Mathematisches Institut
LMU München
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Prof. H. Siedentop
Dr. R. Helling
Dr. S. Morozov

Mathematical Quantum Mechanics

Homework Sheet 4

Exercise 1: For any observable A, we define its expected value and the variance on a pure state $\psi \in D(A)$ as

$$\langle A\rangle_\psi := (\psi, A\psi), \qquad V_\psi(A) := \left\langle \left(A - \langle A\rangle_\psi\right)^2 \right\rangle_\psi = \left(\psi, \left(A - \langle A\rangle_\psi\right)^2 \psi\right).$$

1. Show that for x being the independent variable on $\mathbb{R}$ and $p := -\mathrm{i}\partial_x$ the Heisenberg uncertainty principle
$$V_\psi(p)V_\psi(x) \geqslant \frac{1}{4}\|\psi\|^2 \tag{1}$$
holds for all $\psi \in \mathcal{S}(\mathbb{R})$.
Hint: Start from the case $\langle x\rangle_\psi = \langle p\rangle_\psi = 0$. See Part 2 for an idea of how to pass to the general case.
2. For $x_0, p_0 \in \mathbb{R}$ and $s > 0$ we define the coherent state K as
$$K(x) = K_{x_0,p_0,s}(x) := C_s \exp\left(\mathrm{i}p_0 x - \frac{(x-x_0)^2}{2s^2}\right), \qquad x \in \mathbb{R}.$$
   (a) Find C_s such that
   $$\int_{\mathbb{R}} \left|K(x)\right|^2 \mathrm{d}x = 1. \tag{2}$$
   (b) Calculate the Fourier transform $\hat{K} = \mathcal{F}K$.
   (c) Show that the coherent states minimize the uncertainty, i.e., for any (x_0, p_0, s) and $\psi := K$ there is equality in (1).

Exercise 2: In $\mathbb{C}^4$ consider the anti-linear operator

$$C := \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \Gamma,$$

where Γ is the complex conjugation. Show that $C^2 = -1$ and that C commutes with the Dirac operator $D + V$ for arbitrary electrostatic potential V. Conclude that every eigenvalue of $D + V$ must be twice degenerate, i.e., for every eigenvalue the corresponding eigensubspace has even dimension.

Continues on the next page!

Exercise 3: Determine the essential spectrum of

$$H = -\Delta - \frac{\exp(-|x|)}{|x|},$$

i.e., the Hamiltonian of a particle in a Yukawa potential, in dimension 3.

The solutions should be put to the box marked "Mathematical Quantum Mechanics" on the first floor **by 16:00 on Tuesday, November 12**.

Every solution must be an original work of its single author!
Violations of this rule will be penalized!