Final exam

(12.2.2021)

Surname: ______________________ Given name: ______________________

Birthday: ______________________ Matriculation: ______________________

- There are 3 problems with total $40 + 40 + 50 = 130$ points. You need 50 points to pass and 85 points to get the grade 1.0.
- You have 5 hours from 9:00 to 14:00.
- You can use the lecture notes and solutions of homework sheets.
- Discussion with other people is not allowed.
- Please send your solutions to "nam@math.lmu.de".

Problem 1	Problem 2	Problem 3	$\sum$	GRADE

Problem 1 (10+20+10 points). Here is an alternative proof of the Lieb–Thirring inequality in one dimension. Let $\{u_n\}_{n=1}^N \subset C_c^\infty(\mathbb{R})$ be an orthonormal family in $L^2(\mathbb{R})$ and denote

$$\gamma(x,y) = \sum_{n=1}^N u_n(x)\overline{u_n(y)}, \quad \rho(x) = \sum_{n=1}^N |u_n(x)|^2, \quad \forall x, y \in \mathbb{R}.$$

(a) Prove that for all $y, z \in \mathbb{R}$ we have

$$|\gamma(z,y)|^4 \leq \left(\int_{\mathbb{R}} |\gamma(x,y)|^2 \mathrm{d}x\right)\left(\int_{\mathbb{R}} |\partial_x \gamma(x,y)|^2 \mathrm{d}x\right).$$

Hint: You can use $g(x) = \int_{-\infty}^x g'(t)\mathrm{d}t = -\int_x^\infty g'(t)\mathrm{d}t$ with a suitable function g.

(b) Use (a) to prove that

$$\sum_{n=1}^N \int_{\mathbb{R}} |u_n'(x)|^2 \mathrm{d}x \geq \int_{\mathbb{R}} \rho(x)^3 \mathrm{d}x.$$

Hint: You can use $\rho(y) = \int_{\mathbb{R}} |\gamma(x,y)|^2 \mathrm{d}x$

(c) Use (b) to prove that for every function $0 \leq V \in C_c^\infty(\mathbb{R})$ we have

$$\operatorname{Tr}(-\Delta - V)_- \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} |V(x)|^{3/2} \mathrm{d}x.$$

Here $\operatorname{Tr}(-\Delta - V)_-$ is the sum of all negative eigenvalues of $-\Delta - V$.

Problem 2 (10+10+20 points). Consider the operator

$$A = -\Delta - |x|^{-1/2}$$

on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$. We know that A is self–adjoint and has infinitely many eigenvalues.

(a) Let E_N be the sum of the first N eigenvalues of A. Prove that

$$\lim_{N\to\infty} \frac{E_N}{N^{7/9}} = E^{\mathrm{TF}} := \inf_{\substack{0\le f\in L^1\cap L^{5/3} \\ \int_{\mathbb{R}^3} f \le 1}} \int_{\mathbb{R}^3} \Big(\frac{3}{5}(6\pi^2)^{2/3} f(x)^{5/3} - \frac{f(x)}{|x|^{1/2}} \Big) \mathrm{d}x.$$

Hint: A has the same spectrum with $A_\ell = \ell^2(-\Delta) - \ell^{1/2}|x|^{-1/2}$ for every $\ell > 0$.

(b) Prove that E^{TF} has a unique minimizer f_0 and compute f_0.

(c) Denote $\rho_N(x) = \sum_{i=1}^N |u_n(x)|^2$ with $\{u_n\}_{n=1}^N$ being the first N eigenfunctions of A. Define the 'half–radius' $R_N > 0$ by

$$\int_{|x|<R_N} \rho_N(x)\mathrm{d}x = \int_{|x|>R_N} \rho_N(x)\mathrm{d}x = \frac{N}{2}.$$

Prove that the following limit exists

$$\lim_{N\to\infty} \frac{R_N}{N^{4/9}}.$$

Problem 3 (10+10+10+20 points). Let $N \geq 2$ and consider the operator

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} - |x_j|^{-1/2}\right) + \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1/2}, \quad x_j \in \mathbb{R}^3,$$

on the anti-symmetric space $L^2_a(\mathbb{R}^{3N})$ with the core domain $\mathcal{D}_N = L^2_a(\mathbb{R}^{3N}) \cap C_c^\infty(\mathbb{R}^{3N})$. Let $\Psi_N \in \mathcal{D}_N$ be a normalized function in $L^2_a(\mathbb{R}^{3N})$ and let ρ_N be its one–body density.

(a) Prove that for every $R > 0$ we have

$$\left\langle \Psi_N, \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1/2} \Psi_N \right\rangle \geq \frac{1}{2\sqrt{2R}} (N_R^2 - N_R)$$

with $N_R = \int_{|x| \leq R} \rho_N(x) \mathrm{d}x$.

Hint: You can use $|x - y|^{-1/2} \geq (2R)^{-1/2} \mathbb{1}_{B(0,R)}(x) \mathbb{1}_{B(0,R)}(y)$.

(b) Prove that for every $R > r > 0$ we have

$$\left\langle \Psi_N, \sum_{i=1}^{N} (-\Delta_{x_i} - |x_i|^{-1/2}) \Psi_N \right\rangle \geq -C r^{7/4} - \frac{N_R}{\sqrt{r}} - \frac{N}{\sqrt{R}}$$

with a constant $C > 0$ independent of N, R, r.

Hint: You can split the potential $-|x|^{-1/2}$ into three parts.

(c) Use (a) and (b) to prove that $H_N \geq -CN^{14/25}$ with a constant $C > 0$ independent of N.

Note that $N^{14/25} \ll N^{7/9}$ when $N \to \infty$. Thus the interaction improves significantly the ground state energy (c.f. Problem 2 for the non–interacting case).

(d) Can you prove that $H_N \geq -CN^a$ with a constant $a < 14/25$?