Mathematical Quantum Mechanics – Final Test, 12.02.2011
Mathematische Quantenmechanik – Endklausur, 12.02.2011

Name:/*Name*: ____________________

Matriculation number:/*Matrikelnr.:* ____________ **Semester:**/*Fachsemester:* ____________

Degree course:/*Studiengang:* ❑ Bachelor PO 2007 ❑ Lehramt Gymnasium (modularisiert)
❑ Bachelor PO 2010 ❑ Lehramt Gymnasium (nicht modularisiert)
❑ Diplom ❑ Master ❑ TMP ❑ ____________

Major:/*Hauptfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ________

Minor:/*Nebenfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ________

Credits needed for:/*Anrechnung der Credit Points für das:* ❑ Hauptfach ❑ Nebenfach (Bachelor/Master)

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter:* ❑ Yes ❑ No

problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	$\sum$
total points	**10**	**10**	**15**	**15**	**15**	**15**	**15**	**20**	**25**	140
scored points										

homework performance		**final test performance**		**total performance**		**FINAL MARK**	

INSTRUCTIONS:

- This booklet is made of twenty-two pages, including the cover, numbered from 1 to 22. The test consists of nine problems. Each problem is worth the number of points specified in the table above. 100 points are counted as 100% performance in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one two-sided A4-paper "cheat sheet" (Spickzettel). You cannot use your own paper: should you need more paper, raise your hand and you will be given extra sheets.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 150 minutes.

GOOD LUCK!

Fill in the form here below only if you need the certificate (Schein).

Dieser Leistungsnachweis entspricht auch den Anforderungen
nach § Abs. Nr. Buchstabe LPO I
nach § Abs. Nr. Buchstabe LPO I

UNIVERSITÄT MÜNCHEN

ZEUGNIS

Der / Die Studierende der ________________

Herr / Frau ________________ *aus* ________________

geboren am ________ *in* ________________ *hat im* **WiSe** ________ *–Halbjahr* **2010/2011** ________

meine Übungen **zur Mathematischen Quantenmechanik** ________________

____

mit ________________ *besucht.*

Er / Sie hat ________________

schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden. ________________

____

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MÜNCHEN, *den* **12 Februar 2011**

Name

PROBLEM 1. (10 points) Consider the sequence $\{f_n\}_{n=1}^{\infty}$ in $L^1_{\text{loc}}(\mathbb{R})$ defined by

$$f_n(x) \;:=\; x^{2011} \sin nx\,, \qquad x \in \mathbb{R}\,.$$

Prove that f_n converges in $\mathcal{D}'$ (i.e., in the sense of distributions) as $n \to \infty$ and compute its limit.

SOLUTION:

Name

PROBLEM 2. (10 points) Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. Let $f, g, h \in \mathcal{H}$ be three orthonormal elements in $\mathcal{H}$, that is, $\|f\| = \|g\| = \|h\| = 1$ and $\langle f, g\rangle = \langle f, h\rangle = \langle g, h\rangle = 0$. Show that $f \wedge g$, $f \wedge h$, $g \wedge h$ are three orthonormal elements in the tensor product space $\mathcal{H} \otimes \mathcal{H}$.

SOLUTION:

Name

PROBLEM 3. (15 points) The purpose of this problem is to show that the ground state of a single-well potential has only a single peak. Consider the Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - V(x)$ acting on $L^2(\mathbb{R})$, where $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$, $V \geqslant 0$, $V'(x) > 0\ \forall x \in (-\infty, 0)$, $V'(x) < 0\ \forall x \in (0, +\infty)$.

(i) Show that H admits a ground state ψ_0 with ground state energy $E_0 < 0$. (Recall that in this case ψ_0 can be assumed to be strictly positive.)

(ii) Show that ψ_0 has only one local maximum (which then, of course, is global). I.e., show that ψ_0 cannot have a shape of, e.g., two peaks as in Fig (a), the correct behaviour is depicted in Fig (b).

