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PDE II
SoSe 2017
03. 07. 2017

Excercise Sheet 9 for 10. 07. 2017

Consider the variational problem

$$E := \inf \left\{ \frac{\|\nabla u\|_2 \|u\|_2}{\|u\|_4^2} : \ u \in H^1(\mathbb{R}^2), u \neq 0 \right\}.$$

9.1. Recall the Sobolev's inequality

$$\|\nabla \varphi\|_1 \geq \|\varphi\|_2, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^2).$$

Use this to prove that $E > 0$.

9.2. Show that we can choose a minimizing sequence $\{u_n\}$ for E such that

$$\|\nabla u_n\|_2 = \|u_n\|_2 = 1.$$

Prove that the sequence $\{u_n\}$ is not vanishing, namely there exist a subsequence (still denoted by $\{u_n\}$ for simplicity) and a sequence $\{x_n\} \subset \mathbb{R}^2$ such that $u_n(\cdot + x_n)$ converges weakly in $H^1(\mathbb{R}^2)$ to a function $u_0 \not\equiv 0$.

9.3. Prove that

$$\liminf_{n\to\infty} \left(\|\nabla u_n\|_2^2 \|u_n\|_2^2 - \|\nabla u_0\|_2^2 \|u_0\|_2^2 - \|\nabla (u_n - u_0)\|_2^2 \|u_n - u_0\|_2^2 \right) \geq 0$$

and

$$\lim_{n\to\infty} \left(\|u_n\|_4^4 - \|u_0\|_4^4 - \|u_n - u_0\|_4^4 \right) = 0.$$

Deduce that u_0 is a minimizer for E.

9.4. Prove that there exists $Q \in H^1(\mathbb{R}^2)$, $Q \geq 0$ such that

$$-\Delta Q + Q - Q^3 = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$