0 Recapitulation of basic notions

Let X be a set and $\mathscr{P}(X)$ its power set.

0.1 Topological spaces

0.1 Definition. • $\mathscr{T} \subseteq \mathscr{P}(X)$ is a *topology* $:\Longleftrightarrow$

(1) $\varnothing, X \in \mathscr{T}$.

(2) $\mathscr{T}$ is closed under arbitrary unions (i.e. if I is an arbitrary index set and for every $\alpha \in I$ let a set $A_\alpha \in \mathscr{T}$ be given. Then

$$\bigcup_{\alpha \in I} A_\alpha \in \mathscr{T}$$

holds.).

(3) $\mathscr{T}$ is closed under finite intersections (i.e. if $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathscr{T}$, then

$$\bigcap_{k=1}^{n} A_k \in \mathscr{T}$$

holds.).

- $(X, \mathscr{T})$ is called *topological space* (often just X)
- $A \in \mathscr{P}(X)$ is *open* $:\Longleftrightarrow$ $A \in \mathscr{T}$.
- Let $\mathscr{T}_1, \mathscr{T}_2$ be topologies on X. $\mathscr{T}_1$ is *finer* than $\mathscr{T}_2$ $:\Longleftrightarrow$ $\mathscr{T}_1 \supseteq \mathscr{T}_2$ and *coarser* than $\mathscr{T}_2$ $:\Longleftrightarrow$ $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

0.2 Examples. (a) *Indiscrete topology:* $\mathscr{T} = \{\varnothing, X\}$.

(b) *Discrete topology:* $\mathscr{T} = \mathscr{P}(X)$.

(c) Euclidean (or standard) topology on $\mathbb{R}^d$, $d \in \mathbb{N}$: $A \subseteq \mathbb{R}^d$ is open $:\Longleftrightarrow$ $\forall x \in A \, \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^d : |x - y| < \varepsilon\}$ is the Euclidean ball of radius $\varepsilon > 0$ about $x \in \mathbb{R}^d$.

Induced topology on subsets

0.3 Definition. Let $(X, \mathscr{T})$ be a topological space, $A \in \mathscr{P}(X)$ (not necessarily open!). *Relative topology* on A:

$$\mathscr{T}_A := \{B \subseteq A : \exists C \in \mathscr{T} \text{ with } B = C \cap A\} \subseteq \mathscr{P}(A).$$

0.4 Remark. (a) $\mathscr{T}_A$ is topology on A.

(b) If $A \notin \mathscr{T}$ and $B \in \mathscr{T}_A$ then it may happen that $B \notin \mathscr{T}$.

Example. Let $X = \mathbb{R}$ with standard topology, $A = [0, 1]$. Then $B := [0, 1/2[\in \mathscr{T}_A$ but $B \notin \mathscr{T}$.

0.5 Definition. Let X be a topological space, $A \subseteq X$, $x \in X$.

(a) A is *closed* $:\Longleftrightarrow$ $A^c := X \setminus A \in \mathscr{T}$.

(b) $U \subseteq X$ (not necessarily open) is a *neighbourhood* of x $:\Longleftrightarrow$ $\exists A \in \mathcal{T}$ such that $x \in A$ and $A \subseteq U$.

(c) X is a *Hausdorff space* $:\Longleftrightarrow$ for all $x, y \in X$, $x \neq y$, there exist neighbourhoods U_x of x and U_y of y such that $U_x \cap U_y = \varnothing$.

(d) x is a *limit point* of A (or *accumulation point*) $:\Longleftrightarrow$ for all neighbourhoods U of x

$$U \cap A \neq \varnothing.$$

Note: Every point of A is also a limit point according to this definition.

(e) x is an *interior point* of A $:\Longleftrightarrow$ there exists a neighbourhood U of x such that $U \subseteq A$.

(f) x is a *boundary point* of A $:\Longleftrightarrow$ for every neighbourhood U of x: $U \cap A \neq \varnothing$ and $U \cap A^c \neq \varnothing$.
Boundary of A: $\partial A := \{x \in X : x \text{ boundary point of } A\}$.

(g) *Interior* of A: $\mathring{A} := A \setminus \partial A = \{x \in X : x \text{ interior point of } A\}$
closure of A: $\overline{A} := A \cup \partial A = \{x \in X : x \text{ limit point of } A\}$.

(h) A is *dense* in X $:\Longleftrightarrow$ $X = \overline{A}$.

