Mathematisches Institut
LMU München
WS 2013-14

Prof. H. Siedentop
Dr. R. Helling
Dr. S. Morozov

Mathematical Quantum Mechanics

Homework Sheet 13

Exercise 1:

1. Let V be a central potential given by a real–valued function $V(r)$. Prove that if $V(r)$ has an analytic continuation $V(z)$ to a sector $\{z : |\arg z| < \alpha\}$ with
$$\lim_{\substack{|z|\to\infty \\ |\arg z|<\beta}} V(z) = 0$$
and
$$\sup_{0<|\phi|<\beta} \iint_{|x|,|x'|\leqslant 1} \left|V\left(\mathrm{e}^{\mathrm{i}\varphi}|x|\right)\right|\left|V\left(\mathrm{e}^{\mathrm{i}\varphi}|x'|\right)\right||x-x'|^{-2}\,\mathrm{d}x\,\mathrm{d}x' < \infty$$
for each $\beta < \alpha$, then V is dilation analytic.

2. Prove that the Coulomb potential $V(r) = r^{-1}$ is in $\mathcal{F}_\infty$ and the Yukawa potential $V(r) = \mathrm{e}^{-\mu r}/r$, $\mu > 0$, is in $\mathcal{F}_{\pi/2}$.

Exercise 2: Let (r, θ, ϕ) be the spherical coordinates of $x \in \mathbb{R}^3$. Prove that the resonances $E \in \mathbb{C}$ of the Schrödinger operator with a delta–shell potential of radius $a > 0$
$$H = -\Delta - c\delta(r-a) \quad \text{in} \quad L^2(\mathbb{R}^3)$$
satisfy the equation
$$caI_{l+\frac{1}{2}}(\sqrt{-E}a)K_{l+\frac{1}{2}}(\sqrt{-E}a) = 1$$
for some $l \in \mathbb{N}_0$. Here $I_{l+\frac{1}{2}}$ and $K_{l+\frac{1}{2}}$ are the modified Bessel functions.
Hint: Use that the Green function of the Helmholtz equation can be represented as
$$\frac{\mathrm{e}^{-\sqrt{-E}|x-x'|}}{4\pi|x-x'|} = \frac{1}{\sqrt{rr'}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\overline{Y_{lm}(\theta',\phi')} \begin{cases} I_{l+\frac{1}{2}}(\sqrt{-E}r)K_{l+\frac{1}{2}}(\sqrt{-E}r'), & r < r', \\ I_{l+\frac{1}{2}}(\sqrt{-E}r')K_{l+\frac{1}{2}}(\sqrt{-E}r), & r > r'; \end{cases}$$
where Y_{lm} are the spherical harmonics.

The solutions should be put to the box marked “Mathematical Quantum Mechanics” on the first floor **by 16:00 on Tuesday, January 28**.