Mathematisches Institut
LMU München
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Mathematical Quantum Mechanics

Homework Sheet 5

Let $\mathfrak{G}$, $\mathfrak{H}$ be Hilbert spaces.

Exercise 1: In $\mathfrak{H}^N$ (i.e., the N-fold tensor product of $\mathfrak{H}$ with itself) we consider the natural action of the symmetric group σ_N. Namely, for every permutation $\pi \in \sigma_N$ we define a linear map U_π on pure tensor products:

$$U_\pi(u_1 \otimes \cdots \otimes u_N) := u_{\pi(1)} \otimes \cdots \otimes u_{\pi(N)}. \tag{1}$$

1. Prove that (1) uniquely defines a unitary operator $U_\pi : \mathfrak{H}^N \to \mathfrak{H}^N$.

2. Prove that the two operators

$$P_+ := \frac{1}{N!} \sum_{\pi \in \sigma_N} U_\pi, \qquad P_- := \frac{1}{N!} \sum_{\pi \in \sigma_N} (\operatorname{sgn} \pi) U_\pi$$

   are orthogonal projections ($P_\pm = P_\pm^*$, $P_\pm^2 = P_\pm$) satisfying $P_- P_+ = 0$ if $N \geqslant 2$.

Exercise 2 (Reduced density matrix): For a normalized pure state $\psi \in \mathfrak{H} \otimes \mathfrak{G}$ find a trace-class operator $\gamma_\psi : \mathfrak{H} \to \mathfrak{H}$ such that for every bounded operator A in $\mathfrak{H}$

$$\big(\psi, (A \otimes \mathbb{1}_{\mathfrak{G}})\psi\big) = \operatorname{tr}(\gamma_\psi A).$$

Prove that γ_ψ satisfies $0 \leqslant \gamma_\psi \leqslant \mathbb{1}_{\mathfrak{H}}$ and that $\operatorname{tr} \gamma_\psi = 1$.
Hint: A pure state is not necessarily a pure tensor product!

Exercise 3:

1. Let $f \in \mathfrak{H}$. Starting from their action on pure tensor products, define the operators $a_\pm(f)$ and $a_\pm^*(f)$ densely in the Fock spaces $\mathcal{F}_\pm(\mathfrak{H})$.

2. Show that on their domains these operators satisfy

$$\big(a_\pm(f)\Psi, \Phi\big)_{\mathcal{F}_\pm} = \big(\Psi, a_\pm^*(f)\Phi\big)_{\mathcal{F}_\pm}.$$

3. Prove that for $f \neq 0$ the operators $a_-(f)$, $a_-^*(f)$ are bounded, whereas $a_+(f)$, $a_+^*(f)$ are unbounded.

The solutions should be put to the box marked "Mathematical Quantum Mechanics" on the first floor **by 16:00 on Tuesday, November 19**.

Every solution must be an original work of its single author!
Violations of this rule will be penalized!