)\mathrm{d}x = \frac{1}{6\pi^2} \int_{|x|<R_0} \left[\frac{1}{|x|^{-1/2}} - \mu\right]_+^{3/2} \mathrm{d}x.$$

Finally, note that the function

$$g(R) := \frac{1}{6\pi^2} \int_{|x|<R} \left[\frac{1}{|x|^{-1/2}} - \mu\right]_+^{3/2} \mathrm{d}x$$

is increasing on $R \in [0, \infty)$; moreover, $g(0) = 0, g(\mu^{-2}) = 1$ and g is strictly increasing on $R \in (0, \mu^{-2})$ since $\left[\frac{1}{|x|^{-1/2}} - \mu\right]_+^{3/2} > 0$ for all $|x| \in (0, \mu^{-2})$. Thus there exists a unique R_0 such that $g(R_0) = 1/2$ (we can even compute R_0 explicitly). Since the limit R_0 is unique, we have the convergence of the whole sequence $\ell R_N \to R_0$.

Problem 3 (10+10+10+20 points). Let $N \geq 2$ and consider the operator

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} - |x_j|^{-1/2}\right) + \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1/2}, \quad x_j \in \mathbb{R}^3,$$

on the anti-symmetric space $L_a^2(\mathbb{R}^{3N})$ with the core domain $\mathcal{D}_N = L_a^2(\mathbb{R}^{3N}) \cap C_c^\infty(\mathbb{R}^{3N})$. Let $\Psi_N \in \mathcal{D}_N$ be a normalized function in $L_a^2(\mathbb{R}^{3N})$ and let ρ_N be its one–body density.

(a) Prove that for every $R > 0$ we have

$$\left\langle \Psi_N, \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1/2} \Psi_N \right\rangle \geq \frac{1}{2\sqrt{2R}} (N_R^2 - N_R)$$

with $N_R = \int_{|x| \leq R} \rho_N(x) \mathrm{d}x$.

Hint: You can use $|x - y|^{-1/2} \geq (2R)^{-1/2} \mathbb{1}_{B(0,R)}(x) \mathbb{1}_{B(0,R)}(y)$.

(b) Prove that for every $R > r > 0$ we have

$$\left\langle \Psi_N, \sum_{i=1}^{N} (-\Delta_{x_i} - |x_i|^{-1/2}) \Psi_N \right\rangle \geq -Cr^{7/4} - \frac{N_R}{\sqrt{r}} - \frac{N}{\sqrt{R}}$$

with a constant $C > 0$ independent of N, R, r.

Hint: You can split the potential $-|x|^{-1/2}$ into three parts.

(c) Use (a) and (b) to prove that $H_N \geq -CN^{14/25}$ with a constant $C > 0$ independent of N.

Note that $N^{14/25} \ll N^{7/9}$ when $N \to \infty$. Thus the interaction improves significantly the ground state energy (c.f. Problem 2 for the non–interacting case).

(d) Can you prove that $H_N \geq -CN^a$ with a constant $a < 14/25$?

Solutions: (a) For every $R > 0$, by the triangle inequality we have

$$|x - y| \leq 2R, \quad \forall x, y \in B_R = B(0, R).$$

Hence,

$$|x - y|^{-1/2} \geq (2R)^{-1/2} \mathbb{1}_{B_R}(x) \mathbb{1}_{B_R}(y), \quad \forall x, y \in \mathbb{R}^3.$$

Consequently,

$$\sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1/2} \geq (2R)^{-1/2} \sum_{1 \leq i < j \leq N} \mathbb{1}_{B_R}(x_i) \mathbb{1}_{B_R}(x_j)$$

$$= \frac{1}{2\sqrt{2R}} \left(\left(\sum_{i=1}^{N} \mathbb{1}_{B_R}(x_i) \right)^2 - \sum_{i=1}^{N} \mathbb{1}_{B_R}(x_i) \right).$$

Taking the expectation against Ψ_N and using the Cauchy–Schwarz inequality we find that

$$\begin{aligned}
&\Big\langle \Psi_N, \sum_{1\le i<j\le N} |x_i - x_j|^{-1/2}\Psi_N \Big\rangle \\
&\ge \frac{1}{2\sqrt{2R}} \left(\Big\langle \Psi_N, \Big(\sum_{i=1}^N \mathbb{1}_{B_R}(x_i)\Big)^2 \Psi_N \Big\rangle - \Big\langle \Psi_N, \Big(\sum_{i=1}^N \mathbb{1}_{B_R}(x_i)\Big) \Psi_N \Big\rangle \right) \\
&\ge \frac{1}{2\sqrt{2R}} \left(\Big\langle \Psi_N, \Big(\sum_{i=1}^N \mathbb{1}_{B_R}(x_i)\Big) \Psi_N \Big\rangle^2 - \Big\langle \Psi_N, \Big(\sum_{i=1}^N \mathbb{1}_{B_R}(x_i)\Big) \Psi_N \Big\rangle \right).
\end{aligned}$$

