Dirac Operators Coupled to an External Field: Essential Self-adjointness à la Chernoff

Christian Schmidbauer

Bachelor Thesis
Mathematics Department
LMU Munich

Advisor: Dr. habil. H. Zenk

Munich, September 26, 2011

Declaration of Authorship

I certify that the work presented here is, to the best of my knowledge and belief, the result of my own investigations, except as acknowledged, and has not been submitted, neither in part nor whole, for a degree at the LMU Munich or any other university.

Christian Schmidbauer
Munich, September 26, 2011

Acknowledgements

I want to thank my supervisor H. Zenk for guiding me through the topic of Dirac operators and for his patience during my thesis. It was very nice to get a better picture about Dirac operators. I also want to thank A. Michelangeli for his continous support through the topic of unbounded operators and for his unlimited patience. Furthermore I thank M. Bukov for his initial support and D. Pavlick for proof-reading the thesis several times.

Contents

1 Introduction **1**

1.1 The model 1

1.1.1 The free Dirac operator 1

1.1.2 The Dirac equation 2

1.1.3 Dirac equation coupled to an external field 2

1.2 The main result and the strategy of the proof 2

1.3 Structure of the thesis 4

2 Definitions and symbols **7**

3 Technical preliminaries **9**

3.1 Essential self-adjointness and initial value problems 9

3.2 Data localisation and finite speed of propagation 12

4 Essential self-adjointness à la Chernoff **17**

Chapter 1

Introduction

The object of this thesis is the three-dimensional Dirac operator D_0 in the presence of an external electro-magnetic field, which consists of an electric potential V and a magnetic vector potential A. Following the work of [5] we give the details of the proof of the essential self-adjointness of $D_0+\alpha\cdot A+V$. Such a result turns out to be a Chernoff-type theorem, in the sense that the self-adjointness is not affected by the behaviour of V at ∞.

1.1 The model

1.1.1 The free Dirac operator

The free Dirac operator D_0 is defined by

$$D_0 = c\alpha\cdot(-i\hbar\nabla_x) \, + \, mc^2\beta \tag{1.1}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and β are the Dirac matrices

$$\alpha_i = \begin{bmatrix} \mathbb{O} & \sigma_i \\ \sigma_i & \mathbb{O} \end{bmatrix}, \; \beta = \begin{bmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{bmatrix}, \tag{1.2}$$

where $\mathbb{O}\in\mathbb{C}^{2\times 2}$ is the zero matrix, $\mathbb{1}\in\mathbb{C}^{2\times 2}$ the identity matrix and

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the usual Pauli matrices.

It is well known that D_0 is essentially self-adjoint for instance on the domain $C_0^\infty(\mathbb{R}^3,\mathbb{C}^4)$ and therefore extends naturally to a self-adjoint operator, which we will denote D_0 for convenience, on (a dense subset of) $L^2(\mathbb{R}^3,\mathbb{C}^4)$ (see [6, Theorem 1.1]).

1.1.2 The Dirac equation

The time dependent (free) Dirac equation is the differential equation

$$i\hbar\partial_t\psi(t,x) \;=\; D_0\psi(t,x). \tag{1.3}$$

Here the four component wave-function $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the four component Dirac spinor. Two if its components are positive energy states and the other two components are negative energy states, which describe the particle and its anti-particle, respectively.

1.1.3 Dirac equation coupled to an external field

A spin-$\frac{1}{2}$ particle with charge $-q$, subject to an external electric field described by electric potentials $V_{\ell,m} : \mathbb{R}^3 \to \mathbb{R}$, $\ell, m = 1, 2$ and an external magnetic field described by a magnetic vector potential $A : \mathbb{R}^3 \to \mathbb{R}^3$, is governed by the Hamiltonian

$$H := D(A,V) \;:=\; c\alpha\cdot(-i\hbar\nabla_x) \;+\; mc^2\beta \;-\; q\alpha\cdot A + V \tag{1.4}$$

where

$$V := \begin{bmatrix} (V_{\ell,m})_{\ell,m=1,2} & \mathbb{O} \\ \mathbb{O} & -(V_{\ell,m})_{\ell,m=1,2} \end{bmatrix}.$$

This Hamiltonian is obtained by applying the minimal coupling principle to the free Dirac operator D_0.

