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PDE II
SoSe 2017
10. 07. 2017

Excercise Sheet 10 for 17. 07. 2017

10.1. Assume that $u_n \to u$ weakly in $H^1(\mathbb{R}^d)$. Prove that there exist a subsequence u_{n_k} and a sequence $R_k \to +\infty$ such that $u_{n_k}\mathbb{1}_{B(0,R_k)} \to u$ strongly in $L^2(\mathbb{R}^d)$. Here $\mathbb{1}_{B(0,R)}$ is the characteristic function of the ball $B(0,R)$.

10.2. Let $0 \le f_n \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3)$ such that $\|f_n\|_1 + \|f_n\|_{5/3} \le C$ and $f_n \to f_0$ weakly in $L^{5/3}(\mathbb{R}^3)$ as $n \to \infty$. Prove that

$$\liminf_{n\to\infty} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f_n(x)f_n(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \ge \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f_0(x)f_0(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

10.3. Let $0 \le \rho \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3)$ be a radial solution to the Thomas-Fermi equation

$$\frac{5}{3}\rho^{2/3} = \Big[\frac{Z}{|x|} - \rho * |x|^{-1} + \mu\Big]_+$$

for some constants $Z > 0 \ge \mu$. Prove that $\mu < 0$ if $\int \rho < Z$ and $\mu = 0$ if $\int \rho = Z$.

10.4. Let Ω be an open, bounded set in $\mathbb{R}^d$. Let $\{u_n\}$ be a bounded sequence in $L^p(\Omega)$ for some $p > 1$. Prove that if $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, then $u_n \to u$ strongly in $L^q(\Omega)$ for all $1 \le q < p$.