Retrospective and concluding remarks
on the $C^*$-algebraic formulation of Q.M.
and on the representation of $C^*$-algebras.

[1]. The Dirac-von Neumann axioms of Quantum Mechanics consist of cinematic rules

(1) The pure states of a quantum system are described by rays in a separable complex Hilbert space $H$. More generally, mixed states are density matrices on $H$.

(2) The observables are described by a subset of bounded, self-adjoint operators on $H$.

(3) If a state $\omega$ is described by the vector $\Psi_\omega \in H$, for any observable $A$ the measure $\omega(A) = \langle \Psi_\omega, A \Psi_\omega \rangle$ (or $\omega(A) = \text{Tr}(\rho_\omega A)$ if a state is described by a density matrix $\rho_\omega$).

one dynamical rule

(4) The dynamical evolution of the system is determined by the specification of a self-adjoint operator $\hat{H}$ through either of the algorithms $A \mapsto A_t = e^{it\hat{H}} A e^{-it\hat{H}}$ or $\psi \mapsto \psi_t = e^{-it\hat{H}} \psi$. (In fact there is full complementarity of the two algorithms: $\langle \psi, A_t \psi \rangle = \langle \psi_t, A \psi_t \rangle$.)

and one operational constraint

(5) Whereas for any given observable $A$ it is possible to prepare a state $\omega$ with no limitation on the smallness of the “dispersion” $\Delta_\omega(A) := \omega((A - \omega(A))^2) = \omega(A^2) - \omega(A)$, nevertheless there may be observables $A, B$ for which $\Delta_\omega(A) + \Delta_\omega(B)$ is not arbitrarily small. Equivalently, there may exist observables that do not commute.

The last point is in fact the axiomatization of the celebrated Heisenberg’s indeterminacy principle. It is therefore the translation of an experimental fact and not an axiom of mathematical nature.

[2]. In the $C^*$-algebraic formulation of Q.M. the axioms become:

(1*) The observables that define a quantum system are the self-adjoint elements of a non-commutative $C^*$-algebra $\mathcal{A}$.

(2*) The states of the given system are normalized positive linear functionals on $\mathcal{A}$.

(3*) The measure of the observable $A$ in the state $\omega$ is the number $\omega(A)$.

(4*) The dynamics of the system is described by a weakly continuous group $\{\alpha_t | t \in \mathbb{R}\}$ of *-automorphisms of $\mathcal{A}$ onto itself.

(5*) [incorporated in (1*)]

[3]. There is no Hilbert space in the algebraic axioms. There, the primal object is the $C^*$-algebra. The philosophical point is: I start from the physical properties I can observe in the system (i.e., the observables). The list of possible states in which I can prepare my system comes after.

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1 In the original Dirac-von Neumann formulation any bounded and self-adjoint operator on $H$ was an observable, so that in particular all orthogonal projections describe observables and therefore any ray in $H$ describes a physically realizable state. Later it was realised that $H$ may carry a reducible representation of the algebra of the observables, equivalently there exist bounded self-adjoint operators on $H$ known as superselection operators or “charges” (like the electric charge or the permutation operator of identical particles), which commute with all the observables. In the known cases the spectrum of the superselection operators is discrete so that $H$ decomposes as a direct sum of superselection sectors, each carrying an irreducible representation of the observable algebra. Since observables have zero matrix elements between vectors belonging to different superselection sectors, the relative phase in the superposition of such states are not observable. The important physical implication is that not all projections are observables: it is impossible to prepare physical states which are coherent superposition of states belonging to different superselection sectors.

2 defined to be the average of the outcomes of replicated measurements of $A$ performed on the system in the state $\omega$

3 The constraint itself is an axiom. The equivalence between its formulation for states and its formulation for observables is a theorem by von Neumann.