By the definition of the one–body density ρ_N we have

$$\Big\langle \Psi_N, \Big(\sum_{i=1}^N \mathbb{1}_{B_R}(x_i)\Big) \Psi_N \Big\rangle = \int_{\mathbb{R}^3} \rho_N(x)\mathbb{1}_{B_R}(x)\mathrm{d}x = \int_{B_R} \rho_N(x)\mathrm{d}x = N_R.$$

Thus we conclude that

$$\Big\langle \Psi_N, \sum_{i=1}^N (-\Delta_{x_i} - |x_i|^{-1/2})\Psi_N \Big\rangle \ge \frac{N_R^2 - N_R}{2\sqrt{2R}}.$$

(b) For any $R > r > 0$ we can decompose

$$|x|^{-1/2} = V_1 + V_2 + V_3$$

with

$$V_1 = |x|^{-1/2}\mathbb{1}(|x| \le r), \quad V_2 = |x|^{-1/2}\mathbb{1}(r < |x| \le R), \quad V_3 = |x|^{-1/2}\mathbb{1}(|x| > R).$$

Hence,

$$\begin{aligned}
&\Big\langle \Psi_N, \sum_{i=1}^N (-\Delta_{x_i} - |x_i|^{-1/2})\Psi_N \Big\rangle \\
&= \Big\langle \Psi_N, \sum_{i=1}^N (-\Delta_{x_i} - V_1(x_i))\Psi_N \Big\rangle - \int_{\mathbb{R}^3} V_2(x)\rho_N(x)\mathrm{d}x - \int_{\mathbb{R}^3} V_3(x)\rho_N(x)\mathrm{d}x.
\end{aligned}$$

Let us estimate the right side term by term.

For $k = 1$, using Pauli's exclusion principle and Lieb–Thirring inequality we have

$$\begin{aligned}
\Big\langle \Psi_N, \sum_{i=1}^N (-\Delta_{x_i} - V_1(x_i))\Psi_N \Big\rangle &\ge \mathrm{Tr}(-\Delta - V_1)_- \ge -L_{1,3}\int_{\mathbb{R}^3} |V_1(x)|^{5/2}\mathrm{d}x \\
&= -L_{1,3}\int_{|x|\le r} \frac{1}{|x|^{5/4}}\mathrm{d}x = -L_{1,3}(4\pi)r^{7/4}.
\end{aligned}$$

For $k=2$, using $V_2 \le r^{-1/2}\mathbb{1}(|x|\le R)$ and the definition of N_R we have

$$-\int_{\mathbb{R}^3} V_2(x)\rho_N(x)\mathrm{d}x \ge -\frac{1}{\sqrt{r}}\int_{|x<R|}\rho_N(x)\mathrm{d}x = -\frac{N_R}{\sqrt{r}}.$$

For $k=3$, using $V_3 \le R^{-1/2}$ we have

$$-\int_{\mathbb{R}^3} V_3(x)\rho_N(x)\mathrm{d}x \ge -\frac{1}{\sqrt{R}}\int_{\mathbb{R}^3}\rho_N(x)\mathrm{d}x = -\frac{N}{\sqrt{R}}.$$

Thus in summary,

$$\Big\langle \Psi_N, \sum_{i=1}^N(-\Delta_{x_i} - |x_i|^{-1/2})\Psi_N\Big\rangle \ge -Cr^{7/4} - \frac{N_R}{\sqrt{r}} - \frac{N}{\sqrt{R}}.$$

The constant $C = 4\pi L_{1,3}$ is independent of N, R, r.

(c) From (a) and (b) we find that for every normalized wave function $\Psi_N \in \mathcal{D}_N$ and for every $R > r > 0$

$$\langle \Psi_N, H_N\Psi_N\rangle \ge -Cr^{7/4} - \frac{N_R}{\sqrt{r}} - \frac{N}{\sqrt{R}} + \frac{N_R^2 - N_R}{2\sqrt{2R}}.$$