Henceforth we assume A locally square integrable and $V_{\ell,m}$ locally square integrable and Hermitian for almost all $x \in \mathbb{R}^3$. This Hamiltonian is clearly symmetric on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ and the goal of this thesis will be to find the domain on which $D(A,V)$ is essential self-adjoint.

In analogy to (1.3), the Dirac equation associated with the Dirac operator (1.4) takes the form

$$i\hbar\partial_t\psi(t,x) \;=\; D(A,V)\psi(t,x). \tag{1.5}$$

1.2 The main result and the strategy of the proof

We are interested in the conditions under which the Dirac operator (1.4) is essentially self-adjoint, as it allows us to exponentiate it via the Spectral Theorem, as is well known. Therefore the dynamics associated with (1.4) is well-posed.

As we discuss in detail in Section 3.1, the essential self-adjointness for a symmetric H is equivalent to the existence of a unique global solution

$u \in C(\mathbb{R}, D(H)) \cap C^1(\mathbb{R}, D(H)^*)$ to the Schrödinger equation associated with H and the initial datum u_0, i.e.

$$\begin{cases} i\partial_t u \;=\; Hu \\ u(0,\cdot) = u_0 \in D(H) \end{cases} . \tag{1.6}$$

In our case, i.e. for external As and Vs, one can prove self-adjointness of $D(0,V)$ using the self-adjointness of the free Dirac operator D_0 (see [6, Section 4.3]). To be precise, if each $V_{\ell,m}$ has at most a local Coloumb singularity with a coupling constant smaller than $c/2$ and a bounded tail, we can use Hardy's inequality and the Kato-Rellich Theorem to prove that $D(0,V)$ is self-adjoint on the domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

Additionally, essential self-adjointness of $D(A,0)$, which can be seen as a super-symmetric operator (see [6, Section 5.5.2]), on the self-adjoint domain of $Q := \alpha \cdot (-i\nabla - A)$ can be proven using the Kato-Rellich Theorem and the boundedness of mc^2 (see [6, Theorem 5.12]).

Putting these results together, self-adjointness of $D(A,V)$ on the self-adjoint domain of Q can be shown if V is relatively Q-bounded such that the Q-bound of V is smaller than c, using again the Kato-Rellich Theorem (see [6, Section 6.1.1]).

In this thesis we follow instead the scheme of the work [5], which was originally designed to prove self-adjointness in the case of a quantised radiation field, which we shall present in its simplified version for non-quantised, external fields.

Therefore a finite-speed-of-propagation method is employed. To this aim, one considers the truncated Dirac operator

$$H_R \;:=\; D_0 - q\alpha \cdot A + V_R \tag{1.7}$$

where

$$V_R(x) \;:=\; V(x)\mathbb{1}_{B_R}(x) \tag{1.8}$$

for some ball $B_R \subseteq \mathbb{R}^3$ centred at the origin and with radius $R > 0$.

In revisiting Arai's analysis [1, Section D] one gets, due to the detour via the quantised model, to a result with very strong conditions on V:

Proposition 1.1. *Let $V(x)$ be Hermitian for almost all $x \in \mathbb{R}^3$ and $V \in L^\infty$ or V relatively $-\Delta$-bounded and $\frac{\partial V_{jl}}{\partial x_m} \in L^2_{loc}(\mathbb{R}^3)$ relatively $(-\Delta+1)^{\frac{1}{2}}$-bounded. Then $D(A,V)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$.*

Proposition 1.1 guarantees that the class of Vs such that for all $R > 0$, H_R is essentially self-adjoint, is not empty. Therefore the solution to the initial value problem

$$\begin{cases} i\partial_t u = \overline{H_R}u \\ u(0,\cdot) = u_0 \in D(\overline{H_R}) \end{cases} \tag{1.9}$$

exists and is unique globally in time.

