Mathematisches Institut der LMU
Prof. Dr. P. Müller
Dr. S. Morozov

Mathematical Quantum Mechanics II
SoSe 2016
29. 04. 2016

Homework 3

For Thursday, 12 May 2016

3.1. The class $\mathcal{T}^1$ of trace class operators in the separable Hilbert space $\mathcal{H}$ consists of compact operators T for which

$$\|T\|_1 := \sum_k s_k(T) < \infty$$

holds, where $\{s_k\}$ are the eigenvalues of $|T|$. Prove that:

(a) If for $T \in \mathcal{L}(\mathcal{H})$ and some orthonormal basis (ONB) $\{g_l\}$ we have $\sum_l \|Tg_l\|^2 < \infty$, then $T \in \mathcal{T}^\infty$.

(b) If $T \in \mathcal{L}(\mathcal{H})$, $T > 0$, and for some ONB $\{g_l\}$ the series $\sum_l \langle Tg_l, g_l\rangle$ converges, then $T \in \mathcal{T}^1$ holds. Moreover, for any ONB $\{h_l\}$ we have $\sum_l \langle Th_l, h_l\rangle = \|T\|_1$.

(c) For $T \in \mathcal{T}^1$ and arbitrary orthonormal systems (ONS) $\{g_l\}$, $\{h_l\}$ we have $\sum_l \big|\langle Tg_l, h_l\rangle\big| \leqslant \|T\|_1$, with the equality being attined at $g_l := \varphi_l$, $h_l := \psi_l$ from Homework 2.3.

(d) If $T \in \mathcal{L}(\mathcal{H})$ and $\sum_l \langle Tg_l, h_l\rangle$ converges for any ONS $\{g_l\}$, $\{h_l\}$, then $T \in \mathcal{T}^1$.

(e) If $T \in \mathcal{L}(\mathcal{H})$ and for some ONB $\{g_l\}$ the series $\sum_l \|Tg_l\|$ converges, then $T \in \mathcal{T}^1$.

3.2. Prove Lemma 2.9: The following limits hold strongly as $t \to \infty$:

(a) $\mathrm{e}^{-\mathrm{i}tH} W_+ - \mathrm{e}^{-\mathrm{i}tH_0} P_{M_0} \to 0$;

(b) $\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH} W_+ \to P_{M_0}$;

(c) $\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH} P_M \to W_+^*$;

(d) $(W_+ - \mathbb{I})\mathrm{e}^{-\mathrm{i}tH_0} P_{M_0} \to 0$;

(e) $(W_+^* - \mathbb{I})\mathrm{e}^{-\mathrm{i}tH_0} P_{M_0} \to 0$;

(f) $\mathrm{e}^{\mathrm{i}tH_0} W_+ \mathrm{e}^{-\mathrm{i}tH_0} P_{M_0} \to P_{M_0}$;

(g) $\mathrm{e}^{\mathrm{i}tH_0} W_+^* \mathrm{e}^{-\mathrm{i}tH_0} P_{M_0} \to P_{M_0}$;

(h) $(\mathbb{I} - P_M)\mathrm{e}^{-\mathrm{i}tH} P_{M_0} \to 0$.

3.3. For $d \in \mathbb{N}$ and $t \in \mathbb{R} \setminus \{0\}$ determine the integral kernel of the operator $\mathrm{e}^{\mathrm{i}t\Delta}$ in $L^2(\mathbb{R}^d)$.