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PDE II
SoSe 2017
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Excercise Sheet 3 for 22. 05. 2017

Let $d \in \mathbb{N}$.

3.1. (a) Compute the Fourier transform of $\chi_{[-1,1]} : \mathbb{R} \to \{0, 1\}$.

(b) Compute the Fourier transform of $f : \mathbb{R} \to \mathbb{R}$, $f(x) := (\sin x)^2/x^2$.

3.2. Prove that every $f \in C_c^\infty(\mathbb{R}^3)$ satisfies

$$\left(\iint_{\mathbb{R}^6} \frac{\overline{f(x)} f(y)}{|x - y|} \, \mathrm{d}x \, \mathrm{d}y \right) \|\nabla f\|_2^2 \geqslant 4\pi \|f\|_2^4.$$

3.3. Let $f \in L^2(\mathbb{R}^d)$. Prove that the PDE $u - \Delta u = f$ has a distributional solution $u \in H^2(\mathbb{R}^d)$.
Hint: Using Riesz's representation theorem conclude the existence of $u \in H^1(\mathbb{R}^d)$ satisfying $\langle u, \varphi \rangle_{H^1} = \langle f, \varphi \rangle_{L^2}$ for all $\varphi \in H^1(\mathbb{R}^d)$.

3.4. Let $f \in L^2(\mathbb{R}^d)$ satisfy

$$\sup_{\substack{\varphi \in C_c^\infty(\mathbb{R}^d) \\ \|\varphi\|_2 \leqslant 1}} \left| \int_{\mathbb{R}^d} f(x) (\nabla \varphi)(x) \, \mathrm{d}x \right| < \infty.$$

Prove that $f \in H^1(\mathbb{R}^d)$.