Moreover it can be proved that the time evolution $e^{-it\overline{H_R}}$ evolves a localised initial f into a localised $e^{-it\overline{H_R}}f$, which has a possibly larger support. This is the finite speed of propagation (see details in Theorem 3.4).

Hence, one can check that for sufficiently small times t, $e^{-it\overline{H_R}}f$ solves the initial value problem (1.6) for every $R > 0$, too. An extension argument shows that the solution is global and therefore (1.6) has a unique solution globally in time.

With the strategy sketched above, we shall prove the main theorem:

Theorem 1.2. *Let $V \in L^2_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ be such that $V(x)$ is Hermitian for almost all $x \in \mathbb{R}^3$ and such that for any $R > 0$ the operator H_R is essentially self-adjoint. Then $H = D(A) + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$.*

Theorem 1.2 is in the spirit of Chernoff's analysis [2], [5] where self-adjointness holds irrespectively of the behaviour of V at infinity. The underlying picture is that the particle cannot escape to infinity in a finite amount of time and therefore boundary conditions at infinity are not needed to make H essentially self-adjoint.

1.3 Structure of the thesis

The material is organized as follows:

In Chapter 2 we state important notations and definitions needed throughout the thesis

Chapter 3 contains the two main technical ingredients for the proof of our main theorem 1.2.

First, in Section 3.1, we discuss in abstract terms the relation between essential self-adjointness of an operator A on a separable Hilbert space $\mathcal{H}$ and the existence of solutions to the corresponding initial value problem.

In Section 3.2 we perform the finite-speed-of-propagation analysis for $e^{-it\overline{H_R}}$.

In Chapter 4 we combine the previously proved ingredients in order to show that H is essentially self-adjoint.

Chapter 2

Definitions and symbols

List of symbols:

$\mathcal{H}$	a Hilbert space
$\langle\cdot,\cdot\rangle$	the scalar product of a Hilbert space, linear in the second argument
$L^2(\mathbb{R}^3,\mathbb{C}^4) \equiv L^2$	the space of functions $f:\mathbb{R}^3\to\mathbb{C}^4$ such that $\|f\|_{L^2}<\infty$, where $\|f\|_{L^2} := \big(\int_{\mathbb{R}^3}\|f(x)\|^2\,\mathrm{d}x\big)^{\frac{1}{2}}$
$H^1(\mathbb{R}^3,\mathbb{C}^4) \equiv H^1$	the Sobolev space of functions on $f:\mathbb{R}^3\to\mathbb{C}^4$ such that $\|f\|_{H^1}<\infty$, where $\|f\|_{H^1}^2 := \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$
$C_0^\infty(\mathbb{R}^d)$	the space of smooth functions with compact support on $\mathbb{R}^d$
$C(I,X)$	the space of continuous functions from I to X, $I\subseteq\mathbb{R}$ an interval
$C^1(I,X)$	the space of continuous differentiable functions from I to X, $I\subseteq\mathbb{R}$ an interval
A	an operator
$\overline{A}$	the closure of an operator A
A^*	the adjoint of an operator A
A'	the dual of an operator A
$D(A)$	the domain of an operator A
$D(A)^*$	the dual of the domain of an operator A
$\sigma(A)$	the spectrum of an operator A
B_r	$\equiv \{x\in\mathbb{R}^3 : \|x\| < r\}$ the open ball centred at the origin and with radius r in $\mathbb{R}^3$

Chapter 3

Technical preliminaries

3.1 Essential self-adjointness and initial value problems

Lemma 3.1. *Let A be symmetric on $\mathcal{H}$ and let $I \subset \mathbb{R}$ be an interval of the form $[-T, T]$.*

a) Then there exists at most one solution $u \in C(\mathbb{R}, D(\overline{A})) \cap C^1(\mathbb{R}, D(\overline{A})^*)$ to

$$\begin{cases} i\partial_t u = \overline{A}u \\ u(0) = u_0 \in D(\overline{A}) \; . \end{cases} \tag{3.1}$$

b) If A is also essentially self-adjoint there exists a solution of (3.1) defined $\forall t \in \mathbb{R}$.

