Homework 7

For Thursday, 16 June 2016

7.1. Prove that for a self-adjoint operator H and $E \in \mathbb{R}$ the following statements are equivalent:

(a) $E \in \sigma_{\mathrm{ess}}(H)$.

(b) There exists an orthonormal system $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathfrak{D}(H)$ such that $\big\|(H-E)\varphi_n\big\| \underset{n\to\infty}{\longrightarrow} 0$.

(c) There exists a sequence of normalized vectors $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathfrak{D}(H)$ weakly converging to zero such that $\big\|(H-E)\varphi_n\big\| \underset{n\to\infty}{\longrightarrow} 0$.

7.2. Prove Remark 4.20:

(a) $l^q\big(L^p(\mathbb{R}^d)\big) \supset L^p(\mathbb{R}^d) \cup L^q(\mathbb{R}^d)$ for $q \geqslant p$;

(b) $l^q\big(L^p(\mathbb{R}^d)\big) \subset l^p\big(L^q(\mathbb{R}^d)\big)$ for $q \leqslant p$;

(c) $l^q\big(L^p(\mathbb{R}^d)\big) \subset L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $q \leqslant p$.

7.3. Let $d \in \mathbb{N}$, $\delta > d/2$ and $h := (2\pi)^{d/2}\mathcal{F}^{-1}\langle\cdot\rangle^{-\delta}$. Prove that $\mathrm{e}^{\eta|\cdot|}h \in L^2(\mathbb{R}^d)$ for every $\eta \in [0,1)$ (cf. the proof of Lemma 4.24).