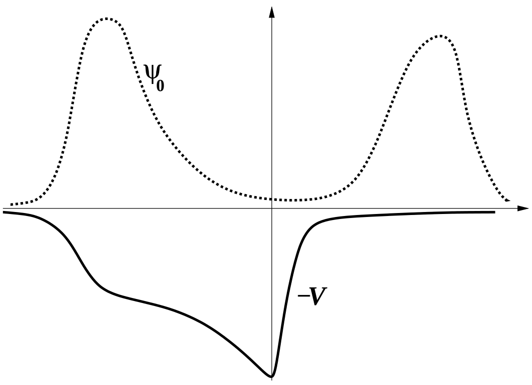

Figure (a)

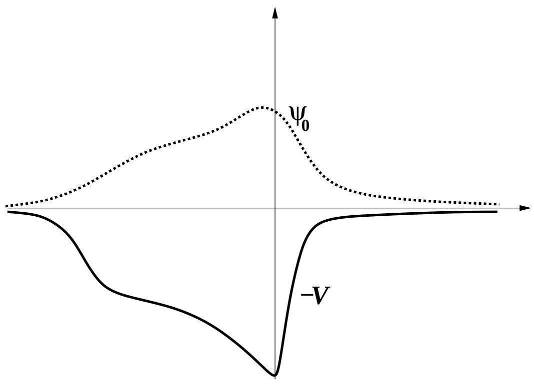

Figure (b)

SOLUTION:

Name

PROBLEM 4 (15 points). Consider the one-body wave-functions $\varphi, \psi \in L^2(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2(\mathbb{R}^d)} = \|\psi\|_{L^2(\mathbb{R}^d)} = 1$ and $\langle \varphi, \psi \rangle_{L^2(\mathbb{R}^d)} = 0$ (d is a given positive integer). For a given integer $N \geqslant 2$ consider the bosonic N-body wave functions $\Phi_N, \Psi_N \in L^2(\mathbb{R}^{Nd})$ defined by

$$\Phi_N := \varphi^{\otimes N}, \qquad \Psi_N := \big(\psi \otimes \varphi^{\otimes (N-1)}\big)_{\mathrm{sym}}.$$

Recall the notation: $\big(\psi \otimes \varphi^{\otimes (N-1)}\big)_{\mathrm{sym}} = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{j=1}^{N} \big(\varphi \otimes \cdots \otimes \varphi \otimes \psi \otimes \varphi \otimes \cdots \otimes \varphi\big)$, where in the j-th summand the function ψ occupies the j-th entry of the tensor product.

(i) Show that $\|\Phi_N\|_{L^2(\mathbb{R}^{Nd})} = \|\Psi_N\|_{L^2(\mathbb{R}^{Nd})} = 1$.

(ii) Show that $\langle \Phi_N, \Psi_N \rangle_{L^2(\mathbb{R}^{Nd})} = 0$ and compute $\|\Phi_N - \Psi_N\|_{L^2(\mathbb{R}^{Nd})}$.

(iii) Compute $\mathrm{Tr}\,\big|\gamma^{(1)}_{\Phi_N} - \gamma^{(1)}_{\Psi_N}\big|$, where $\gamma^{(1)}_{\Theta}$ denotes the one-body reduced density matrix associated with an N-body state $\Theta \in L^2(\mathbb{R}^{Nd})$, $|A|$ denotes the absolute value of an operator A, and Tr is the trace of non-negative operators on $L^2(\mathbb{R}^d)$.

SOLUTION:

Name

PROBLEM 5. (15 points) Let $V = (V_1, V_2, V_3) \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

(i) Prove that there exists a constant C, depending on V, such that

$$\left\| \frac{1}{1-\Delta} V \cdot \nabla f \right\|_2 \leqslant C \, \|f\|_2$$

for all $f \in C_0^\infty(\mathbb{R}^3)$. Recall the notation: $V \cdot \nabla = \sum_{j=1}^3 V_j \partial_{x_j}$, where $(x_1, x_2, x_3) \in \mathbb{R}^3$.