Proof. a) As A is symmetric and therefore closable we assume it to be closed, without loss of generality.
Let $u_1, u_2 \in C(I, D(\overline{A}))$ be solutions to (3.1), $u := u_1 - u_2$ and $s \neq 0$. From

$$\frac{d}{dt}\|u(t)\|^2 = \lim_{s\to 0} \frac{1}{s}\left(\|u(t+s)\|^2 - \|u(t)\|^2\right)$$

and

$$\begin{aligned} &\frac{1}{s}\left(\|u(t+s)\|^2 - \|u(t)\|^2\right) = \\ &= \frac{1}{s}\Big(\langle u(t+s), u(t+s)\rangle - \langle u(t+s), u(t)\rangle \\ &\qquad + \langle u(t+s), u(t)\rangle - \langle u(t), u(t)\rangle\Big) \\ &= \left\langle u(t+s), \frac{u(t+s) - u(t)}{s}\right\rangle + \left\langle \frac{u(t+s) - u(t)}{s}, u(t)\right\rangle \; , \end{aligned}$$

we get, in the limit $s \to 0$:

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 &= \langle u(t), u'(t)\rangle + \langle u'(t), u(t)\rangle \\ &\overset{(3.1)}{=} \langle u(t), -i\overline{A}u(t)\rangle + \langle -i\overline{A}u(t), u(t)\rangle \\ &= \langle i\overline{A}u(t), u(t)\rangle - \langle i\overline{A}u(t), u(t)\rangle \\ &= 0\,.\end{aligned} \tag{3.2}$$

So $\|u_1(t) - u_2(t)\|$ is constant and from $u_1(0) = u_2(0) = u_0$ we can conclude that $u_1 \equiv u_2$.

It remains to prove that $u(t) \in C^1(I, D(\overline{A})^*)$, i.e.

$$\langle u'(t) - u'(s), g\rangle_{\mathcal{D}^*,\mathcal{D}} \overset{t\to s}{\longrightarrow} 0 \qquad \forall g \in \mathcal{D} \tag{3.3}$$

with $\mathcal{D} := D(\overline{A})$ and its dual space $\mathcal{D}^* := D(\overline{A})^*$, equipped with the weak-* topology induced by $\mathcal{D}$. Notice, in particular, that

$$\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{D}^*.$$

Denote the dual of $\overline{A}$ by

$$\overline{A}' : \mathcal{H} \to \mathcal{D}^*.$$

Let $g \in D(\overline{A})$. Then

$$\begin{aligned}\langle u'(t) - u'(s), g\rangle_{\mathcal{D}^*,\mathcal{D}} &= \langle -i\overline{A}'\, u(t) + i\overline{A}'\, u(s),\ g\rangle_{\mathcal{D}^*,\mathcal{D}} \\ &= \langle i\overline{A}'(-u(t) + u(s)),\ g\rangle_{\mathcal{D}^*,\mathcal{D}} \\ &= \langle i(-u(t) + u(s)),\ \overline{A}g\rangle_{\mathcal{D}^*,\mathcal{D}} \\ &\underset{t\to s}{\longrightarrow} 0\,.\end{aligned} \tag{3.4}$$

So $u \in C^1(I, D(\overline{A})^*)$.

b) Assume $\overline{A}$ is self-adjoint. Let $\{T(t)\}_t$ be the group of isometries on $\mathcal{H}$ generated by $\overline{A}$

$$T(t) := e^{-it\overline{A}}, \quad \|T(t)\| = 1, \quad T(t)^* = T(-t). \tag{3.5}$$

From [4, Theorem VIII.7] we know that

$$\forall \psi \in D(\overline{A}) \qquad \frac{T(s)\psi - \psi}{s} \underset{s\to 0}{\overset{\|\cdot\|}{\longrightarrow}} -i\overline{A}\psi\,.$$

Let now $u_0 \in D(\overline{A})$ be the initial datum of the initial value problem

(3.1). Define $u(t) := T(t)u_0$, then

$$\begin{aligned}\frac{u(t+s)-u(t)}{s} &= \frac{T(t+s)u_0 - T(t)u_0}{s} \\ &= T(t)\,\frac{T(s)u_0 - u_0}{s} \\ &\xrightarrow[s\to 0]{\|\cdot\|} -T(t)\,i\overline{A}u_0 \\ &= -i\overline{A}\,T(t)u_0 \\ &= -i\overline{A}u(t)\;.\end{aligned}$$