0.6 Lemma. *Let X be a topological space, $A \subseteq X$.*

(a) *A is open $\Longleftrightarrow \forall x \in A : x$ is an interior point of A.*

(b) *A is closed $\Longleftrightarrow A = \overline{A}$.*

(c) *$\overline{A}, \partial A$ are closed.*

Proof. Exercise. ∎

0.7 Definition. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subseteq \mathcal{T}$ a family of open sets.

(a) $\mathcal{B}$ is a *base* for $\mathcal{T}$ $:\Longleftrightarrow$ $\mathcal{T}$ consists of unions of sets from $\mathcal{B}$.

(b) $\mathcal{S}$ is a *subbase* for $\mathcal{T}$ $:\Longleftrightarrow$ finite intersecions of sets from $\mathcal{S}$ form a base.

(c) $\mathcal{N} \subseteq \mathcal{T}$ is a *neighbourhood base* at $x \in X$ $:\Longleftrightarrow$ every $N \in \mathcal{N}$ is a neighbourhood of x and for every neighbourhood U of x there exists $N \in \mathcal{N}$ with $N \subseteq U$.

0.8 Remark.

Let $\mathcal{S} \subseteq \mathcal{P}(X)$. Then there exists a topology $\mathcal{T}$ on X such that $\mathcal{S}$ is a subbase for $\mathcal{T}$ and $\mathcal{T}$ is the coarsest topology containing $\mathcal{S}$. Jargon: $\mathcal{T}$ is generated by $\mathcal{S}$.

0.9 Example. Consider $\mathbb{R}^d$ with standard topology. Let $x \in \mathbb{R}^d$.

(a) $\{B_{1/n}(x) : n \in \mathbb{N}\}$ is a neighbourhood base at x.

(b) $\{B_{1/n}(q) : n \in \mathbb{N},\ q \in \mathbb{Q}^d\}$ is a base for the standard topology (see the proof of Thm. 1.9 later).

0.10 Definition. Let $I \neq \varnothing$ be an arbitrary index set. For every $\alpha \in I$ let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. Cartesian product space

$$\bigtimes_{\alpha \in I} X_\alpha := \Big\{ f : J \to \bigcup_{\alpha \in I} X_\alpha \text{ with } f(\alpha) \in X_\alpha \Big\}$$

has base of the *product topology*

$$\Big\{ \bigtimes_{\alpha \in I} A_\alpha : A_\alpha \in \mathcal{T}_\alpha \ \forall \alpha \in I,\ A_\alpha \neq X_\alpha \text{ for at most finitely many } \alpha\text{'s} \Big\}.$$

0.11 Remark. If I is finite, then the condition "$A_\alpha \neq X_\alpha$ for at most finitely many α's" is always fulfilled.

0.2 Metric spaces

0.12 Definition. $d : X \times X \longrightarrow [0, \infty[$ is a *metric* $:\Longleftrightarrow$

- $d(x, y) \geqslant 0 \quad \forall x, y \in X$ with $d(x, y) = 0 \Longleftrightarrow x = y$.
- $d(x, y) = d(y, x) \quad \forall x, y \in X$.
- $d(x, y) \leqslant d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

(X, d) is called a *metric space* (often just X)

0.13 Definition. Let X be a metric space.

- *Induced metric* on $Y \subseteq X$: $d|_{Y \times Y}$ (metric on Y).
- *Open metric ball* of radius $\varepsilon > 0$ about $x \in X$:

$$B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}.$$

- $A \subseteq X$ *open* $:\Longleftrightarrow$ for every $x \in A$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$.
- *Cauchy sequence* $(x_n)_{n \in \mathbb{N}} \subset X$ $:\Longleftrightarrow$

  for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n, m \geqslant N : d(x_n, x_m) < \varepsilon$.

- $(x_n)_{n \in \mathbb{N}} \subset X$ converges to $x \in X$ $:\Longleftrightarrow \lim_{n \to \infty} d(x_n, x) = 0$
  $\Longleftrightarrow \forall \varepsilon > 0\, \exists n_0 \in \mathbb{N} \colon \forall n \geqslant n_0 \colon x_n \in B_\varepsilon(x)$.
- X is *complete* $:\Longleftrightarrow$ every Cauchy sequence in X converges.

0.14 Remark. Completeness is not a topological notion! See Exercise.

0.15 Definition. Let $A \subseteq X$, $x \in X$.

- $\operatorname{diam}(A) := \sup_{a, a' \in A} d(a, a')$ *diameter* of A.
- $\operatorname{dist}(x, A) := \inf_{a \in A} d(x, a)$ *distance* of x to A.

0.16 Lemma. *Let X be a complete metric space and $A \subseteq X$.*
Then: A closed $\Longleftrightarrow$ A complete.

Proof. See Analysis II. ■