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Tutorial Sheet 1
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Functional Analysis

T1 [6 points]. Let (X, d) be a metric space. Let $x \in X$, $\varepsilon > 0$ and let $B_\varepsilon(x)$ denote the *open ball* of radius ε centred at x, i.e. $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$.

(*i*) Show that the *closed ball* $\bar{B}_\varepsilon(x) := \{y \in X \mid d(x, y) \leqslant \varepsilon\}$ is closed.

(*ii*) Prove that $\overline{B_\varepsilon(x)} \subseteq \bar{B}_\varepsilon(x)$.

(*iii*) Give an example for a metric space where, in general, $\overline{B_\varepsilon(x)} \neq \bar{B}_\varepsilon(x)$.

T2. Let $X = Y := \mathbb{R}$ and consider the two topological spaces $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$, where $\mathcal{T}_2$ is the standard topology on $\mathbb{R}$ induced by the Euclidean metric and

$$\mathcal{T}_1 := \{\emptyset, \mathbb{R}\} \cup \{\mathbb{R}\backslash A \mid A \subseteq \mathbb{R} \text{ is countable}\}.$$

Prove the following:

(*i*) The pair $(X, \mathcal{T}_1)$ is in fact a topological space.

(*ii*) Every mapping $F : X \to Y$ is sequentially continuous.

(*iii*) The mapping $G : X \to Y$, $x \mapsto x$ is not continuous.

T3. Let $1 \leq p \leq q < \infty$. Define

$$c_c := \{x \in \ell^\infty : x_j = 0 \text{ for all but finitely many } j \in \mathbb{N}\},$$
$$c_0 := \{x \in \ell^\infty : x_j \to 0 \text{ for } j \to \infty\}.$$

Moreover if $A \subseteq X$ and d is a metric on X we denote by $\overline{A}^d$ the closure of A in X with respect to the metric d. Show:

(*i*) $c_c \subseteq \ell^p \subseteq \ell^q \subseteq c_0$.

(*ii*) $\overline{c_c}^{d_q} = \ell^q$.

(*iii*) $\overline{c_c}^{d_\infty} = c_0$.

T4. Show: If a topological space $(X, \mathcal{T})$ has a countable subbase, it has a countable base.