Since $N_R \le N$ we have

$$-\frac{N_R}{2\sqrt{2R}} \ge -\frac{N}{2\sqrt{2R}}.$$

Moreover, by the Cauchy–Schwarz inequality

$$\frac{N_R^2}{2\sqrt{2R}} - \frac{N_R}{\sqrt{r}} \ge -\frac{R}{\sqrt{2}r}.$$

Thus

$$\langle \Psi_N, H_N\Psi_N\rangle \ge -C\Big(r^{7/4} + \frac{R}{r} + \frac{N}{\sqrt{R}}\Big)$$

with a constant C independent of N, R, r. It remains to choose R and r to optimize the right side. We can take them such that

$$r^{7/4} = \frac{R}{r} = \frac{N}{\sqrt{R}} = \left((r^{7/4})^4\left(\frac{R}{r}\right)^7\left(\frac{N}{\sqrt{R}}\right)^{14}\right)^{1/(4+7+14)} = N^{14/25}.$$

Thus we conclude that

$$\langle \Psi_N, H_N\Psi_N\rangle \ge -CN^{14/25}.$$

Since it holds for every normalized function $\Psi_N \in \mathcal{D}_N$, we get the operator lower bound $H_N \ge -CN^{14/25}$.

(d) We can improve the bound to $H_N \ge -C_\varepsilon N^\varepsilon$ for every $\varepsilon > 0$. The idea is that we can decompose $V(x) = |x|^{-1/2}$ into several pieces. Introducing the parameters

$$r_1 < r_2 < ... < r_M$$

we can write

$$\begin{aligned} V(x) = |x|^{-1/2} &= V(x)\mathbb{1}(|x| < r_1) + \sum_{m=2}^{M} V(x)\mathbb{1}(r_{m-1} \le |x| < r_m) + V(x)\mathbb{1}(|x| \ge r_M) \\ &\le V(x)\mathbb{1}(|x| < r_1) - \sum_{m=2}^{M} \frac{\mathbb{1}(|x| < r_m)}{\sqrt{r_{m-1}}} - \frac{1}{\sqrt{r_M}}. \end{aligned}$$

Therefore, arguing as in (c) we obtain for every normalized wave function $\Psi_N \in \mathcal{D}_N$

$$\langle \Psi_N, H_N \Psi_N \rangle \ge -C r_1^{7/4} - \sum_{m=2}^{M} \frac{N_{r_m}}{\sqrt{r_{m-1}}} - \frac{N}{\sqrt{r_M}}.$$

Moreover, from (a) we can write, for every $m = 1, 2, ..., M$,

$$\left\langle \Psi_N, \sum_{1 \le i < j \le N} |x_i - x_j|^{-1/2} \Psi_N \right\rangle \ge \frac{(N_{r_m}^2 - N_{r_m})_+}{2\sqrt{2 r_m}} \ge \frac{(N_{r_m}^2 - 1)}{2\sqrt{2 r_m}}$$

since the left side is finite. Thus in summary,

$$\langle \Psi_N, H_N \Psi_N \rangle \ge -C r_1^{7/4} - \sum_{m=2}^{M} \frac{N_{r_m}}{\sqrt{r_{m-1}}} - \frac{N}{\sqrt{r_M}} + \frac{1}{M} \sum_{m=2}^{M} \frac{(N_{r_m}^2 - 1)}{2\sqrt{2 r_m}}.$$

We will choose all $r_m \ge 1$, so that

$$-\frac{1}{2\sqrt{2 r_m}} \ge -1.$$

Moreover, by the Cauchy–Schwarz inequality

$$\frac{N_{r_m}^2}{M 2\sqrt{2 r_m}} - \frac{N_{r_m}}{\sqrt{r_{m-1}}} \ge -C_M \frac{r_m}{r_{m-1}}, \quad \forall m = 2, ..., M.$$

Hence,

$$\langle \Psi_N, H_N \Psi_N \rangle \ge -C_M \left(r_1^{7/4} + \sum_{m=2}^{M} \frac{r_m}{r_{m-1}} + \frac{N}{\sqrt{r_M}} + 1 \right).$$

It remains to optimize the right side by choosing $1 \le r_1 \le r_2 \le ... \le r_M$. We can take

$$\begin{aligned} 1 \le r_1^{7/4} = \frac{r_2}{r_1} = ... = \frac{r_M}{r_{M-1}} = \frac{N}{\sqrt{r_M}} &= \left((r_1^{7/4})^4 \prod_{m=2}^{M} \left(\frac{r_m}{r_{m-1}} \right)^7 \left(\frac{N}{\sqrt{r_M}} \right)^{14} \right)^{1/(4+7(M-1)+14)} \\ &= N^{14/(4+7(M-1)+14)}. \end{aligned}$$

Thus

$$H_N \geq -C_M N^{14/(4+7(M-1)+14)}.$$

Since M can be arbitrarily large, we find that $H_N \geq -C_\varepsilon N^\varepsilon$ for every $\varepsilon > 0$.