4 As emerged in Exercise 18(iii), the often quoted example of Heisenberg’s commutation relation $[Q, P] = i$ is not suited for a formulation within the algebra of bounded observables. But this is just technical: these relations can be re-formulated conveniently for bounded operators.
The algebraic axioms were inferred from the Dirac-von Neumann’s axioms by identifying in $\mathcal{B}(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$, those structural, algebraic, and analytical properties that constitute the definition of a $C^*$-algebra. This was quite a long process, in particular it was not clear a priori why the algebra containing observables should be closed (complete) in norm. Thus, when the Hilbert space operator theory began in the 1930s (Murray and von Neumann), the first physically meaningful structures to be identified were weakly-closed algebras of operators (nowadays referred to as von Neumann’s algebras, or $W^*$-algebras). $C^*$ came with Gelfand and Naimark in 1943, but its relevance to Q.M. was not fully appreciated for more than twenty years. Despite that, there has been a subsequent fruitful period of interplay between maths and physics which has instigated both interesting structural analysis of operator algebras and significant physical applications, notably to quantum statistical mechanics and relativistic quantum field theory.

Recall the definition from class. A $C^*$-algebra $\mathcal{A}$ is an algebra over $\mathbb{C}$ containing the identity (this is called $*$-algebra), with an involution $*$, with a norm that makes it complete (i.e., a Banach space) and satisfies the product inequality $\|AB\| \leq \|A\|\|B\|$ and the identity $\|A\| = \|A^*\|$ $\forall A, B \in \mathcal{A}$ (so far this is a Banach *-algebra), and such that the so-called $C^*$-condition holds: $\|A^*A\| = \|A\|^2$ $\forall A \in \mathcal{A}$. Note: our convention is that $\mathcal{A}$ is unital, i.e., it has an identity (which is necessarily unique). The self-adjoint elements of $\mathcal{A}$ are those $A$'s such that $A = A^*$.

A linear map $\omega : \mathcal{A} \rightarrow \mathbb{C}$ on the $C^*$-algebra $\mathcal{A}$ is said a functional. A functional is positive if $\omega(A^*A) \geq 0$ $\forall A \in \mathcal{A}$. A linear functional $\omega$ is positive iff: $\omega$ is bounded and $\|\omega\| := \sup_{\|A\|=1} |\omega(A)| = \omega(1)$. A state $\omega$ on $\mathcal{A}$ is a positive linear functional such that $\omega(1) = 1$ (that is, $\omega$ has norm equal to 1). A state $\omega$ is said pure if the only positive linear functionals $\rho$ majorized by $\omega$ (in the sense $0 \leq \rho(A) \leq \omega(A)$ $\forall A \in \mathcal{A}$) are of the form $\rho = \lambda \omega$ with $\lambda \in [0, 1]$. Equivalently, a state is pure if it cannot be written as a convex combination of other states. All this nomenclature is clearly borrowed from the usual Hilbert-space language where the action of a state $\Psi \in \mathcal{H}$ on $\mathcal{B}(\mathcal{H})$ is $A \mapsto \langle \Psi, A\Psi \rangle$. The meaning of axiom $(4^*)$ in [2] is: for every state $\omega$, $t \mapsto \omega(\alpha_t(A))$ is continuous.

As a consequence of the Hahn-Banach theorem (therefore: axiom of choice) there exist loads of states on a $C^*$-algebra in the following precise sense: if $\mathcal{A}$ is an arbitrary element of a $C^*$-algebra $\mathcal{A}$ then there exists a pure state $\omega$ on $\mathcal{A}$ such that $\omega(A^*A) = \|A\|^2$. Moreover (again from Hahn-Banach), the states separate the elements of $\mathcal{A}$, i.e., $\forall A, B \in \mathcal{A}$ with $A \neq B$ there exists a state $\omega$ such that $\omega(A) \neq \omega(B)$.

Having recognised that the observables of a quantum system generate a non-commutative $C^*$-algebra and that the states of the system are normalised positive linear functionals on it, there comes the question of how such an abstract structure can be used for concrete physical problems, for calculations, predictions, etc. We have therefore to find concrete realisations of the above structure. It is not a priori obvious what the concrete realisations of $C^*$-algebras are. The tool is representation theory.