This can be rewritten as

$$i\partial_t u(t) = \overline{A}u(t) \tag{3.6}$$

So $u \in C(\mathbb{R}, D(\overline{A}))$ and

$$\begin{cases} i\partial_t u \;=\; \overline{A}u \\ u(0) = u_0 \in D(\overline{A})\;. \end{cases} \tag{3.7}$$

By repeating the calculation (3.4), we show that $u \in C^1(\mathbb{R}, D(\overline{A})^*)$.

□

Lemma 3.2. *Let A be a symmetric operator on $\mathcal{H}$. If for any $u_0 \in D(A)$ the initial value problem (3.1) has a solution $u : \mathbb{R} \to D(\overline{A})$, then A is essentially self-adjoint.*

Proof. Assume $\overline{A}$ is not self-adjoint, i.e. $\exists w_\pm \in D(\overline{A}^*)$, $w_\pm \neq 0$ such that

$$(\overline{A}^* \pm i)w_\pm = 0. \tag{3.8}$$

Let u be a solution to the initial value problem (3.1). In particular for any $t \in \mathbb{R}$ $u(t) \in D(A)$ and $u(0) = u_0$. Hence

$$\begin{aligned}\frac{d}{dt}\langle u(t), w_\pm\rangle \;&=\; \langle \partial_t u(t), w_\pm\rangle \;=\; i\langle \overline{A}u(t), w_\pm\rangle \\ &=\; i\langle u(t), \overline{A}^* w_\pm\rangle \;=\; i\langle u(t), \mp i w_\pm\rangle \\ &=\; \pm\langle u(t), w_\pm\rangle\end{aligned} \tag{3.9}$$

First we applied (3.1) to $\overline{A}u$ and then we used (3.8).

Therefore $\alpha(t) := \langle u(t), w_\pm\rangle$ satisfies $\frac{d}{dt}\alpha(t) = \pm\alpha(t)$, hence

$$\alpha(t) = \alpha(0)\cdot e^{\pm t}\;. \tag{3.10}$$

Owing to (3.2) and the fact that $u_1 - u_2$ is again a solution if u_1 and u_2 are solutions, we know that $\frac{d}{dt}\|u(t)\| = 0$. So $\|u(t)\|$ is constant and therefore for any $t \in \mathbb{R}$

$$|\alpha(t)| \;=\; |\langle u(t), w_\pm\rangle| \;\leq\; \|u(t)\|\,\|w_\pm\| \;=\; \|u_0\|\,\|w_\pm\| \;<\; \infty\,. \tag{3.11}$$

The uniform bound (3.11) is consistent with (3.10) only if $\alpha(0) = 0$, that is $\langle u(0), w_\pm\rangle = 0$. Since $u(0) \in D(A)$ is arbitrary and $D(A)$ is dense in $\mathcal{H}$, $w_\pm = 0$ and we conclude that

$$\operatorname{Ker}(\overline{A} \mp i) = 0, \tag{3.12}$$

which is a contradiction to the assumption. Therefore $\overline{A}$ is self-adjoint. □

3.2 Data localisation and finite speed of propagation

Lemma 3.3. *Let* $0 \leq R$, $f \in D(\overline{H_R})$ *such that* $\operatorname{supp} f \subseteq B_R$. *Then*

$$\begin{aligned} \forall \rho \geq R \qquad & f \in D(\overline{H_\rho}) \cap D(\overline{H}) \text{ and} \\ & \overline{H_R} f = \overline{H_\rho} f = \overline{H} f\,. \end{aligned}$$

Proof. Assume $f \in D(\overline{H_R})$. By the definition of the closure we can find a $(f_j)_{j\in\mathbb{N}} \subseteq D(H_R)$ such that

$$f_j \xrightarrow{\|\cdot\|} f, \quad H_R f_j \xrightarrow{\|\cdot\|} \overline{H_R} f\,. \tag{3.13}$$

