Homework 9

For Thursday, 30 June 2016

9.1. Prove Lemma 5.13: Let D be the closure of the symmetric operator $D_0 : C_c^\infty\big(\mathbb{R}^d \setminus \{0\}\big) \to L^2(\mathbb{R}^d)$ defined by

$$(D_0\psi)(x) := \frac{-\mathrm{i}}{2}\sum_{\nu=1}^{d}\Big(x_\nu \frac{\partial}{\partial x_\nu} + \frac{\partial}{\partial x_\nu} x_\nu\Big)\psi(x), \qquad \forall x \in \mathbb{R}^d.$$

Then

(a) $D = D^*$;

(b) $\mathcal{F} D \mathcal{F}^* = -D$;

(c) For $\tau \in \mathbb{R}$ let $\mathcal{U}(\tau) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be given by

$$\big(\mathcal{U}(\tau)\psi\big)(x) := \mathrm{e}^{d\tau/2}\psi(\mathrm{e}^\tau x), \qquad \forall x \in \mathbb{R}^d.$$

Then $\big\{\mathcal{U}(\tau)\big\}_{\tau\in\mathbb{R}}$ is a strongly continuous unitary group with generator D, i.e. $\mathcal{U}(\tau) = \mathrm{e}^{\mathrm{i}D\tau}$ for all $\tau \in \mathbb{R}$.

9.2. Prove Lemma 5.15: Define the Mellin transfrom

$$\mathcal{M} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R} \times \mathcal{S}^{d-1})$$

by

$$(\mathcal{M}\psi)(\lambda, \omega) := \frac{1}{\sqrt{2\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int_{1/N}^{N} r^{-\mathrm{i}\lambda-1+d/2}\psi(r\omega)\mathrm{d}r, \qquad \forall(\lambda, \omega) \in \mathbb{R} \times \mathcal{S}^{d-1}.$$

Then

(a) $\mathcal{M}$ is unitary with inverse

$$(\mathcal{M}^{-1}\varphi)(r\omega) := \frac{r^{-d/2}}{\sqrt{2\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int_{-N}^{N} r^{-\mathrm{i}\lambda}\varphi(\lambda, \omega)\mathrm{d}\lambda, \qquad \forall(r, \omega) \in \mathbb{R}_{>0} \times \mathcal{S}^{d-1}.$$

(b) For every $\tau \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R} \times \mathcal{S}^{d-1})$ we have

$$\big(\mathcal{M}\,\mathcal{U}(\tau)\mathcal{M}^{-1}\varphi\big)(\lambda, \omega) = \mathrm{e}^{\mathrm{i}\tau\lambda}\varphi(\lambda, \omega), \qquad \forall(\lambda, \omega) \in \mathbb{R} \times \mathcal{S}^{d-1}$$

and

$$\big(\mathcal{M} D \mathcal{M}^{-1}\varphi\big)(\lambda, \omega) = \lambda\varphi(\lambda, \omega), \qquad \forall(\lambda, \omega) \in \mathbb{R} \times \mathcal{S}^{d-1}.$$