To this aim, recall: a $*$-homomorphism $\pi$ between two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is a map $\pi : \mathcal{A} \rightarrow \mathcal{B}$ which preserves all the algebraic relations including the $*$, namely $\pi(\alpha A + b B) = a \pi(A) + b \pi(B)$, $\pi(A^*) = \pi(A)^*$, $\pi(AB) = \pi(A) \pi(B)$, $\pi(1_A) = \pi(1_B)$ $\forall A, B \in \mathcal{A}$, $\forall a, b \in \mathbb{C}$. If $\pi$ is bijective, it is called $*$-isomorphism. A $*$-isomorphism of $\mathcal{A}$ onto itself is called $*$-automorphism.

A representation $\pi$ of a $C^*$-algebra $\mathcal{A}$ on a ("target") Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$. In symbols: $(\pi, \mathcal{H})$.  

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3As mentioned in class, absence of 1 can complicate the structural analysis of $\mathcal{A}$ and one has to appeal to the existence of approximate identities. Nevertheless, complications can to a large extent be avoided by suitably embedding $\mathcal{A}$ in a larger algebra $A$ with identity.

4Since we consider only unital algebras and representations, necessarily $\pi$ is non-degenerate, which means that $\cap_{A \in \mathcal{A}} \ker \pi(A) = \{ \psi \in \mathcal{H} \mid \pi(A)\psi = 0 \ \forall A \in \mathcal{H} \} = \{0\}$. Note also that $\mathcal{H} = \cap_{A \in \mathcal{A}} \ker \pi(A) \oplus \text{Span}\{ \pi(A)\psi \mid A \in \mathcal{A}, \psi \in \mathcal{H} \}$.
[10]. Although the definition of a representation $\pi$ of a $C^*$-algebra $\mathcal{A}$ does not require continuity (i.e., boundedness), this is automatically true: $\|\pi(A)\| \leq \|A\| \forall A \in \mathcal{A}$. Moreover, $\pi$ is said faithful if it is injective ($\ker \pi = \{0\}$), equivalently if $\|\pi(A)\| = \|A\| \forall A \in \mathcal{A}$.

[11]. There are two most relevant types of representation of a $C^*$-algebra: cyclic and irreducible. Let us start with the first one. Let $\pi$ be a representation of a $C^*$-algebra $\mathcal{A}$ with target space $\mathcal{H}$. $\pi$ is called a cyclic representation if there exists a vector $\Omega \in \mathcal{H}$ which is cyclic for $\pi$ in $\mathcal{H}$, which means that $\{\pi(A)\Omega \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$. In symbols: $(\pi, \mathcal{H}, \Omega)$. Thus, the sole knowledge of $\Omega$ allows to reconstruct the whole $\mathcal{H}$ by application of representatives of $\mathcal{A}$.

[12]. Let $\pi$ be a representation of a $C^*$-algebra $\mathcal{A}$ with target space $\mathcal{H}$. $\pi$ is said to be an irreducible representation ("irrep") if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ invariant under $\pi(\mathcal{A})$. Equivalently, $\pi$ is irreducible if every non-zero vector $\psi \in \mathcal{H}$ is cyclic for $\pi$. Equivalently, $\pi$ is irreducible if the commutant $\pi(\mathcal{A})' := \{T \in \mathcal{B(\mathcal{H})} \mid [\pi(A), T] = 0 \forall A \in \mathcal{A}\}$ consists of multiples of the identity operator. (In Exercise 19(i) you proved some of the directions of the iff above.) Thus, an irrep is cyclic (not vice versa, in general).

[13]. Although cyclic representations are only special cases of representation of a $C^*$-algebra, they turn out to be a sort of "building blocks" of generic representations. In order to make this precise, let us first define the direct sum of representations. Let $\{(\pi_a, \mathcal{H}_a)\}_{a \in \mathcal{I}}$ be a collection of representations of a given $C^*$-algebra $\mathcal{A}$ (the index set $\mathcal{I}$ being countable or uncountable). The Hilbert direct sum of the target spaces is the space $\mathcal{H} := \oplus_{a \in \mathcal{I}} \mathcal{H}_a$ constructed as follows. The finite subsets $F$ of $\mathcal{I}$ form a directed set when ordered by inclusion. Consider the set of finite linear combinations of elements $\psi := \{\psi_a\}_{a \in F}$, where each $\psi_a$ belongs to $\mathcal{H}_a$, such that $\lim_F \sum_{a \in F} \|\psi_a\|_{\mathcal{H}_a}^2 < \infty$. Equip this set with the obvious vector space structure inherited component-wise. On such a vector space define $\langle \psi, \phi \rangle \equiv \sum_{a \in \mathcal{I}} \langle \psi_a, \phi_a \rangle_{\mathcal{H}_a} \equiv \lim_F \sum_{a \in F} \langle \psi_a, \phi_a \rangle_{\mathcal{H}_a}$. This turns out to be a scalar product, and take a completion with respect to it. This is $\oplus_{a \in \mathcal{I}} \mathcal{H}_a$.

The direct sum of representations $\pi_a$ is the map $\pi: \mathcal{A} \to \mathcal{B}(\oplus_{a \in \mathcal{I}} \mathcal{H}_a)$ defined by setting $\pi(A)$ equal to the operator $\pi_a(A)$ on the component subspace $\mathcal{H}_a$. In symbols: $\pi = \oplus_{a \in \mathcal{I}} \pi_a$. This definition yields bounded operators $\pi(A)$ on $\mathcal{H}$ because by [10] $\|\pi_a(A)\| \leq \|A\| \forall A \in \mathcal{A} \forall a \in \mathcal{I}$. One checks that $(\pi, \mathcal{H})$ is indeed a representation with $\|\pi(A)\| = \sup_{a \in \mathcal{I}} \|\pi_a(A)\|$.

[14]. Any representation $(\pi, \mathcal{H})$ of a $C^*$-algebra $\mathcal{A}$ is the direct sum of a family $\{(\pi_a, \mathcal{H}_a, \Omega_a)\}_{a \in \mathcal{I}}$ of cyclic representations. That is, there exists $\{\{\pi_a, \mathcal{H}_a, \Omega_a\}\}_{a \in \mathcal{I}}$ such that $\mathcal{H} = \oplus_{a \in \mathcal{I}} \mathcal{H}_a$, $\pi = \oplus_{a \in \mathcal{I}} \pi_a$, and each $\mathcal{H}_a$ is cyclic for the corresponding $\pi_a$. This is a fact that follows by standard transfinite induction (Zorn’s Lemma). Theoretically this result is of importance because it reduces the discussion of generic representations to that of cyclic representations and there is in fact a canonical manner for constructing representations (GNS).

[15]. In [9]–[14] the issue of the existence of representations was not touched. In fact, a $C^*$-algebra $\mathcal{A}$ has loads of representations. First of all, there is the following canonical one. Let $\mathcal{A}$ be a $C^*$-algebra and let $\omega$ be a state on it (as seen in [7], $\omega$ exists always). Then there exists a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ such that $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \forall A \in \mathcal{A}$, and consequently $\|\Omega_\omega\|^2 = \|\omega\| = 1$. Any other cyclic representation $(\pi, \mathcal{H}, \Omega)$ such that $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle \forall A \in \mathcal{A}$ is unitarily equivalent to $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$, that is, there exists a unitary operator $U: \mathcal{H} \to \mathcal{H}_\omega$ such that $U\Omega = \Omega_\omega$ and $U\pi(A)U^{-1} = \pi_\omega(A) \forall A \in \mathcal{A}$. Such a $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is called the GNS-representation of $\mathcal{A}$ associated with $\omega$.

\*\*A directed set is an index set $\mathcal{I}$ together with an ordering $\prec$ that satisfies: (a) If $\alpha, \beta \in \mathcal{I}$ then there exists $\gamma \in \mathcal{I}$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$. (b) $\prec$ is a partial ordering in $\mathcal{I}$, i.e., $\prec$ is reflexive, transitive, and anti-symmetric.
The GNS representation of a $C^*$-algebra $\mathcal{A}$ associated with a state $\omega$ on $\mathcal{A}$ is irreducible if and only if $\omega$ is pure. This result has two relevant consequences. (1) Since (as seen in [7]) for any element $A$ of a $C^*$-algebra $\mathcal{A}$ there exists a pure state $\omega$ on $\mathcal{A}$ such that $\omega(A^*A) = \|A\|^2$, then for any $A \in \mathcal{A}$ there exists an irreducible representation $(\pi, \mathcal{H})$ such that $\|\pi(A)\| = \|A\|$. (A $\rightarrow$ pure state $\omega$ $\rightarrow$ irreducible GNS representation $\pi_\omega$. The norm identity follows by $\|A\|^2 = \omega(A^*A) = \|\pi_\omega(A)\Omega\|^2 \leq \|\pi_\omega(A)\|^2 \leq \|A\|^2$.) (2) Further consequence: for any $A \neq \emptyset$ in $\mathcal{A}$ there exists a representation $\pi$ such that $\pi(A) \neq \emptyset$, i.e., the representations of a $C^*$-algebra separate points (this is usually referred to by saying that a $C^*$-algebra is REDUCED or SEMI-SIMPLE).

The GNS construction is mathematically important because it reduces the existence of Hilbert space representations of a $C^*$-algebra to the existence of states, which in turn is guaranteed by general (Hahn-Banach) existence theory. It is also physically relevant since it says that the (experimental) set of expectations of the observables given by a state have the customary Hilbert space interpretation. Thus, the basis of the mathematical description of Q.M. systems need not be postulated as in the Dirac-von Neumann axiomatic setting but it is merely a consequence of the $C^*$-algebra structure of the observables.

Although every state defines a concrete realisation of a $C^*$-algebras as operators on a Hilbert space, this realisation may not be faithful (in the sense of [10]). Nevertheless, the following holds. Given a $C^*$-algebra $\mathcal{A}$, for each state $\omega$ on $\mathcal{A}$ construct the associated GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ and form the direct sum representation $(\pi, \mathcal{H})$, $\mathcal{H} := \oplus_\omega \mathcal{H}_\omega$, $\pi := \oplus_\omega \pi_\omega$. Owing to the fact that the states separate (see [7]), the representation $\pi$ is faithful. Explicitly: for each $A \in \mathcal{A}$ there exists a (pure) state $\omega_A$ such that $\|\pi_\omega(A)\| = \|A\|$ (as see in [16]), and from $\|A\| \geq \|\pi(A)\| \geq \|\pi_\omega(A)\| \geq \|A\|$ one has $\|\pi(A)\| = \|A\|$ (\forall A \in \mathcal{A}). So $\pi$ is faithful. This result is also quoted as: a $C^*$-algebra $\mathcal{A}$ is $*$-isomorphic to a norm-closed self-adjoint sub-algebra of bounded linear operators on a Hilbert space. This is the celebrated GELFAND-NAIMARK THEOREM. Mathematically it provides the basic structure theorem for $C^*$-algebras. Physically it completes the path from the $C^*$-algebraic formulation of Q.M. back to the Hilbert space formulation.

For us the true criterion for a “good” representation is faithfulness, i.e., the case where each element of the represented $C^*$-algebra is identified with one and only one Hilbert space operator. Cyclic representations are in some sense the building blocks (as seen in [14]) and the knowledge of the cyclic vector is enough to reconstruct the whole space. Also, by [15], given a cyclic representation $(\pi, \mathcal{H}, \Omega)$ of $\mathcal{A}$, this is (unitarily equivalent to) the GNS representation associated with the state $(\Omega, (\cdot)\Omega)$ on $\mathcal{A}$. Thus, any direct sum of cyclic representations is in fact a direct sum of GNS representations. As for irreps, they have the virtue of identifying invariant target spaces, but one single irrep in general brings only a very partial information on $\mathcal{A}$, the full information being encoded in the knowledge of all invariant subspaces (think of the irreps of the angular momentum, for example). We did not go in this direction, our interest was rather in the path from the $C^*$-algebraic axioms to the Dirac-von Neumann ones.

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