Since $\operatorname{supp} f$ is a closed and bounded subset of B_R, we can find an open neighbourhood Ω with $\operatorname{supp} f \subseteq \Omega \subsetneq \overline{\Omega} \subseteq B_R$. By Urysohn's lemma [4, Theorem IV.7] there exists a $\chi \in C_0^\infty(\mathbb{R}^3)$ with

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^3 \backslash B_R \end{cases}.$$

We now evaluate the action of H_R on the approximated χf_js:

$$\begin{aligned} H_R(\chi f_j) &= (-i\hbar c\alpha \cdot \nabla_x)(\chi f_j) + \chi(-q\alpha A + mc^2\beta + V_R) f_j \\ &= \chi H_R f_j - i\hbar c\alpha \cdot (\nabla_x \chi) f_j\,. \end{aligned}$$

Hence,

$$H_R(\chi f_j) \xrightarrow{\|\cdot\|} \chi \overline{H_R} f - i\hbar c\alpha (\nabla_x \chi) f = \chi \overline{H_R} f \tag{3.14}$$

as $(\nabla_x \chi)(x) = 0 \;\; \forall x \in \operatorname{supp} f \subseteq \Omega$. We also see that

$$V_\rho \chi f_j \;=\; V_R \chi f_j \;=\; V \chi f_j$$

hence we get together with (3.14)

$$H_R\chi f_j \;=\; H_\rho \chi f_j \;=\; H\chi f_j \xrightarrow{\|\cdot\|} \chi \overline{H_R} f \;. \tag{3.15}$$

We also know that $\chi f = f$ and

$$\|\chi f - \chi f_j\| = \|\chi(f - f_j)\| \leq \|f - f_j\| \;\to 0 \;,$$

therefore $\chi f_j \xrightarrow{\|\cdot\|} \chi f = f$.

Let $H_\# \in \{H, H_\rho, H_R\}$. We can easily see that $\chi f_j \in D(H_\#)$. Thus, we can replace H_R with $H_\#$ in (3.14) and again due to the definition of the closure similiar to (3.13) we get

$$\chi f_j \xrightarrow{\|\cdot\|} f, \quad H_\#(\chi f_j) \xrightarrow{\|\cdot\|} \overline{H_\#} f \;.$$

Together with (3.15) we see that $\overline{H_\#} f = \chi \overline{H_R} = \overline{H_R} f$, so

$$\overline{H_R} f = \overline{H_\#} f, \quad f \in D(\overline{H_\#}) \cap D(\overline{H_R}) \;.$$

□

Theorem 3.4. *For $R > 0$, assume that H_R is essentially self-adjoint. For a fixed $r > 0$ choose $f \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, such that* $\operatorname{supp} f \subseteq B_r = \{x \in \mathbb{R}^3 : \|x\| < r\}$. *Then*

$$\forall t \in \mathbb{R}^3 \qquad \operatorname{supp}\left(e^{-it\overline{H_R}} f\right) \subseteq B_{r+c|t|} \;.$$

Proof. Let $B \subseteq \mathbb{R}^3$ be measurable and define

$$\mathbb{P}_B : L^2 \to L^2, \quad (\mathbb{P}_B f)(x) := \mathbb{1}_B(x) f(x) \tag{3.16}$$

where $\mathbb{1}_B$ denotes the indicator function on B.

Note that both V_R and $\alpha \cdot A$ are multiplication operators. Let $W \in \{V_R,\, \alpha \cdot A\}$. Its operator domain is

$$D(W) \;=\; \left\{ f \in L^2(\mathbb{R}^3, \mathbb{C}^4) : \int_{\mathbb{R}^3} |W(x)\; f(x)|^2 dx \;< \infty \right\}.$$

We also know that W is essentially self-adjoint with $\sigma(W) = \operatorname{ess\,ran} W(x)$ [4, Chapter VIII.3, Prop 1]. Therefore we can define e^{-itW} via the Spectral Theorem [4, Theorem VIII.5] and

$$(e^{-itW} f)(x) = e^{-itW(x)} f(x) \;.$$

Hence e^{-itW} is a multiplication operator and it commutes with $\mathbb{P}_B$.