(ii) Show that the operator $\dfrac{1}{1-\Delta} V \cdot \nabla$ extends to a bounded operator T on $L^2(\mathbb{R}^3)$.

(iii) Show that T is compact.

SOLUTION:

Name

PROBLEM 6. (15 points) The purpose of this problem is to show that there exist L^2-normalisable approximate eigenstates to any positive energy. Consider the Hamiltonian $H = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$, where $V \in L^\infty(\mathbb{R})$ and $\lim_{x\to\pm\infty} V(x) = 0$. For any given $E > 0$ construct a sequence $\{\psi_n\}_{n=1}^\infty$ in $C_0^\infty(\mathbb{R})$ such that $\|\psi_n\|_2 = 1$ for any n and

$$\lim_{n\to\infty} \|(H-E)\psi_n\|_2 \;=\; 0\,.$$

SOLUTION:

Name

PROBLEM 7. (15 points) Consider the Hamiltonian $H = -\Delta - V(x)$ in three dimensions, where

$$V(x) \;=\; Z\,|x|^{-1}\,e^{-|x|}\,, \qquad x \neq 0\,,$$

and Z is a positive parameter. (You may think of H as the Hamiltonian of an hydrogenic atom where the Coulomb interaction is exponentially suppressed at large distances.)

(i) Prove that there exists a universal constant $Z_0 > 0$ such that if $Z < Z_0$ then the ground state energy E_0 of H is non negative (i.e., V does not bind any electron).

(ii) Prove that there exists a universal constant $C_0 > 0$ such that the ground state energy $E_0^f(N)$ of a system of N non-interacting fermions, each subject to the same potential V, is bounded below by $E_0^f(N) \geqslant -C_0\,Z^{5/2}$ uniformly in N.

SOLUTION:

Name

PROBLEM 8. (20 points) The purpose of this problem is to establish certain properties of the non-linear Hartree potential. Throughout the problem p shall be a fixed index in $[1,\infty]$, $V \in L^p(\mathbb{R}^3)$, and $q := \frac{4p}{2p-1}$.

(i) Show that for any $f, g, h \in L^q(\mathbb{R}^3)$ one has

$$\|(V * (fg))\, h\|_{q'} \;\leqslant\; C_q\, \|V\|_p\, \|f\|_q \|g\|_q \|h\|_q\,,$$

the constant C_q depending on q only. Here q' is the Hölder conjugate of q, that is, $q' = (1-\frac{1}{q})^{-1}$, fg is the pointwise product of f and g, and $*$ is the convolution.

(ii) Show that $G(f) := (V * |f|^2)f$ defines a continuous map $G : L^q(\mathbb{R}^3) \to L^{q'}(\mathbb{R}^3)$. (*Warning:* G is not a linear map, so it is useless to check whether G is bounded.)

SOLUTION:

Name

PROBLEM 9. (25 points) Consider a potential $V \in C(\mathbb{R}^3)$ such that $V \geqslant 0$, $V(x) \to 0$ as $|x| \to \infty$, and the ground state energy E_0 of the Hamiltonian $h = -\Delta - V$ is negative. Let $U \in C(\mathbb{R}^3)$ be such that $U \geqslant 0$, $U(x) = U(-x)$, and $U(x) \to 0$ as $|x| \to \infty$. Consider the two-body Hamiltonian

$$H \ = \ -\Delta_1 - \Delta_2 - V(x_1) - V(x_2) + U(x_1 - x_2)$$

acting on wave-functions of two variables $(x_1, x_2) \in \mathbb{R}^3 \times \mathbb{R}^3$.

(i) Prove that the fermionic ground state energy of H is bounded above as $E_0^f \leqslant E_0$.

(ii) Prove that the bosonic ground state energy of H is bounded above as $E_0^b \leqslant E_0$.

SOLUTION: