Mathematisches Institut
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Mathematical Quantum Mechanics

Homework Sheet 2

Exercise 1: Let $d \geqslant 3$.

1. Use ground state transform to show that

$$\int_{\mathbb{R}^d} \big|\nabla\phi(x)\big|^2 \mathrm{d}x - \Big(\frac{d-2}{2}\Big)^2 \int_{\mathbb{R}^d} \frac{\big|\phi(x)\big|^2}{|x|^2} \mathrm{d}x \geqslant 0 \qquad \text{for all } \phi \in C_0^1(\mathbb{R}^d).$$

   This result is widely known as Hardy inequality, or the uncertainty principle.

2. For which values of $C \in \mathbb{R}$ is the quadratic form of the operator $-\Delta - C/|\cdot|^2$ defined on $C_0^\infty(\mathbb{R}^d)$ bounded below?

3. For $d = 3$ use ground state transform to show that

$$\Big(\phi, \big(|x|p^2 + p^2|x|\big)\phi\Big) \geqslant 0 \qquad \text{for every } \phi \in C_0^\infty(\mathbb{R}^3).$$

   Here, as usual, $p = -\mathrm{i}\nabla$ is the momentum operator.

Exercise 2:

1. From the Sobolev inequality

$$\Big(\int_{\mathbb{R}^d} \big|\phi(x)\big|^{\frac{2d}{d-2}} \mathrm{d}x\Big)^{\frac{d-2}{d}} \leqslant C_d \int_{\mathbb{R}^d} \big|\nabla\phi(x)\big|^2 \mathrm{d}x \qquad \forall \phi \in C_0^1(\mathbb{R}^d)$$

   deduce a lower bound for Schrödinger operators, i.e., find a constant $B_d \in \mathbb{R}$ such that for all $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with $V_- := \max\{-V, 0\} \in L^{\frac{2+d}{2}}(\mathbb{R}^d)$ the inequality

$$\int_{\mathbb{R}^d} \big|\nabla\phi(x)\big|^2 \mathrm{d}x + \int_{\mathbb{R}^d} V(x)\big|\phi(x)\big|^2 \mathrm{d}x \geqslant B_d \int_{\mathbb{R}^d} V_-^{\frac{d+2}{2}}(x)\mathrm{d}x \int_{\mathbb{R}^d} \big|\phi(x)\big|^2 \mathrm{d}x \tag{1}$$

   holds for every $\phi \in C_0^1(\mathbb{R}^d)$.

2. For the case of a Hydrogen-like atom ($V(x) = -Z/|x|$) deduce a lower bound for the quantum energy.

Continues on the next page!

3. As another application of the Sobolev inequality, prove that for $d \geqslant 3$ inequality (1) holds with $B_d = 0$ if $\int_{\mathbb{R}^d} V_-^{d/2}(x)\mathrm{d}x$ is small enough.

The solutions should be put to the box marked "Mathematical Quantum Mechanics" on the first floor **by 16:00 on Tuesday, October 29**.

Every solution must be an original work of its single author! Violations of this rule will be penalized!