$$\begin{aligned} \mathbb{P}_B e^{-itV_R} \;&=\; e^{-itV_R} \mathbb{P}_B \;, \\ \mathbb{P}_B e^{-it(\alpha\cdot A)} \;&=\; e^{-it(\alpha\cdot A)} \mathbb{P}_B \;. \end{aligned} \tag{3.17}$$

Considering e^{-iD_0}, we know from [6, Section 1.5] that

$$\operatorname{supp}(e^{-itD_0}\psi) \subseteq B_{r'+c|t|} \quad \text{for} \quad \operatorname{supp}\psi \subseteq B_{r'}. \tag{3.18}$$

We want to prove that the same holds for $D(A) := D(A,0)$ and for $D(A,V)$.

First we are going to apply the Trotter product formula for $\overline{D(A)}$ which is self-adjoint due to [3, Appendix B, C]:

$$e^{-it\overline{D(A)}}f = \lim_{n\to\infty}\left(e^{-i\frac{t}{n}D_0}\,e^{-i\frac{t}{n}(\alpha\cdot A)}\right)^n f. \tag{3.19}$$

We then have

$$\begin{aligned}\operatorname{supp} e^{-i\frac{t}{n}(\alpha\cdot A)}f &= \operatorname{supp} e^{-i\frac{t}{n}(\alpha\cdot A)}\,(\mathbb{P}_{B_r}f)\\ &= \operatorname{supp} \mathbb{P}_{B_r}\left(e^{-i\frac{t}{n}(\alpha\cdot A)}f\right)\\ &\subseteq B_r\end{aligned} \tag{3.20}$$

The last inclusion follows from the definition (3.16) of $\mathbb{P}_{B_r}$, whereas the equalities follow from (3.17). Due to (3.18),

$$\begin{aligned}\operatorname{supp}\left(e^{-i\frac{t}{n}D_0}\,e^{-i\frac{t}{n}(\alpha\cdot A)}f\right) &\subseteq B_{r+c\frac{|t|}{n}}\\ \operatorname{supp}\left(e^{-i\frac{t}{n}D_0}\,e^{-i\frac{t}{n}(\alpha\cdot A)}\,e^{-i\frac{t}{n}(\alpha\cdot A)}f\right) &\subseteq B_{r+c\frac{|t|}{n}}\\ \operatorname{supp}\left(e^{-i\frac{t}{n}D_0}\,e^{-i\frac{t}{n}D_0}\,e^{-i\frac{t}{n}2(\alpha\cdot A)}f\right) &\subseteq B_{r+c\frac{2|t|}{n}}\,.\end{aligned} \tag{3.21}$$

First we replaced f with $e^{-i\frac{t}{n}(\alpha\cdot A)}f$, as the support is not changed (by (3.20)) and then we applied to it $e^{-i\frac{t}{n}D_0}$ which enlarged our support to $B_{r+c\frac{2|t|}{n}}$ (by (3.18)).

These steps can be repeated until we end up with

$$\operatorname{supp}\left(e^{-i\frac{t}{n}D_0}\,e^{-i\frac{t}{n}(\alpha\cdot A)}f\right)^n \subseteq B_{r+c|t|}. \tag{3.22}$$

The ball $B_{r+c|t|}$ is independent of n, so we can apply (3.19) and get

$$\operatorname{supp}\left(e^{-it\overline{D(A)}}f\right) \subseteq B_{r+c|t|}. \tag{3.23}$$

By assumption, $H_R = D(A) + V_R$ defines an essentially self-adjoint operator, whence we can apply the Trotter product formula of (3.19) again for H_R

$$e^{-it\overline{H_R}}f = \lim_{n\to\infty}\left(e^{-i\frac{t}{n}\overline{D(A)}}\,e^{-i\frac{t}{n}V_R}\right)^n f. \tag{3.24}$$

Using (3.17), we can replace $e^{-i\frac{t}{n}(\alpha\cdot A)}$ with $e^{-i\frac{t}{n}V_R}$ in (3.20) and get

$$\operatorname{supp} e^{-i\frac{t}{n}V_R}f \subseteq B_r \tag{3.25}$$

We repeat the iteration (3.21), (3.22) with $\overline{D(A)}$ instead of D_0 using (3.23) and obtain together with (3.24)

$$\operatorname{supp}\left(e^{-it\overline{H_R}}f\right) \subseteq B_{r+c|t|}\,. \tag{3.26}$$

□

Chapter 4

Essential self-adjointness à la Chernoff

We are now going to prove Theorem 1.2.

Proof. Let $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ and $r > 0$, such that $\operatorname{supp} f \subseteq B_r$.
Let also $R, T_1 > 0$ be such that

$$R > r + cT_1 \ . \tag{4.1}$$

Define

$$v_1(t) := e^{-it\overline{H_R}} f \ . \tag{4.2}$$

As $f \in C_0^\infty \subseteq D(\overline{H_R})$, we can apply Lemma 3.1 and conclude that $v_1(t) \in D(\overline{H_R})$ and

$$\partial_t v_1(t) \ = \ -i\overline{H_R} v_1(t) \quad \forall t \in \mathbb{R} \ . \tag{4.3}$$

We also know by Theorem 3.4 and (4.1) that

$$\operatorname{supp} v_1(t) \ \subseteq \ B_{r+c|t|} \ \subseteq \ B_R$$

for all $t \in [-T_1, T_1]$. Therefore, by Lemma 3.3,

$$\partial_t v_1(t) \ = \ -i\overline{H_R} v_1(t) \ = \ -i\overline{H} v_1(t)$$

and $v_1(t) \in D(\overline{H})$ for all $t \in [-T_1, T_1]$. Thus, we found a local solution to (4.3), which is unique by Lemma 3.1.

We shall now extend this solution to the whole $\mathbb{R}$.
Assume by contradiction that there exist $T_{max}, r > 0$, and $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ with $\operatorname{supp} f \subseteq B_r$, such that the corresponding solution $u : [0, T_{max}) \to \mathcal{H}$ to the initial value problem (1.9) cannot be extended to $t > T_{max}$. The case $T_{max} < 0$ can be treated analogously.

We choose $\tau > T_{max}$ and $\rho > r + c\tau > r + cT_{max}$ and define

$$v_2(t) := e^{-it\overline{H_\rho}} f \quad \forall t \in [0, \tau) \ . \tag{4.4}$$

According to Theorem 3.4

$$\operatorname{supp} v_2(t) \subseteq B_{r+c|t|} \subseteq B_\rho \quad \forall t \in [0,\tau) \ .$$

Hence, by Lemma 3.3,

$$\partial_t v_2(t) = -i\overline{H_\rho} v_2(t) = -i\overline{H} v_2(t) \ .$$

Therefore $v_2 : [0,\tau) \to \mathcal{H}$ is a solution to (4.3). However, by Lemma 3.1 the solution u is unique, so $u = v_2|_{[0,T_{max})}$, which contradicts our assumption.

This leads to the conclusion that $\forall f \in D(H)$ there exists a global solution $u : \mathbb{R} \to \mathcal{H}$ to (1.6). Hence, by Lemma 3.2, H is essentially self-adjoint. □

Bibliography

[1] Asao Arai, *A particle-field hamiltonian in relativistic quantum electrodynamics*, Journal of Mathematical Physics **41** (2000), 4271–4283.

[2] Paul R. Chernoff, *Schrödinger and Dirac operators with singular potentials and hyperbolic equations*, Pac. J. Math. **72** (1977), 361–382.

[3] Elliott H. Lieb and Michael Loss, *Stability of a model of relativistic quantum electrodynamics*, Commun. Math. Phys. **228** (2002), 561–588.

[4] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press Inc., New York, 1980, Functional analysis.

[5] Edgardo Stockmeyer and Heribert Zenk, *Dirac operators coupled to the quantized radiation field: Essential self-adjointness à la Chernoff*, Letters in Mathematical Physics **83** (2008), 59–68.

[6] Bernd Thaller, *The Dirac equation*, Texts and monographs in physics, Springer-Verlag, 1992.